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# NOVEL BOUNDS FOR THE EUCLIDEAN OPERATOR RADIUS OF HILBERT SPACE OPERATOR PAIRS

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*Abstract.* This paper aims to establish new upper bounds for the Euclidean operator radius concerning pairs of bounded linear operators in a complex Hilbert space. To achieve this objective, we utilize some Boas-Bellman type inequalities as proof tools. Furthermore, we extend our findings to derive novel upper bounds for the numerical radius of operators in Hilbert spaces. These results contribute to advancing our understanding and analytical capabilities regarding operator properties within the framework of Hilbert spaces.

## 1. Introduction

Throughout this paper, we work within a complex Hilbert space $\mathscr{H}$ equipped with an inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. The $C^*$-algebra $\mathscr{B}(\mathscr{H})$ encompasses all bounded linear operators on $\mathscr{H}$, including the identity operator $I$. An operator $A\in\mathscr{B}(\mathscr{H})$ is considered positive if $\langle Ax,x\rangle \geqslant 0$ for all $x\in\mathscr{H}$, denoted as $A\geqslant 0$. Moreover, for any positive bounded linear operator $A$, a unique positive bounded linear operator $A^{\frac{1}{2}}$ exists such that $A=(A^{\frac{1}{2}})^2$. We also introduce the absolute value of $A$ defined as $|A|=(A^*A)^{\frac{1}{2}}$.

Let $A\in\mathscr{B}(\mathscr{H})$ be a bounded linear operator. We define the operator norm $\|A\|$ as

$$\|A\| = \sup\{\|Ax\|\,;\, x\in\mathscr{H},\, \|x\|=1\},$$

and the numerical radius of $A$, denoted by $\omega(A)$, as

$$\omega(A) = \sup\left\{\left|\langle Ax,x\rangle\right|\,;\, x\in\mathscr{H},\, \|x\|=1\right\}.$$

It is readily verified that $\omega(A)\leqslant\|A\|$. It is well known that the numerical radius function $w(\cdot)$ defines a norm on $\mathscr{B}(\mathscr{H})$ that is equivalent to the operator norm. For any $A\in\mathscr{B}(\mathscr{H})$, the following inequality holds:

$$\frac{1}{2}\|A\| \leqslant w(A) \leqslant \|A\|.$$

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Additional details and properties of the numerical radius can be explored in references [3–8, 11–16, 20].

Following the pioneering work of Popescu [17], we explore the notion of the *Euclidean operator radius* for a pair $(C,D)$ of bounded linear operators defined on a Hilbert space $\mathscr{H}$. It is noteworthy that in [17], the author introduced this concept for an $n$-tuple of operators and highlighted its key properties. Consider a pair of bounded linear operators $(C,D)$ on $\mathscr{H}$. The *Euclidean operator radius* is defined by:

$$w_e(C,D) := \sup_{\substack{x\in\mathscr{H},\\ \|x\|=1}} \sqrt{\left|\langle Cx,x\rangle\right|^2 + \left|\langle Dx,x\rangle\right|^2}.$$

As established in [17], $w_e : \mathscr{B}(\mathscr{H}) \times \mathscr{B}(\mathscr{H}) \to [0,\infty)$ is a norm, and the following inequality holds:

$$\frac{\sqrt{2}}{4}\sqrt{\|C^*C + D^*D\|} \leqslant w_e(C,D) \leqslant \sqrt{\|C^*C + D^*D\|}, \tag{1}$$

where the constants $\frac{\sqrt{2}}{4}$ and 1 are best possible in (1). The study of the Euclidean operator radius of a 2-tuple of operators is important because it helps us understand how pairs of operators affect vectors in Hilbert spaces. This understanding has applications in different fields of mathematics and physics. By finding new upper bounds for the Euclidean operator radius, this research improves our ability to analyze and compare operators. Mathematical inequalities, such as the Boas-Bellman type, play a significant role in establishing these bounds. For more information on the Euclidean operator radius and related inequalities, readers can refer to the works [18, 19] and the references therein.

This paper is structured as follows: In Section 2, we provide a summary of several useful inequalities. Some of these inequalities will play a crucial role in proving our main results.

Section 3 focuses on our main results. Utilizing some types of Boas-Bellman inequalities, we establish multiple upper bounds for the Euclidean radius of a pair of Hilbert space operators. Additionally, we explore the practical applications of our main findings. Specifically, we present a set of inequalities for the numerical radius of a Hilbert space operator.

## 2. Preliminary results

In this section, we present an overview of several important inequalities. Some of them will be used in the proof of our main results.

Let us begin by revisiting an inequality obtained by the second author in [10]. It asserts that for $B,C \in \mathscr{B}(\mathscr{H})$,

$$w_e(B,C) \leqslant \sqrt{\max\left\{\|B\|^2, \|C\|^2\right\} + w(C^*B)}. \tag{2}$$

The inequality (2) is known to be sharp.

