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IMPROVED JENSEN–DRAGOMIR TYPE INEQUALITIES AND APPLICATIONS

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Abstract. In this paper we establish refinements of the Jensen-Dragomir type inequalities for convex and log-convex functions. Some further generalizations of these types of inequalities via the theory of weak submajorization are also given. Several applications of the obtained inequalities for refining and reversing of the majorization inequality and the generalized triangle inequality in Banach spaces are also presented.

1. Introduction

The theory of convex functions plays an important role in different fields of pure and applied mathematics. Recall that a valued-real function f defined a convex set C in a normed space is said to be convex if the following inequality

$$f(\alpha x+(1-\alpha)y) \leqslant \alpha f(x)+(1-\alpha)f(y) \tag{1}$$

holds for all $\alpha \in [0,1]$ and $x,y \in C$. If the inequality (1) is reversed, it is then called concave on C. The general form of (1) is the famous Jensen inequality, which says that

$$J_n(f,x,\alpha) := \sum_{i=1}^{n} \alpha_i f(x_i) - f\Big(\sum_{i=1}^{n} \alpha_i x_i\Big) \geqslant 0, \tag{2}$$

holds for all convex functions f, all $x=(x_1,\dots,x_n) \in C^n$ and all $\alpha=(\alpha_1,\dots,\alpha_n) \in [0,1]^n$ with $\sum_{i=1}^{n} \alpha_i = 1$. Here, $J_n(f,x,\alpha)$ is called the normalised Jensen functional. In particular, if $\alpha_i = \frac{1}{n}$ for every $i=1,\dots,n$, we then write

$$J_n(f,x) = \frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\Big(\frac{1}{n}\sum_{i=1}^{n} x_i\Big). \tag{3}$$

The Jensen inequality is also one of the most significant features of the class of convex functions. It is usually used in settings of inequalities; and hence, it has been extended and generalized to many different frameworks, see [1, 3, 5, 6, 12] and the

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references therein. For convenience, hereafter we always use the notations $\mathscr{P}_n = \{\alpha = (\alpha_1,\dots,\alpha_n) \in [0,1]^n : \sum_{i=1}^n \alpha_i = 1\}$ and $\mathscr{P}_n^* = \{\alpha = (\alpha_1,\dots,\alpha_n) \in (0,1)^n : \sum_{i=1}^n \alpha_i = 1\}$.

In 2006, Dragomir [4] established a remarkable refinement and reverse of the Jensen inequality as follows.

THEOREM 1.1. *Let f be a convex function defined a convex set C in a normed space. Let $\alpha = (\alpha_i)_{i=1}^n \in \mathscr{P}_n$, $\beta = (\beta_i)_{i=1}^n \in \mathscr{P}_n^*$ be two weight sequences and denote by*

$$m = \min\left\{\frac{\alpha_i}{\beta_i} : i = 1,\dots,n\right\} \quad \textit{and} \quad M = \max\left\{\frac{\alpha_i}{\beta_i} : i = 1,\dots,n\right\}.$$

For any sequence of vectors $x = \{x_i\}_{i=1}^n \subset C^n$, we have

$$mJ_n(f,x,\beta) \leqslant J_n(f,x,\alpha) \leqslant MJ_n(f,x,\beta), \tag{4}$$

A direct consequence of the above theorem is as follows.

COROLLARY 1.2. *Under the hypotheses and notations of Theorem 1.1, we have*

$$n\alpha_{\min}J_n(f,x) \leqslant J_n(f,x,\alpha) \leqslant n\alpha_{\max}J_n(f,x), \tag{5}$$

where $\alpha_{\min} = \min\limits_{1\leqslant i\leqslant n} \alpha_i$ and $\alpha_{\max} = \max\limits_{1\leqslant i\leqslant n} \alpha_i$

These two inequalities are so-called Jensen-Dragomir inequalities; and, they were extended to the class of (M_φ,A)-convex functions in 2010 by F. C. Mitroi [11] and to the class of (p,h)-convex functions in 2024 by Ighachane and Bouchangour [7]. Recently, Y. Sayyari et al. [13] established new bounds for the Jensen-Dragomir functional and obtained a refinement and reverse of the Jensen-Dragomir inequality as follows.

THEOREM 1.3. *Let $\alpha = (\alpha_1,\dots,\alpha_n)$, $\beta = (\beta_1,\dots,\beta_n)$, $\gamma = (\gamma_1,\dots,,\gamma_n) \in \mathscr{P}_n$ satisfy that $\beta_i+\gamma_i > 0$ for each $i = 1,\dots,n$. If f is a convex function on an interval $I := [a,b]$ and $x = (x_1,\dots,x_n) \in I^n$, then*

$$\begin{aligned} &\min_{1\leqslant j\leqslant n}\left\{\frac{\alpha_i}{\beta_i+\gamma_i}\right\}[J_n(f,x,\beta)+J_n(f,x,\gamma)] \\ &\qquad \leqslant J_n(f,x,\alpha) \leqslant 2\max_{1\leqslant j\leqslant n}\left\{\frac{\alpha_i}{\beta_i+\gamma_i}\right\}J_n(f,x,(\beta+\gamma)/2). \end{aligned} \tag{6}$$

The first inequality in (6) is a refinement of the first inequality in (4) because

$$\min\left\{\frac{\alpha_i}{\beta_i}J_n(f,x,\beta), \frac{\alpha_i}{\gamma_i}J_n(f,x,\gamma)\right\} \leqslant \frac{\alpha_i}{\beta_i+\gamma_i}[J_n(f,x,\beta)+J_n(f,x,\gamma)],$$

for all $i \in \{1,\dots,n\}$, see [13, Remark 2.5] for the details. However, this is not a proper refinement of the first inequality in (4) because when $\beta = \gamma$, the first inequality in (6) coincides with the first inequality in (4). Moreover, the second inequalities in (6) and (4), in fact, are the same ones.