Also, we have the following inequality:

$$w_e(B,C) \leqslant \frac{\sqrt{2}}{2}\sqrt{\left[\left\||B|^2+|C|^2\right\|+\left\||B|^2-|C|^2\right\|\right]+w(C^*B)}. \tag{3}$$

The inequality (3) is known to be sharp as well.

It is noteworthy that both of these inequalities were established through the use of the following vector inequality which is stated in [10, Eq. (2.26)]:

$$\left|\langle x,y\rangle\right|^2+\left|\langle x,z\rangle\right|^2 \leqslant \|x\|^2\left[\max\left\{\|y\|^2,\|z\|^2\right\}+\left|\langle y,z\rangle\right|\right] \tag{4}$$

for all $x,y,z \in \mathscr{H}$.

In 1941, R. P. Boas [2] and independently in 1944, R. Bellman [1] established a generalization of Bessel's inequality. For vectors $x,y_1,\ldots,y_n$ in $\mathscr{H}$, the inequality given by:

$$\sum_{i=1}^{n}\left|\langle x,y_i\rangle\right|^2 \leqslant \|x\|^2\left[\max_{1\leqslant i\leqslant n}\|y_i\|^2+\left(\sum_{1\leqslant i\neq j\leqslant n}\left|\langle y_i,y_j\rangle\right|^2\right)^{\frac{1}{2}}\right], \tag{5}$$

holds.

By substituting $n=2$, $y_1=y$, and $y_2=z$ into equation (5), we can deduce the following inequality:

$$\left|\langle x,y\rangle\right|^2+\left|\langle x,z\rangle\right|^2 \leqslant \|x\|^2\left[\max\left\{\|y\|^2,\|z\|^2\right\}+\sqrt{2}\left|\langle y,z\rangle\right|\right]. \tag{6}$$

This inequality holds for every $x,y,z \in \mathscr{H}$. However, it is evident that inequality (4) is stronger and provides a better bound than (6).

In [9], the second author derived a similar result:

$$\sum_{i=1}^{n}\left|\langle x,y_i\rangle\right|^2 \leqslant \|x\|\max_{1\leqslant i\leqslant n}\left|\langle x,y_i\rangle\right|\left\{\sum_{i=1}^{n}\|y_i\|^2+\sum_{1\leqslant i\neq j\leqslant n}\left|\langle y_i,y_j\rangle\right|\right\}^{\frac{1}{2}}, \tag{7}$$

for any $x,y_1,\ldots,y_n$ in $\mathscr{H}$.

The inequalities (4) and the special case when $n=2$ in (7) play a crucial role in the upcoming section. These inequalities are important tools that we will use to derive meaningful power bounds for the square of the Euclidean operator radius, denoted as $\omega_e^2(B,C)$. In particular, we will explore how these inequalities contribute to understanding and bounding the Euclidean operator radius in the context of the operators $B$ and $C$. The analysis aims to uncover the behavior of $\omega_e^2(B,C)$ under certain conditions and shed light on its properties based on the established inequalities (4) and (7).

## 3. Main results

In this section, we will discuss our main findings. One of our important contributions is presented in the upcoming theorem, where the main tool we use is the inequality (4).

THEOREM 1. *For all $B,C \in \mathscr{B}(\mathscr{H})$ and $r \geqslant 1$, we have*

$$\omega_e^2(B,C) \leqslant 2^{1-\frac{1}{r}} \left[\max\left\{\|B\|^{2r}, \|C\|^{2r}\right\} + \omega^r(C^*B)\right]^{\frac{1}{r}}, \tag{8}$$

*and*

$$\omega_e^2(B,C) \leqslant 2^{1-\frac{1}{r}} \left[\frac{\left\||B|^2+|C|^2\right\|^r + \left\||B|^2-|C|^2\right\|^r}{2} + \omega^r(C^*B)\right]^{\frac{1}{r}}. \tag{9}$$

*Proof.* From (4), we get

$$\begin{aligned}\left|\langle x,y\rangle\right|^2 + \left|\langle x,z\rangle\right|^2 &\leqslant \|x\|^2\left[\max\left\{\|y\|^2,\|z\|^2\right\} + \left|\langle y,z\rangle\right|\right] \\ &= 2\|x\|^2\left[\frac{\max\left\{\|y\|^2,\|z\|^2\right\} + \left|\langle y,z\rangle\right|}{2}\right]\end{aligned}$$

for all $x,y,z \in \mathscr{H}$.

If we take the power $r \geqslant 1$ and use the convexity of the power $r$, then we get

$$\begin{aligned}\left(\left|\langle x,y\rangle\right|^2 + \left|\langle x,z\rangle\right|^2\right)^r &\leqslant 2^r\|x\|^{2r}\left[\frac{\max\left\{\|y\|^2,\|z\|^2\right\} + \left|\langle y,z\rangle\right|}{2}\right]^r \\ &\leqslant 2^r\|x\|^{2r}\frac{\left(\max\left\{\|y\|^2,\|z\|^2\right\}\right)^r + \left|\langle y,z\rangle\right|^r}{2},\end{aligned}$$

for all $x,\ y,\ z \in \mathscr{H}$. This implies that

$$\left(\left|\langle x,y\rangle\right|^2 + \left|\langle x,z\rangle\right|^2\right)^r \leqslant 2^{r-1}\|x\|^{2r}\left[\max\left\{\|y\|^{2r},\|z\|^{2r}\right\} + \left|\langle y,z\rangle\right|^r\right] \tag{10}$$

for every $x,\ y,\ z \in \mathscr{H}$. If we take $y = Bx,\ z = Cx$ with $x \in \mathscr{H}$ and $\|x\| = 1$, then we get from (10) that

$$\begin{aligned}\left(\left|\langle x,Bx\rangle\right|^2 + \left|\langle x,Cx\rangle\right|^2\right)^r &\leqslant 2^{r-1}\left[\left(\max\left\{\|Bx\|^2,\|Cx\|^2\right\}\right)^r + \left|\langle Bx,Cx\rangle\right|^r\right] \\ &= 2^{r-1}\left[\left(\max\left\{\|Bx\|^2,\|Cx\|^2\right\}\right)^r + \left|\langle C^*Bx,x\rangle\right|^r\right].\end{aligned} \tag{11}$$

If we take the supremum over all $x \in \mathscr{H}$ with $\|x\| = 1$, then we get

$$\begin{aligned}
\omega_e^{2r}(B,C) &= \sup_{\|x\|=1} \left( \left|\left\langle x, Bx \right\rangle\right|^2 + \left|\left\langle x, Cx \right\rangle\right|^2 \right)^r \\
&\leqslant 2^{r-1} \sup_{\|x\|=1} \left[ \left( \max\left\{ \|Bx\|^2, \|Cx\|^2 \right\} \right)^r + \left|\left\langle C^*Bx, x \right\rangle\right|^r \right] \\
&\leqslant 2^{r-1} \left[ \left( \sup_{\|x\|=1} \max\left\{ \|Bx\|^2, \|Cx\|^2 \right\} \right)^r + \sup_{\|x\|=1} \left|\left\langle C^*Bx, x \right\rangle\right|^r \right] \\
&= 2^{r-1} \left[ \left( \max\left\{ \sup_{\|x\|=1} \|Bx\|^2, \sup_{\|x\|=1} \|Cx\|^2 \right\} \right)^r + \sup_{\|x\|=1} \left|\left\langle C^*Bx, x \right\rangle\right|^r \right] \\
&= 2^{r-1} \left[ \left( \max\left\{ \|B\|^2, \|C\|^2 \right\} \right)^r + \omega^r(C^*B) \right] \\
&= 2^{r-1} \left[ \max\left\{ \|B\|^{2r}, \|C\|^{2r} \right\} + \omega^r(C^*B) \right],
\end{aligned}$$

which proves (8).

Observe also that

$$\begin{aligned}
\max\left\{ \|Bx\|^2, \|Cx\|^2 \right\} &= \frac{1}{2}\left( \|Bx\|^2 + \|Cx\|^2 \right) + \frac{1}{2}\left| \|Bx\|^2 - \|Cx\|^2 \right| \\
&= \frac{1}{2}\left( \left\langle |B|^2 x, x \right\rangle + \left\langle |C|^2 x, x \right\rangle \right) + \frac{1}{2}\left| \left\langle |B|^2 x, x \right\rangle - \left\langle |C|^2 x, x \right\rangle \right| \\
&= \frac{1}{2}\left( \left\langle \left( |B|^2 + |C|^2 \right) x, x \right\rangle + \left| \left\langle \left( |B|^2 - |C|^2 \right) x, x \right\rangle \right| \right)
\end{aligned}$$

and by the convexity of power function we have

$$\begin{aligned}
\left( \max\left\{ \|Bx\|^2, \|Cx\|^2 \right\} \right)^r &= \left( \frac{\left\langle \left( |B|^2 + |C|^2 \right) x, x \right\rangle + \left| \left\langle \left( |B|^2 - |C|^2 \right) x, x \right\rangle \right|}{2} \right)^r \\
&\leqslant \frac{\left\langle \left( |B|^2 + |C|^2 \right) x, x \right\rangle^r + \left| \left\langle \left( |B|^2 - |C|^2 \right) x, x \right\rangle \right|^r}{2}
\end{aligned}$$

and by (11) we obtain

$$\begin{aligned}
&\left( \left|\left\langle x, Bx \right\rangle\right|^2 + \left|\left\langle x, Cx \right\rangle\right|^2 \right)^r \\
&\leqslant 2^{r-1} \left[ \frac{\left\langle \left( |B|^2 + |C|^2 \right) x, x \right\rangle^r + \left| \left\langle \left( |B|^2 - |C|^2 \right) x, x \right\rangle \right|^r}{2} + \left|\left\langle C^*Bx, x \right\rangle\right|^r \right]
\end{aligned} \tag{12}$$

for $x \in \mathscr{H}$ and $\|x\| = 1$.