Motivated by the above mentioned results, in the present paper we give some proper refinements and reverses of the celebrated Jensen-Dragomir inequality (4). The main idea for doing this is to apply the inequalities in (5) to the difference between the quantities $\sum_{i=1}^{n}\alpha_i f(x_i)$ and $mJ_n(f,x,\beta)$ above. This idea can be repeated to get more rigorous inequalities as we wish. Next, we provide some applications of the obtained inequalities to refine and reverse the majorization inequality and the generalized triangle inequality in Banach spaces.

The paper is organized as follows. In Section 2, we establish some improved Jensen-Dragomir type inequalities for the class of convex and log-convex functions. Relying on the theory of weak submajorization, these just obtained inequalities are further generalized. In Section 3, we supply some applications of the obtained results in the previous section to the majorization inequality and the generalized triangle inequality.

2. Improved Jensen-Dragomir type inequalities for convex and log-convex functions

The main goal of this section is to establish some proper refinements and reverses of the well-known Jensen-Dragomir type inequalities for convex and log-convex functions. Firstly, we give some improved Jensen-Dragomir type inequalities for convex functions. Next, some further generalizations of these just obtained inequalities via the theory of weak submajorization is also given. Finally, we deduce some general inequalities of log-convex functions. These contents are presented respectively in three subsections below.

2.1. Improved Jensen-Dragomir type inequalities for convex functions

THEOREM 2.1. *Under the hypotheses and notations as in Theorem 1.1, we have the Jensen-Dragomir type inequalities*

$$mJ_n(f,x,\beta)+\mathfrak{m}(|J|+1)H_J \leqslant J_n(f,x,\alpha) \leqslant mJ_n(f,x,\beta)+\mathfrak{M}(|J|+1)H_J, \tag{7}$$

where $J=\{i:\alpha_i-m\beta_i\neq 0\}$, $|J|$ *is the cardinal of* J, $\mathfrak{m}=\min\limits_{i\in J}\{m,\alpha_i-m\beta_i\}$, $\mathfrak{M}=\max\limits_{i\in J}\{m,\alpha_i-m\beta_i\}$, *and*

$$H_J := \frac{1}{|J|+1}\Big[\sum_{i\in J} f(x_i)+f\Big(\sum_{i=1}^{n}\beta_i x_i\Big)\Big]-f\Big(\frac{1}{|J|+1}\Big(\sum_{i\in J}x_i+\sum_{i=1}^{n}\beta_i x_i\Big)\Big).$$

Proof. We first find that

$$\begin{aligned}\sum_{i=1}^{n}\alpha_i f(x_i)-mJ_n(f,x,\beta) &= \sum_{i=1}^{n}(\alpha_i-m\beta_i)f(x_i)+mf\Big(\sum_{i=1}^{n}\beta_i x_i\Big)\\ &= \sum_{i\in J}(\alpha_i-m\beta_i)f(x_i)+mf\Big(\sum_{i=1}^{n}\beta_i x_i\Big) =: H.\end{aligned}$$

Using the first inequality in (5) for H, we get

$$
\begin{aligned}
H &\geqslant (|J|+1)\min_{i\in J}\{m,\alpha_i-m\beta_i\}H_J+f\Big(\sum_{i\in J}(\alpha_i-m\beta_i)x_i+m\sum_{i=1}^{n}\beta_i x_i\Big)\\
&=(|J|+1)\min_{i\in J}\{m,\alpha_i-m\beta_i\}H_J+f\Big(\sum_{i=1}^{n}\alpha_i x_i\Big)\\
&=\mathfrak{m}(|J|+1)H_J+f\Big(\sum_{i=1}^{n}\alpha_i x_i\Big).
\end{aligned}
$$

This implies that

$$J_n(f,x,\alpha)\geqslant mJ_n(f,x,\beta)+\mathfrak{m}(|J|+1)H_J,$$

which is the first desired inequality.

Similarly, by the other inequality in (5), we deduce that

$$
\begin{aligned}
H &\leqslant (|J|+1)\max_{i\in J}\{m,\alpha_i-m\beta_i\}H_J+f\Big(\sum_{i\in J}(\alpha_i-m\beta_i)x_i+m\sum_{i=1}^{n}\beta_i x_i\Big)\\
&=(|J|+1)\max_{i\in J}\{m,\alpha_i-m\beta_i\}H_J+f\Big(\sum_{i=1}^{n}\alpha_i x_i\Big)\\
&=\mathfrak{M}(|J|+1)H_J+f\Big(\sum_{i=1}^{n}\alpha_i x_i\Big).
\end{aligned}
$$

Hence, we obtain

$$J_n(f,x,\alpha)\leqslant mJ_n(f,x,\beta)+\mathfrak{M}(|J|+1)H_J,$$

this completes the proof. $\square$

By taking $\beta_i=\frac{1}{n}$ for all $i=1,\ldots,n$ in the above theorem, we get the following.