If we take the supremum in (12) over all $x \in \mathscr{H}$ with $\|x\| = 1$, then we get

$$\begin{aligned}
\omega_e^{2r}(B,C) &= \sup_{\|x\|=1} \left( \left|\left\langle x, Bx \right\rangle\right|^2 + \left|\left\langle x, Cx \right\rangle\right|^2 \right)^r \\
&\leqslant 2^{r-1} \sup_{\|x\|=1} \left[ \frac{\left\langle \left(|B|^2+|C|^2\right)x,x\right\rangle^r + \left|\left\langle \left(|B|^2-|C|^2\right)x,x\right\rangle\right|^r}{2} + \left|\left\langle C^*Bx,x\right\rangle\right|^r \right] \\
&\leqslant 2^{r-1} \left[ \frac{\sup_{\|x\|=1} \left\langle \left(|B|^2+|C|^2\right)x,x\right\rangle^r + \sup_{\|x\|=1} \left|\left\langle \left(|B|^2-|C|^2\right)x,x\right\rangle\right|^r}{2} \right] \\
&\quad + 2^{r-1} \sup_{\|x\|=1} \left|\left\langle C^*Bx,x\right\rangle\right|^r \\
&= 2^{r-1} \left[ \frac{\left\||B|^2+|C|^2\right\|^r + \left\||B|^2-|C|^2\right\|^r}{2} + \omega^r\left(C^*B\right) \right],
\end{aligned}$$

which proves (9). $\square$

REMARK 1. (1) By setting $r = 1$ in Theorem 1, we can derive the inequalities (2) and (3) mentioned in Section 2.

(2) When we substitute $r = 2$ in Theorem 1, we can derive the following inequalities:

$$\omega_e^2(B,C) \leqslant \sqrt{2}\left[\max\left\{\|B\|^4, \|C\|^4\right\} + \omega^2\left(C^*B\right)\right]^{\frac{1}{2}}$$

and

$$\omega_e^2(B,C) \leqslant \sqrt{2}\left[ \frac{\left\||B|^2+|C|^2\right\|^2 + \left\||B|^2-|C|^2\right\|^2}{2} + \omega^2\left(C^*B\right) \right]^{\frac{1}{2}}.$$

As a consequence of Theorem 1, we can establish the following corollary, which provides two numerical radius inequalities for an operator $A \in \mathscr{B}(\mathscr{H})$.

COROLLARY 1. *Let $A \in \mathscr{B}(\mathscr{H})$. Then, the following inequalities*

$$w^2(A) \leqslant \frac{1}{2^{\frac{1}{r}}}\left[\|A\|^{2r} + \omega^r\left(A^2\right)\right]^{\frac{1}{r}}$$

*and*

$$w^2(A) \leqslant \frac{1}{2^{\frac{1}{r}}}\left[ \frac{\left\||A|^2+|A^*|^2\right\|^r + \left\||A|^2-|A^*|^2\right\|^r}{2} + \omega^r\left(A^2\right) \right]^{\frac{1}{r}}.$$

*hold for all $r \geqslant 1$.*

*Proof.* Follows immediately by taking $B = A$ and $C = A^*$ in Theorem 1 and then using the fact that

$$\omega_e^2(A, A^*) = 2w^2(A). \tag{13}$$

□

The special cases $r = 1$ and $r = 2$ have particular significance and are stated in the following remark:

REMARK 2. (1) For $r = 1$ in Corollary 1, we obtain the following inequalities:

$$w^2(A) \leqslant \frac{1}{2}\left[\|A\|^2 + \omega\left(A^2\right)\right] \tag{14}$$

and

$$w^2(A) \leqslant \frac{1}{2}\left[\frac{\left\||A|^2 + |A^*|^2\right\| + \left\||A|^2 - |A^*|^2\right\|}{2} + \omega\left(A^2\right)\right].$$

(2) For $r = 2$ in Corollary 1, we have:

$$w^2(A) \leqslant \frac{\sqrt{2}}{2}\left[\|A\|^4 + \omega^2\left(A^2\right)\right]^{\frac{1}{2}}$$

and

$$w^2(A) \leqslant \frac{\sqrt{2}}{2}\left[\frac{\left\||A|^2 + |A^*|^2\right\|^2 + \left\||A|^2 - |A^*|^2\right\|^2}{2} + \omega^2\left(A^2\right)\right]^{\frac{1}{2}}.$$

Another consequence of Theorem 1 can be observed in the Cartesian decomposition of $A = B + iC$, where $B$ and $C$ are defined as self-adjoint operators:

$$B = \frac{A + A^*}{2} \quad \text{and} \quad C = \frac{A - A^*}{2i}. \tag{15}$$

This leads us to the following corollary:

COROLLARY 2. *Let $A \in \mathscr{B}(\mathscr{H})$. Then, for all $r \geqslant 1$, we have:*

$$w^2(A) \leqslant \frac{1}{2^{1+\frac{1}{r}}}\left[\max\left\{\|A + A^*\|^{2r}, \|A - A^*\|^{2r}\right\} + \omega^r\left((A - A^*)(A + A^*)\right)\right]^{\frac{1}{r}} \tag{16}$$