COROLLARY 2.2. *Under the hypotheses and notations as in Theorem 2.1, we have*

$$n\alpha_{\min}J_n(f,x)+\mathfrak{m}_\alpha(|J|+1)U_J\leqslant J_n(f,x,\alpha)\leqslant n\alpha_{\min}J_n(f,x)+\mathfrak{M}_\alpha(|J|+1)U_J, \quad (8)$$

where $\alpha_{\min}=\min\{\alpha_1,\ldots,\alpha_n\}$, $J=\{i:\alpha_i\neq\alpha_{\min}\}$, $\mathfrak{m}_\alpha=\min_{i\in J}\{n\alpha_{\min},\alpha_i-\alpha_{\min}\}$, $\mathfrak{M}_\alpha=\max_{i\in J}\{n\alpha_{\min},\alpha_i-\alpha_{\min}\}$, *and*

$$U_J(f,x):=\frac{1}{|J|+1}\Big[\sum_{i\in J}f(x_i)+f\Big(\frac{1}{n}\sum_{i=1}^{n}x_i\Big)\Big]-f\Big(\frac{1}{|J|+1}\Big(\sum_{i\in J}x_i+\frac{1}{n}\sum_{i=1}^{n}x_i\Big)\Big).$$

REMARK 2.3. The first two inequalities in (7) and (8) are refinements of the first two inequalities in (4) and (5), respectively. However, to see that the other inequalities in (7) and (8) are better than those in (4) and (5) in some situation. For simplicity, we consider the case $J=\{1,2,\ldots,n-1\}$ and $C=I\subset\mathbb{R}$ being an interval, namely,

$\alpha_{\min} = \alpha_n$ and $x_i \in I$ for each i. Using the Jensen inequality (4), it is not difficult to check that $U_J(f,x) \leqslant J_n(f,x)$ in this case. On the other hand, observe that

$$n\alpha_{\min} + n\mathfrak{M}_\alpha \leqslant n\alpha_{\max}$$

holds if $(n+1)\alpha_{\min} \leqslant \alpha_{\max}$. Hence, in this situation we get a refinement of the second inequality in (5), that is,

$$n\alpha_{\min} J_n(f,x) + \mathfrak{M}_\alpha(|J|+1)U_J(f,x) \leqslant n\alpha_{\max} J_n(f,x).$$

Thus, these results are much better than those of [13].

REMARK 2.4. In order to establish the inequalities (7) and (8), we have already used the Jensen inequality (5) for the quantity H in the proof of Theorem 2.1. However, if we apply the inequalities in (8) for the quantity H, we will then get further refinements of the Jensen-Dragomir type inequalities, and the details are left for interesting readers.

2.2. Further generalizations via the theory of weak submajorization

THEOREM 2.5. *Let $f : C \to [0,\infty)$ be a convex function, where C is a convex set in a normed space containing vectors $\{x_i\}_{i=1}^n$. If weights $\alpha = (\alpha_i)_{i=1}^n \in \mathscr{P}_n$, $\beta = (\beta_i)_{i=1}^n \in \mathscr{P}_n^*$ and $\phi : [0,\infty) \to \mathbb{R}$ is an increasing convex function, we then have*

$$\begin{aligned}\phi\Big(\sum_{i=1}^n \alpha_i f(x_i)\Big) - \phi\circ f\Big(\sum_{i=1}^n \alpha_i x_i\Big) \geqslant\ & \phi\Big(m\sum_{i=1}^n \beta_i f(x_i)\Big) - \phi\Big(mf\Big(\sum_{i=1}^n \beta_i x_i\Big)\Big)\\ & + \phi\Big(\mathfrak{m}\Big(\sum_{i\in J} f(x_i) + f\Big(\sum_{i=1}^n \beta_i x_i\Big)\Big)\Big)\\ & - \phi\Big(\mathfrak{m}(|J|+1)f\Big(\frac{1}{|J|+1}\Big(\sum_{i\in J} x_i + \sum_{i=1}^n \beta_i x_i\Big)\Big)\Big),\end{aligned}$$

where $m, \mathfrak{m}, J$ are as in Theorem 2.1.

The main idea for proving this theorem is to utilize the theory of weak submajorization. To this end, we recall some necessary notions and features. In the whole section, we use the notation $x^* = (x_1^*, \ldots, x_n^*)$ to indicate the vector generated from the vector $x = (x_1, \ldots, x_n)$ with its components in decreasing order. Then, we say that x is weak submajorization of y, written $x \prec_w y$, if

$$\sum_{i=1}^k x_i^* \leqslant \sum_{i=1}^k y_i^* \tag{9}$$

for all $k = 1, \ldots, n$. This relation is characterized by the following result.

LEMMA 2.6. ([9, pp. 13]) *Let $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ be two vectors in $\mathbb{R}^n$ and $I\subset\mathbb{R}$ be an interval containing components of x and y. The following inequality*

$$\sum_{i=1}^{n} f(x_i) \leqslant \sum_{i=1}^{n} f(y_i)$$

holds for every continuously increasing convex function $f: I\to\mathbb{R}$ if and only if $x\prec_w y$.