*and*

$$w^2(A) \leqslant \frac{1}{2^{\frac{1}{r}}}\left[\frac{\left\||A|^2 + |A^*|^2\right\|^r + \left\|A^2 + (A^*)^2\right\|^r}{2} + \frac{1}{2^r}\omega^r\left((A - A^*)(A + A^*)\right)\right]^{\frac{1}{r}}. \tag{17}$$

*Proof.* Let $A = B + iC$ be the Cartesian decomposition of $A$ as given in (15). It can be observed that:

$$w_e^2(B,C) = w^2(A)$$

and

$$B^2 + C^2 = \frac{|A|^2 + |A^*|^2}{2}.$$

So, the desired inequalities can be derived by applying Theorem 1 to the pair $(B,C)$. □

REMARK 3. (1) If we substitute $r = 1$ in (16) and (17), we obtain the following inequalities, as stated in [10, Eq. (2.29)] and [10, Eq. (2.36)]:

$$w^2(A) \leqslant \frac{1}{4}\left[\max\left\{\|A+A^*\|^2, \|A-A^*\|^2\right\} + \omega\left((A-A^*)(A+A^*)\right)\right]$$

and

$$w^2(A) \leqslant \frac{1}{2}\left[\frac{\left\||A|^2+|A^*|^2\right\| + \left\|A^2+(A^*)^2\right\|}{2} + \frac{1}{2}\omega\left((A-A^*)(A+A^*)\right)\right].$$

(2) For $r = 2$ in (16) and (17) we obtain

$$w^2(A) \leqslant \frac{\sqrt{2}}{4}\left[\max\left\{\|A+A^*\|^4, \|A-A^*\|^4\right\} + \omega^2\left((A-A^*)(A+A^*)\right)\right]^{\frac{1}{2}}$$

and

$$w^2(A) \leqslant \frac{\sqrt{2}}{2}\left[\frac{\left\||A|^2+|A^*|^2\right\|^2 + \left\|A^2+(A^*)^2\right\|^2}{2} + \frac{1}{4}\omega^2\left((A-A^*)(A+A^*)\right)\right]^{\frac{1}{2}}.$$

Our next result reads as follows.

THEOREM 2. *For all $B, C \in \mathscr{B}(\mathscr{H})$ and $r \geqslant 2$, we have:*

$$\omega_e^2(B,C) \leqslant 2^{1-\frac{1}{r}} \max\{\omega(B), \omega(C)\}\left\{\left\|\frac{|B|^2+|C|^2}{2}\right\|^{\frac{r}{2}} + \omega^{\frac{r}{2}}(C^*B)\right\}^{\frac{1}{r}}. \tag{18}$$

*Proof.* By applying (7) for $n = 2$, $y_1 = y$ and $y_2 = z$, we obtain

$$\begin{aligned} &\left|\langle x,y\rangle\right|^2 + \left|\langle x,z\rangle\right|^2 \qquad (19)\\ &\leqslant \|x\| \max\left\{\left|\langle x,y\rangle\right|, \left|\langle x,z\rangle\right|\right\}\left\{\|y\|^2 + \|z\|^2 + 2\left|\langle y,z\rangle\right|\right\}^{\frac{1}{2}} \\ &= \sqrt{2}\|x\| \max\left\{\left|\langle x,y\rangle\right|, \left|\langle x,z\rangle\right|\right\}\left\{\frac{\|y\|^2+\|z\|^2}{2} + \left|\langle y,z\rangle\right|\right\}^{\frac{1}{2}} \end{aligned}$$

for all $x,\ y,\ z \in \mathscr{H}$.

If we take $y = Bx,\ z = Cx$ with $x \in \mathscr{H}$ and $\|x\| = 1$, then we get from (19) that

$$\begin{aligned}
&\left|\langle x,Bx\rangle\right|^2 + \left|\langle x,Cx\rangle\right|^2 \\
&\leqslant \sqrt{2}\max\left\{\left|\langle x,Bx\rangle\right|,\left|\langle x,Cx\rangle\right|\right\}\left\{\frac{\|Bx\|^2+\|Cx\|^2}{2}+\left|\langle Bx,Cx\rangle\right|\right\}^{\frac{1}{2}} \\
&= \sqrt{2}\max\left\{\left|\langle x,Bx\rangle\right|,\left|\langle x,Cx\rangle\right|\right\}\left\{\left\langle\left(\frac{|B|^2+|C|^2}{2}\right)x,x\right\rangle+\left|\langle C^*Bx,x\rangle\right|\right\}^{\frac{1}{2}}.
\end{aligned}$$