Proof of Theorem 2.5. Firstly, we show vectors $X=(X_1,X_2,X_3)$ and $Y=(Y_1,Y_2,Y_3)$ with components

$$\begin{aligned}
&X_1=\sum_{i=1}^{n}\alpha_i f(x_i), \quad X_2=mf\Big(\sum_{i=1}^{n}\beta_i x_i\Big),\\
&X_3=(|J|+1)\min_{i\in J}\{m,\alpha_i-m\beta_i\}f\Big(\frac{1}{|J|+1}\Big(\sum_{i\in J}x_i+\sum_{i=1}^{n}\beta_i x_i\Big)\Big),\\
&Y_1=f\Big(\sum_{i=1}^{n}\alpha_i x_i\Big), \quad Y_2=m\sum_{i=1}^{n}\beta_i f(x_i),\\
&Y_3=\min_{i\in J}\{m,\alpha_i-m\beta_i\}\Big(\sum_{i\in J}f(x_i)+f\Big(\sum_{i=1}^{n}\beta_i x_i\Big)\Big)
\end{aligned}$$

satisfying that $Y\prec_w X$, this means that

$$\begin{aligned}
X_1^* &\geqslant Y_1^*,\\
X_1^*+X_2^* &\geqslant Y_1^*+Y_2^*,\\
X_1^*+X_2^*+X_3^* &\geqslant Y_1^*+Y_2^*+Y_3^*.
\end{aligned}$$

Clearly, by Jensen's inequality, $X_1\geqslant Y_1$, $Y_2\geqslant X_2$ and $Y_3\geqslant X_3$. Similarly, by the non-negativity of the function f and $m=\min\limits_{1\leqslant i\leqslant n}\{\frac{\alpha_i}{\beta_i}\}$, we find that

$$X_1-Y_2=\sum_{i=1}^{n}(\alpha_i-m\beta_i)f(x_i)\geqslant 0,$$

namely, $X_1\geqslant Y_2$. Next, we have

$$\begin{aligned}
\sum_{i=1}^{n}\alpha_i f(x_i)&-\min_{j\in J}\{m,\alpha_j-m\beta_j\}\sum_{i\in J}f(x_i)\\
&=\sum_{i\in J}(\alpha_i-\min_{j\in J}\{m,\alpha_j-m\beta_j\})f(x_i)+\sum_{i\notin J}\alpha_i f(x_i)\\
&\geqslant\sum_{i\in J}m\beta_i f(x_i)+\sum_{i\notin J}\alpha_i f(x_i)\\
&=m\sum_{i=1}^{n}\beta_i f(x_i)
\end{aligned}$$

$$\geqslant mf\Big(\sum_{i=1}^{n}\beta_i x_i\Big)$$
$$\geqslant \min_{j\in J}\{m,\alpha_j - m\beta_j\} f\Big(\sum_{i=1}^{n}\beta_i x_i\Big),$$

which implies that $X_1 \geqslant Y_3$.

By the first inequality in Theorem 2.1, we have

$$X_1 + X_2 + X_3 \geqslant Y_1 + Y_2 + Y_3. \tag{10}$$

It follows from this and $Y_3 \geqslant X_3$ that $Y_1 + Y_2 \leqslant X_1 + X_2 + X_3 - Y_3 \leqslant X_1 + X_2$. Also, from (10) and $Y_2 \geqslant X_2$, we infer that $Y_1 + Y_3 \leqslant X_1 + X_2 + X_3 - Y_2 \leqslant X_1 + X_3$. Finally, we have

$$X_1 + X_2 - Y_2 = \sum_{i\in J}(\alpha_i - m\beta_i) f(x_i) + mf\Big(\sum_{i=1}^{n}\beta_i x_i\Big)$$
$$\geqslant \min_{i\in J}\{m,\alpha_i - m\beta_i\}\Big(\sum_{i\in J} f(x_i) + f\Big(\sum_{i=1}^{n}\beta_i x_i\Big)\Big) = Y_3,$$

that is, $Y_2 + Y_3 \leqslant X_1 + X_2$. These facts show that $Y \prec_w X$. This, together with Lemma 2.6, yields that

$$f(X_1) + f(X_2) + f(X_3) \geqslant f(Y_1) + f(Y_2) + f(Y_2),$$

which is equivalent to the claimed inequality. □

COROLLARY 2.7. *Under the hypotheses as in Theorem 2.5 and $\lambda \geqslant 1$, we have*

$$\Big(\sum_{i=1}^{n}\alpha_i f(x_i)\Big)^{\lambda} - f^{\lambda}\Big(\sum_{i=1}^{n}\alpha_i x_i\Big) \geqslant \Big(m\sum_{i=1}^{n}\beta_i f(x_i)\Big)^{\lambda} - m^{\lambda} f^{\lambda}\Big(\sum_{i=1}^{n}\beta_i x_i\Big)$$
$$+ \mathfrak{m}^{\lambda}\Big(\sum_{i\in J} f(x_i) + f\Big(\sum_{i=1}^{n}\beta_i x_i\Big)\Big)^{\lambda}$$
$$- \mathfrak{m}^{\lambda}(|J|+1)^{\lambda} f^{\lambda}\Big(\frac{1}{|J|+1}\Big(\sum_{i\in J} x_i + \sum_{i=1}^{n}\beta_i x_i\Big)\Big),$$

where $m, \mathfrak{m}, J$ are as in Theorem 2.5.

2.3. Some results for log-convex functions

Recall that a positive function f defined on a convex set C in a normed space is called log-convex if $\log f$ is convex on C. In this subsection, by replacing f with $\log f$ in Theorem 2.5, we obtain the following results.

THEOREM 2.8. *Let $f : C \to (0,\infty)$ be a log-convex function defined on the convex set C in a normed space with vectors $\{x_i\}_{i=1}^n \subset C$. For $\alpha = (\alpha_i)_{i=1}^n \in \mathscr{P}_n$, $\beta = (\beta_i)_{i=1}^n \in \mathscr{P}_n^*$ and $\phi : [0,\infty) \to \mathbb{R}$ as in Theorem 2.5, we have*

$$\begin{aligned}
&\phi \circ \log\Big(\prod_{i=1}^n f^{\alpha_i}(x_i)\Big) - \phi \circ \log \circ f\Big(\sum_{i=1}^n \alpha_i x_i\Big)\\
&\quad \geqslant \phi \circ \log\Big(\prod_{i=1}^n f^{m\beta_i}(x_i)\Big) - \phi \circ \log\Big(f^m\Big(\sum_{i=1}^n \beta_i x_i\Big)\Big)\\
&\quad + \phi \circ \log\Big(\Big(f^{\mathfrak{m}}\Big(\sum_{i=1}^n \beta_i x_i\Big)\prod_{i\in J} f^{\mathfrak{m}}(x_i)\Big)\Big)\\
&\quad - \phi \circ \log\Big(f^{(|J|+1)\mathfrak{m}}\Big(\frac{1}{|J|+1}\Big(\sum_{i\in J} x_i + \sum_{i=1}^n \beta_i x_i\Big)\Big)\Big),
\end{aligned}$$

where $\mathfrak{m}, m, J$ are as in Theorem 2.1.

By taking $\phi(x) = \exp(\lambda x)$ with $\lambda > 0$, we obtain the following consequence.

COROLLARY 2.9. *Under the hypotheses and notations as in Theorem 2.8, we have*

$$\begin{aligned}
\prod_{i=1}^n f^{\lambda\alpha_i}(x_i) - f^{\lambda}\Big(\sum_{i=1}^n \alpha_i x_i\Big) &\geqslant \prod_{i=1}^n f^{m\lambda\beta_i}(x_i) - f^{m\lambda}\Big(\sum_{i=1}^n \beta_i x_i\Big)\\
&\quad + f^{\lambda\mathfrak{m}}\Big(\sum_{i=1}^n \beta_i x_i\Big)\prod_{i\in J} f^{\lambda\mathfrak{m}}(x_i)\\
&\quad - f^{\lambda(|J|+1)\mathfrak{m}}\Big(\frac{1}{|J|+1}\Big(\sum_{i\in J} x_i + \sum_{i=1}^n \beta_i x_i\Big)\Big),
\end{aligned}$$

where $m, \mathfrak{m}, J$ are as in Theorem 2.1.

3. Some applications to the majorization inequality and the generalized triangle inequality

This section has two main goals. The first is to give some applications of the obtained results to establish refinements and reverses of the famous majorization inequality by Hardy, Littlewood and Pólya. The second is to provide a new refinement of the generalized triangle inequality by M. Kato et al.

3.1. Refinement and reverse of majorization inequalities

In the previous section we have just seen the weak submajorization relation between vectors in a space $\mathbb{R}^n$. For two vectors $x, y \in \mathbb{R}^n$, if the equality sign in (9) is valid for $k = n$, we say that the vector x is majorized by the vector y, written $x \prec y$. The majorization is a preorder relation between vectors on $\mathbb{R}^n$, which has an important feature by Hardy, Littlewood and Pólya as follows.

THEOREM 3.1. (see [2]) *The following statements are equivalent for* $x, y \in \mathbb{R}^n$.

(i) $x \prec y$;

(ii) $\sum_{i=1}^n \phi(x_i) \leqslant \sum_{i=1}^n \phi(y_i)$ *for all continuous convex function* ϕ *defined on* $\mathbb{R}$;

(iii) x *is in the convex hull of the set* $\{z : z^* = x^*\}$ *in* $\mathbb{R}^n$;

(iv) *There exists a doubly stochastic matrix* A *of order* n *such that* $x = Ay$.

Here, a doubly stochastic matrix $A = (a_{ij})$ of order n is a square matrix of order n satisfying that each entry a_{ij} is non-negative and the sum of each row or of each columm is unit. The inequality in the statement (ii) of Theorem 3.1 is called the majorization inequality. In 2020, Duc and Hue gave a refinement of the majorization inequality of the form

$$\sum_{i=1}^n \phi(x_i) \leqslant c \sum_{i=1}^n \phi(y_i),$$

where the non-negative convex function ϕ obeys some given condition and $c \in (0,1)$ is generated from the vector x and the doubly stochastic matrix A, see [3] for details. Now, using Corollary 2.2, we establish an additive refinement and reverse of the majorization inequality.