Furthermore, if we consider the power $r \geqslant 2$ and utilize the convexity of the power function, we obtain:

$$\begin{aligned}
&\left(\left|\langle x,Bx\rangle\right|^2 + \left|\langle x,Cx\rangle\right|^2\right)^r \\
&\leqslant 2^{\frac{r}{2}}\max\left\{\left|\langle x,Bx\rangle\right|^r,\left|\langle x,Cx\rangle\right|^r\right\}\left\{\left\langle\left(\frac{|B|^2+|C|^2}{2}\right)x,x\right\rangle+\left|\langle C^*Bx,x\rangle\right|\right\}^{\frac{r}{2}} \\
&= 2^{\frac{r}{2}}\max\left\{\left|\langle x,Bx\rangle\right|^r,\left|\langle x,Cx\rangle\right|^r\right\}2^{\frac{r}{2}}\left\{\frac{\left\langle\left(\frac{|B|^2+|C|^2}{2}\right)x,x\right\rangle+\left|\langle C^*Bx,x\rangle\right|}{2}\right\}^{\frac{r}{2}} \\
&\leqslant 2^r\max\left\{\left|\langle x,Bx\rangle\right|^r,\left|\langle x,Cx\rangle\right|^r\right\}\frac{\left\langle\left(\frac{|B|^2+|C|^2}{2}\right)x,x\right\rangle^{\frac{r}{2}}+\left|\langle C^*Bx,x\rangle\right|^{\frac{r}{2}}}{2},
\end{aligned}$$

for all $x \in \mathscr{H},\ \|x\| = 1$. This implies that

$$\begin{aligned}
&\left(\left|\langle x,Bx\rangle\right|^2 + \left|\langle x,Cx\rangle\right|^2\right)^r \\
&\leqslant 2^{r-1}\max\left\{\left|\langle x,Bx\rangle\right|^r,\left|\langle x,Cx\rangle\right|^r\right\}\left[\left\langle\left(\frac{|B|^2+|C|^2}{2}\right)x,x\right\rangle^{\frac{r}{2}}+\left|\langle C^*Bx,x\rangle\right|^{\frac{r}{2}}\right]
\end{aligned} \tag{20}$$

for all $x \in \mathscr{H},\ \|x\| = 1$. If we take the supremum in (20) over all $x \in \mathscr{H}$ with $\|x\| = 1$, then we get (18). $\square$

The following special cases are derived from Theorem 2.

REMARK 4. If we substitute $r = 2$ in Theorem 2, we obtain the inequality:

$$\omega_e^2(B,C) \leqslant \sqrt{2}\max\{\omega(B),\omega(C)\}\left\{\left\|\frac{|B|^2+|C|^2}{2}\right\|+\omega(C^*B)\right\}^{\frac{1}{2}}.$$

Similarly, for $r = 4$, we have:

$$\omega_e^2(B,C) \leqslant 2\sqrt{2}\max\{\omega(B),\omega(C)\}\left\{\left\|\frac{|B|^2+|C|^2}{2}\right\|^2+\omega^2(C^*B)\right\}^{\frac{1}{4}}.$$

Theorem 2 has an important application, which is derived in the following corollary.

COROLLARY 3. *Let $A \in \mathscr{B}(\mathscr{H})$. Then, for all $r \geqslant 2$, we have:*

$$w^2(A) \leqslant \frac{1}{4^{\frac{1}{r}}}\left\{\left\|\frac{|A|^2+|A^*|^2}{2}\right\|^{\frac{r}{2}}+\omega^{\frac{r}{2}}\left(A^2\right)\right\}^{\frac{2}{r}}.$$

*Proof.* This result follows by taking $B = A$ and $C = A^*$ in Theorem 2 and then proceeding as in Corollary 1. □

REMARK 5. For $r = 2$ in Corollary 3, we obtain the following inequality:

$$w^2(A) \leqslant \frac{1}{2}\left\{\left\|\frac{|A|^2+|A^*|^2}{2}\right\|+\omega\left(A^2\right)\right\}.$$

This inequality holds for all operators $A \in \mathscr{B}(\mathscr{H})$. Similarly, for $r = 4$, the following inequality

$$w^2(A) \leqslant \frac{\sqrt{2}}{2}\left\{\left\|\frac{|A|^2+|A^*|^2}{2}\right\|^{2}+\omega^{2}\left(A^2\right)\right\}^{\frac{1}{2}}.$$

holds for every $A \in \mathscr{B}(\mathscr{H})$.

Another application of Theorem 2 can be seen through the Cartesian decomposition of $A$, which is stated in the following corollary. The proof follows a similar approach as in Corollary 2.

COROLLARY 4. *Let $A \in \mathscr{B}(\mathscr{H})$. For all $r \geqslant 2$, the inequality is given by:*

$$\begin{aligned} &w^2(A) \\ &\leqslant \frac{1}{2^{1+\frac{1}{r}}}\max\left\{\|A+A^*\|,\|A-A^*\|\right\}\left\{\left\||A|^2+|A^*|^2\right\|^{\frac{r}{2}}+\omega^{\frac{r}{2}}\left(\left(A-A^*\right)\left(A+A^*\right)\right)\right\}^{\frac{1}{r}}. \end{aligned}$$

Some special cases of interest are also stated in the following remark.