THEOREM 3.2. *Let* $A = (a_{ij})$ *be a doubly stochastic matrix of order* n *and two vectors* $x, y \in \mathbb{R}^n$ *such that* $x = Ay$. *For each* $i = 1, \ldots, n$, *we denote by* $a_i = \min\{a_{i1}, \ldots, a_{in}\}$, $I_i := \{j : a_{ij} \neq a_i\}$, $m_i = \min_{j \in I_i}\{na_i, a_{ij} - a_i\}$ *and* $M_i = \max_{j \in I_i}\{na_i, a_{ij} - a_i\}$. *If we let* $|I_i|$ *be the cardinal of the set* I_i *for each* $i = 1, \ldots, n$, *the following series of inequalities hold*

$$\begin{aligned}
\sum_{i=1}^n \phi(x_i) &\leqslant \sum_{i=1}^n \phi(x_i) + n \sum_{i=1}^n a_i J_n(\phi, y) \\
&\leqslant \sum_{i=1}^n \phi(x_i) + n \sum_{i=1}^n a_i J_n(\phi, y) + \sum_{i=1}^n m_i(|I_i| + 1) U_{I_i}(\phi, y) \\
&\leqslant \sum_{i=1}^n \phi(y_i) \\
&\leqslant \sum_{i=1}^n \phi(x_i) + n \sum_{i=1}^n a_i J_n(\phi, y) + \sum_{i=1}^n M_i(|I_i| + 1) U_{I_i}(\phi, y),
\end{aligned}$$

where $J_n(\phi, y)$ *is defined as in* (3) *and*

$$U_{I_i}(\phi, y) := \frac{1}{|I_i| + 1}\Big[\sum_{i \in I_i} \phi(y_i) + \phi\Big(\frac{1}{n}\sum_{i=1}^n y_i\Big)\Big] - \phi\Big(\frac{1}{|I_i| + 1}\Big(\sum_{j \in I_i} y_j + \frac{1}{n}\sum_{i=1}^n y_i\Big)\Big).$$

Proof. Since $J_n(\phi, y)$, $U_{I_i}(\phi, y)$ and a_{ij}'s are non-negative, the first two inequalities are obvious. Hence, it is sufficient to prove the other inequalities. For the third inequality, we can write $x_i = \sum_{j=1}^{n} a_{ij} y_j$ by $x = Ay$ for all $i = 1, \ldots, n$. By the first inequality in (8), we have

$$\phi(x_i) = \phi\Big(\sum_{j=1}^{n} a_{ij} y_j\Big) \leqslant \sum_{j=1}^{n} a_{ij}\phi(y_j) - n a_i J_n(\phi, y) - m_i(|I_i|+1)U_{I_i}(\phi, y)$$

for all $i = 1, \ldots, n$. Adding these inequalities and noting that $\sum_{i=1}^{n} a_{ij} = 1$ for all $j = 1, \ldots, n$, we obtain

$$\begin{aligned}
\sum_{i=1}^{n} \phi(x_i) &\leqslant \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\phi(y_j) - n\sum_{i=1}^{n} a_i J_n(\phi, y) - \sum_{i=1}^{n} m_i(|I_i|+1)U_{I_i}(\phi, y) \\
&= \sum_{j=1}^{n}\Big(\sum_{i=1}^{n} a_{ij}\Big)\phi(y_j) - n\sum_{i=1}^{n} a_i J_n(\phi, y) - \sum_{i=1}^{n} m_i(|I_i|+1)U_{I_i}(\phi, y) \\
&= \sum_{j=1}^{n} \phi(y_j) - n\sum_{i=1}^{n} a_i J_n(\phi, y) - \sum_{i=1}^{n} m_i(|I_i|+1)U_{I_i}(\phi, y),
\end{aligned}$$

which yields

$$\sum_{i=1}^{n} \phi(x_i) + n\sum_{i=1}^{n} a_i J_n(\phi, y) + \sum_{i=1}^{n} m_i(|I_i|+1)U_{I_i}(\phi, y) \leqslant \sum_{j=1}^{n} \phi(y_j).$$

The last inequality is proved similarly, but using the second inequality in (8). Indeed, relying on this inequality, we have

$$\phi(x_i) = \phi\Big(\sum_{j=1}^{n} a_{ij} y_j\Big) \geqslant \sum_{j=1}^{n} a_{ij}\phi(y_j) - n a_i J_n(\phi, y) - M_i(|I_i|+1)U_{I_i}(\phi, y)$$

for all $i = 1, \ldots, n$. It follows from these inequalities that

$$\sum_{i=1}^{n} \phi(x_i) \geqslant \sum_{i=1}^{n} \phi(y_i) - n\sum_{i=1}^{n} a_i J_n(\phi, y) - \sum_{i=1}^{n} M_i(|I_i|+1)U_{I_i}(\phi, y),$$

or equivalently,

$$\sum_{i=1}^{n} \phi(y_i) \leqslant \sum_{i=1}^{n} \phi(x_i) + n\sum_{i=1}^{n} a_i J_n(\phi, y) + \sum_{i=1}^{n} M_i(|I_i|+1)U_{I_i}(\phi, y).$$

This completes the proof. □

REMARK 3.3. Under the hypotheses and notations as in Theorem 3.2, we have

$$\sum_{i=1}^{n} \phi(y_i) \leqslant \sum_{i=1}^{n} \phi(x_i) + n\sum_{i=1}^{n} A_i J_n(\phi, y), \tag{11}$$

where $A_i = \max\{a_{i1}, \ldots, a_{in}\}$ for each $i = 1, \ldots, n$. Indeed, by the second inequality in (5) and arguments as in the proof of the above theorem, we have

$$\phi(x_i) = \phi\Big(\sum_{j=1}^{n} a_{ij} y_j\Big) \geqslant \sum_{j=1}^{n} \phi(y_j) - n A_i J_n(\phi, y)$$

for all $i = 1,\ldots,n$. Hence, the desired inequality follows from these inequalities. By the inequality (11) and the last inequality in Theorem 3.2, we deduce

$$\sum_{i=1}^{n}\phi(y_i) \leqslant \sum_{i=1}^{n}\phi(x_i)+L,$$

where

$$L = \min\left\{n\sum_{i=1}^{n}A_iJ_n(\phi,y);n\sum_{i=1}^{n}a_iJ_n(\phi,y)+\sum_{i=1}^{n}M_i(|I_i|+1)U_{I_i}(\phi,y)\right\}.$$

3.2. Refinement and reverse of generalized triangle inequalities

The triangle inequality is one of the fundamental inequalities, which is equivalent to convexity. In 2007, M. Kato, K. S. Saito, T. Tamura [8] showed the sharp triangle and its reverse inequality with n elements in a Banach space, which is called the generalized triangle inequality as follows. For all nonzero elements $x_1,\ldots,x_n$ in a Banach space X, we have the following inequalities

$$\begin{aligned}\left(n-\left\|\sum_{i=1}^{n}\frac{x_i}{\|x_i\|}\right\|\right)\min_{1\leqslant i\leqslant n}\|x_i\| &\leqslant \sum_{i=1}^{n}\|x_i\|-\left\|\sum_{i=1}^{n}x_i\right\| \\ &\leqslant \left(n-\left\|\sum_{i=1}^{n}\frac{x_i}{\|x_i\|}\right\|\right)\max_{1\leqslant i\leqslant n}\|x_i\|.\end{aligned} \tag{12}$$

These inequalities were rediscovered by Dragomir [4] via Theorem 1.1. After that, the inequalities in (12) were further refined by Mitani, Saito, Kato and Tamura [10] as follows.

THEOREM 3.4. *For all nonzero elements $x_1,\ldots,x_n$ in a Banach space X satisfying that $\|x_1\| \geqslant \cdots \geqslant \|x_n\|, n \geqslant 2$, we have*

$$\begin{aligned}&\left(n-\left\|\sum_{i=1}^{n}\frac{x_i}{\|x_i\|}\right\|\right)\|x_n\|+\sum_{k=2}^{n-1}\left(k-\left\|\sum_{i=1}^{k}\frac{x_i}{\|x_i\|}\right\|\right)(\|x_k\|-\|x_{k+1}\|)\\ &\leqslant \sum_{i=1}^{n}\|x_i\|-\left\|\sum_{i=1}^{n}x_i\right\|\\ &\leqslant \left(n-\left\|\sum_{i=1}^{n}\frac{x_i}{\|x_i\|}\right\|\right)\|x_1\|-\sum_{k=2}^{n-1}\left(k-\left\|\sum_{i=n-(k-1)}^{n}\frac{x_i}{\|x_i\|}\right\|\right)(\|x_{n-k}\|-\|x_{n-(k-1)}\|),\end{aligned}$$

where x_0 and x_{n+1} are zero vectors.

In this subsection, we prove another refinements for the upper bound of (12) in the following theorem.

THEOREM 3.5. *Let $x_1,\ldots,x_n$ be $n \geqslant 2$ nonzero vectors in a Banach space. If these vectors satisfy $\|x_1\| \geqslant \|x_2\| \geqslant \ldots \geqslant \|x_{n-1}\| > \|x_n\|$, we then have*

$$\begin{aligned}\sum_{i=1}^{n}\|x_i\| - \Big\|\sum_{i=1}^{n}x_i\Big\| \leqslant \Big(n - \Big\|\sum_{i=1}^{n}\frac{x_i}{\|x_i\|}\Big\|\Big)\|x_1\| \\ - \min\Big\{n, \frac{\|x_1\|}{\|x_2\|} - 1\Big\}\Big(\sum_{i=1}^{n-1}\|x_i\| - \Big\|\sum_{i=1}^{n-1}x_i\Big\|\Big).\end{aligned} \tag{13}$$

Proof. By applying the first inequality in (8) for the function $f(x) = \|x\|$ from the Banach space X into $\mathbb{R}$ and the vectors $x_i \in X$, we have

$$\begin{aligned}n\alpha_{\min}\Big(\frac{1}{n}\sum_{i=1}^{n}\|x_i\| - \Big\|\frac{1}{n}\sum_{i=1}^{n}x_i\Big\|\Big) + n\min_{1\leqslant i\leqslant n-1}\{n\alpha_{\min}, \alpha_i - \alpha_{\min}\}U_J \\ \leqslant \sum_{i=1}^{n}\alpha_i\|x_i\| - \Big\|\sum_{i=1}^{n}\alpha_i x_i\Big\|,\end{aligned}$$

where

$$nU_J = \sum_{i=1}^{n-1}\|x_i\| + \left\|\frac{1}{n}\sum_{i=1}^{n}x_i\right\| - \left\|\sum_{i=1}^{n-1}x_i + \frac{1}{n}\sum_{i=1}^{n}x_i\right\| \geqslant \sum_{i=1}^{n-1}\|x_i\| - \left\|\sum_{i=1}^{n-1}x_i\right\|.$$