REMARK 6. For $r = 2$ in the above corollary, we obtain

$$w^2(A) \leqslant \frac{\sqrt{2}}{4}\max\left\{\|A+A^*\|,\|A-A^*\|\right\}\left\{\left\||A|^2+|A^*|^2\right\|+\omega\left(\left(A-A^*\right)\left(A+A^*\right)\right)\right\}^{\frac{1}{2}},$$

while for $r = 4$, we get

$$\begin{aligned} &w^2(A) \\ &\leqslant \frac{\sqrt[4]{8}}{4}\max\left\{\|A+A^*\|,\|A-A^*\|\right\}\left\{\left\||A|^2+|A^*|^2\right\|^{2}+\omega^{2}\left(\left(A-A^*\right)\left(A+A^*\right)\right)\right\}^{\frac{1}{4}}. \end{aligned}$$

Another upper bound for $\omega_e(B,C)$ is stated as follows:

THEOREM 3. *Let $B,C \in \mathscr{B}(\mathscr{H})$. Then, we have*

$$\omega_e^2(B,C) \leqslant \max\{\omega(B),\omega(C)\}\left(\|B \pm C\|^2 + 4\omega(C^*B)\right)^{\frac{1}{2}}. \tag{21}$$

*Proof.* Observe that

$$\begin{aligned}\|y\|^2 + \|z\|^2 + 2|\langle y,z\rangle| &= \|y\|^2 \pm 2\mathrm{Re}\langle y,z\rangle + \|z\|^2 + 2(|\langle y,z\rangle| \mp \mathrm{Re}\langle y,z\rangle) \\ &= \|y \pm z\|^2 + 2(|\langle y,z\rangle| \mp \mathrm{Re}\langle y,z\rangle)\end{aligned}$$

and

$$\mp\mathrm{Re}\langle y,z\rangle \leqslant |\langle y,z\rangle|$$

for all $y,\ z \in \mathscr{H}$.

Then

$$\|y\|^2 + \|z\|^2 + 2|\langle y,z\rangle| \leqslant \|y \pm z\|^2 + 4|\langle y,z\rangle|$$

for all $y,\ z \in \mathscr{H}$.

From (18) we then get

$$|\langle x,y\rangle|^2 + |\langle x,z\rangle|^2 \leqslant \|x\| \max\{|\langle x,y\rangle|,|\langle x,z\rangle|\}\left(\|y \pm z\|^2 + 4|\langle y,z\rangle|\right)^{\frac{1}{2}}$$

for all $y,\ z \in \mathscr{H}$.

If we take $y = Bx,\ z = Cx$ with $x \in \mathscr{H}$ and $\|x\| = 1$, then we obtain

$$|\langle x,Bx\rangle|^2 + |\langle x,Cx\rangle|^2 \leqslant \max\{|\langle x,Bx\rangle|,|\langle x,Cx\rangle|\}\left(\|(B \pm C)x\|^2 + 4|\langle C^*Bx,x\rangle|\right)^{\frac{1}{2}}$$

and by taking the supremum over all $x \in \mathscr{H}$ with $\|x\| = 1$, we derive the desired result (21). □

As an application of Theorem 3, we derive the following corollary.

COROLLARY 5. *Let $B,C \in \mathscr{B}(\mathscr{H})$ and $\alpha,\ \beta \in \mathbb{C}$ with $|\alpha| = |\beta| = 1$. Then*

$$\omega_e^2(B,C) \leqslant \max\{\omega(B),\omega(C)\}\left(\|\alpha B + \beta C\|^2 + 4\omega(C^*B)\right)^{\frac{1}{2}}. \tag{22}$$

*Proof.* By replacing $B$ with $\alpha B$ and $C$ with $\beta C$ in equation (21), we obtain the desired result. □

Two important applications of equation (22) are stated in the following corollaries.

COROLLARY 6. *Let $A \in \mathscr{B}(\mathscr{H})$. For every $\alpha,\beta \in \mathbb{C}$ with $|\alpha| = |\beta| = 1$, the inequality holds:*

$$\omega^2(A) \leqslant \left\|\frac{\alpha A + \beta A^*}{2}\right\|^2 + \omega\left(A^2\right). \tag{23}$$

*Proof.* If $A = 0$, then (23) follows trivially. Assume that $A \neq 0$. By taking $B = A$ and $C = A^*$ in equation (22), and then using equation (13), we obtain the following inequality:

$$2\omega^2(A) \leqslant \omega(A)\left(\|\alpha A + \beta A^*\|^2 + 4\omega\left(A^2\right)\right)^{\frac{1}{2}}.$$

This implies that

$$2\omega(A) \leqslant \left(\|\alpha A + \beta A^*\|^2 + 4\omega\left(A^2\right)\right)^{\frac{1}{2}}.$$

This proves (23) as requested. □

REMARK 7. It follows immediately from (23) that the following inequality holds for all $A \in \mathscr{B}(\mathscr{H})$:

$$\omega^2(A) \leqslant \left\|\frac{A \pm A^*}{2}\right\|^2 + \omega\left(A^2\right).$$

COROLLARY 7. *Let $A \in \mathscr{B}(\mathscr{H})$. Then*

$$\omega^2(A) \leqslant \frac{1}{2}\max\{\|A + A^*\|, \|A - A^*\|\}\left(\|A\|^2 + \omega((A - A^*)(A + A^*))\right)^{\frac{1}{2}}.$$

*Proof.* Let $A = B + iC$ be the Cartesian decomposition of $A$. The result follows by applying equation (22) to the pair $(B, C)$ with $\alpha = 1$ and $\beta = i$. □

Our final theorem in this paper reads as follows.