Hence, we obtain

$$\begin{aligned}&\alpha_{\min}\Big(\sum_{i=1}^{n}\|x_i\| - \Big\|\sum_{i=1}^{n}x_i\Big\|\Big) \\ &\qquad + \min_{1\leqslant i\leqslant n-1}\{n\alpha_{\min}, \alpha_i - \alpha_{\min}\}\left(\sum_{i=1}^{n-1}\|x_i\| - \left\|\sum_{i=1}^{n-1}x_i\right\|\right) \\ &\leqslant \sum_{i=1}^{n}\alpha_i\|x_i\| - \Big\|\sum_{i=1}^{n}\alpha_i x_i\Big\|.\end{aligned} \tag{14}$$

Since x_i's are nonzero vectors with their norms in decreasing order, by taking $\alpha_i = \frac{1}{\|x_i\|}\big(\sum_{j=1}^{n}\frac{1}{\|x_j\|}\big)^{-1}$ for all $i = 1,\ldots,n$ in the inequality (14), it follows that

$$\begin{aligned}&\frac{1}{\|x_1\|}\Big(\sum_{i=1}^{n}\|x_i\| - \Big\|\sum_{i=1}^{n}x_i\Big\|\Big) \\ &\qquad + \min_{1\leqslant i\leqslant n-1}\left\{\frac{n}{\|x_1\|}, \frac{1}{\|x_2\|} - \frac{1}{\|x_1\|}\right\}\left(\sum_{i=1}^{n-1}\|x_i\| - \left\|\sum_{i=1}^{n-1}x_i\right\|\right) \\ &\leqslant n - \Big\|\sum_{i=1}^{n}\frac{x_i}{\|x_i\|}\Big\|,\end{aligned}$$

or equivalently,

$$\sum_{i=1}^{n}\|x_i\|-\Big\|\sum_{i=1}^{n}x_i\Big\|+\min_{1\leqslant i\leqslant n-1}\left\{n,\frac{\|x_1\|}{\|x_2\|}-1\right\}\left(\sum_{i=1}^{n-1}\|x_i\|-\Big\|\sum_{i=1}^{n-1}x_i\Big\|\right)$$
$$\leqslant\left(n-\Big\|\sum_{i=1}^{n}\frac{x_i}{\|x_i\|}\Big\|\right)\|x_1\|.$$

Clearly, this inequality is equivalent to the desired inequality. □

REMARK 3.6. In the below arguments, we prove that under the condition

$$\begin{aligned}&\left((n-1)-\Big\|\sum_{i=2}^{n}\frac{x_i}{\|x_i\|}\Big\|\right)(\|x_1\|-\|x_{n-1}\|)\\&\qquad\leqslant\min\left\{n,\frac{\|x_1\|}{\|x_2\|}-1\right\}\left((n-1)-\Big\|\sum_{i=1}^{n-1}\frac{x_i}{\|x_i\|}\Big\|\right)\|x_{n-1}\|,\end{aligned}\tag{15}$$

the inequality (13) is better than the second inequality in Theorem 3.4, which is equivalent to

$$\begin{aligned}&\sum_{k=2}^{n-1}\left(k-\Big\|\sum_{i=n-(k-1)}^{n}\frac{x_i}{\|x_i\|}\Big\|\right)(\|x_{n-k}\|-\|x_{n-(k-1)}\|)\\&\qquad\leqslant\min\left\{n,\frac{\|x_1\|}{\|x_2\|}-1\right\}\left(\sum_{i=1}^{n-1}\|x_i\|-\Big\|\sum_{i=1}^{n-1}x_i\Big\|\right).\end{aligned}$$

Indeed, by the triangle inequality, it is easy to check that,

$$k-\Big\|\sum_{i=n-(k-1)}^{n}\frac{x_i}{\|x_i\|}\Big\|\leqslant(k+1)-\Big\|\sum_{i=n-k}^{n}\frac{x_i}{\|x_i\|}\Big\|\quad\text{for all } k=2,\ldots,n-2.$$

Hence, combining with (15) and (12), we get

$$\begin{aligned}&\sum_{k=2}^{n-1}\left(k-\Big\|\sum_{i=n-(k-1)}^{n}\frac{x_i}{\|x_i\|}\Big\|\right)(\|x_{n-k}\|-\|x_{n-(k-1)}\|)\\&\qquad\leqslant\left((n-1)-\Big\|\sum_{i=2}^{n}\frac{x_i}{\|x_i\|}\Big\|\right)\sum_{k=2}^{n-1}(\|x_{n-k}\|-\|x_{n-(k-1)}\|)\\&\qquad=\left((n-1)-\Big\|\sum_{i=2}^{n}\frac{x_i}{\|x_i\|}\Big\|\right)(\|x_1\|-\|x_{n-1}\|)\\&\qquad\leqslant\min\left\{n,\frac{\|x_1\|}{\|x_2\|}-1\right\}\left((n-1)-\Big\|\sum_{i=1}^{n-1}\frac{x_i}{\|x_i\|}\Big\|\right)\|x_{n-1}\|\\&\qquad\leqslant\min\left\{n,\frac{\|x_1\|}{\|x_2\|}-1\right\}\left(\sum_{i=1}^{n-1}\|x_i\|-\Big\|\sum_{i=1}^{n-1}x_i\Big\|\right).\end{aligned}$$

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