THEOREM 4. *For all $B, C \in \mathscr{B}(\mathscr{H})$ and $t \in [0, 1]$ we have that*

$$\begin{aligned}\omega_e(B,C) &\leqslant \omega_e((1-t)B, tC) + \omega_e(tB, (1-t)C) \\ &\leqslant \left\|(1-t)^2|B|^2 + t^2|C|^2\right\|^{\frac{1}{2}} + \left\|t^2|B|^2 + (1-t)^2|C|^2\right\|^{\frac{1}{2}}\end{aligned} \tag{24}$$

*and*

$$\begin{aligned}\omega_e(B,C) &\leqslant \omega_e((1-t)B, tC) + \omega_e(tB, (1-t)C) \\ &\leqslant \left[\max\left\{(1-t)^2\|B\|^2, t^2\|C\|^2\right\} + (1-t)t\,\omega(C^*B)\right]^{\frac{1}{2}} \\ &\quad + \left[\max\left\{t^2\|B\|^2, (1-t)^2\|C\|^2\right\} + (1-t)t\,\omega(C^*B)\right]^{\frac{1}{2}}.\end{aligned} \tag{25}$$

*Proof.* Using the elementary Minkowski type inequality

$$\sqrt{|a+b|^2 + |c+d|^2} \leqslant \sqrt{|a|^2 + |c|^2} + \sqrt{|b|^2 + |d|^2}$$

where $a,b,c,d,$ are complex numbers, we have that

$$\begin{aligned}
&\left(\left|\left\langle x,Bx\right\rangle\right|^2+\left|\left\langle x,Cx\right\rangle\right|^2\right)^{\frac{1}{2}}\\
&=\left(\left|(1-t)\left\langle x,Bx\right\rangle+t\left\langle x,Bx\right\rangle\right|^2+\left|(1-t)\left\langle x,Cx\right\rangle+t\left\langle x,Cx\right\rangle\right|^2\right)^{\frac{1}{2}}\\
&=\left(\left|\left\langle x,(1-t)Bx\right\rangle+\left\langle x,tBx\right\rangle\right|^2+\left|\left\langle x,(1-t)Cx\right\rangle+\left\langle x,tCx\right\rangle\right|^2\right)^{\frac{1}{2}}\\
&=\left(\left|\left\langle x,(1-t)Bx\right\rangle+\left\langle x,tBx\right\rangle\right|^2+\left|\left\langle x,tCx\right\rangle+\left\langle x,(1-t)Cx\right\rangle\right|^2\right)^{\frac{1}{2}}\\
&\leqslant\left(\left|\left\langle x,(1-t)Bx\right\rangle\right|^2+\left|\left\langle x,tCx\right\rangle\right|^2\right)^{\frac{1}{2}}+\left(\left|\left\langle x,tBx\right\rangle\right|^2+\left|\left\langle x,(1-t)Cx\right\rangle\right|^2\right)^{\frac{1}{2}}
\end{aligned}$$

for all $x\in\mathscr{H},\ \|x\|=1$ and $t\in[0,1]$.

If we take the supremum over $x\in\mathscr{H}$ with $\|x\|=1$, then we get

$$\omega_e(B,C)\leqslant\omega_e((1-t)B,tC)+\omega_e(tB,(1-t)C).$$

By (1) we derive

$$w_e((1-t)B,tC)\leqslant\left\|(1-t)^2|B|^2+t^2|C|^2\right\|^{\frac{1}{2}}$$

and

$$\omega_e(tB,(1-t)C)\leqslant\left\|t^2|B|^2+(1-t)^2|C|^2\right\|^{\frac{1}{2}},$$

which proves (24).

Also, by (2) we have

$$w_e((1-t)B,tC)\leqslant\left[\max\left\{(1-t)^2\|B\|^2,t^2\|C\|^2\right\}+(1-t)tw(C^*B)\right]^{\frac{1}{2}}$$

and

$$\omega_e(tB,(1-t)C)\leqslant\left[\max\left\{t^2\|B\|^2,(1-t)^2\|C\|^2\right\}+(1-t)tw(C^*B)\right]^{\frac{1}{2}},$$

which proves (25). □

REMARK 8. For $t=\frac{1}{2}$ in equation (24), we obtain equation (1), and from equation (25), we get equation (2).

By taking $B=A$ and $C=A^*$ in Theorem 4, then we get the following corollary.

COROLLARY 8. *If $A\in\mathscr{B}(\mathscr{H})$, then we have*

$$w(A)\leqslant\left\|\frac{(1-t)^2|A|^2+t^2|A^*|^2}{2}\right\|^{\frac{1}{2}}+\left\|\frac{t^2|A|^2+(1-t)^2|A^*|^2}{2}\right\|^{\frac{1}{2}}\tag{26}$$

*and*

$$w(A) \leqslant \sqrt{2}\left[\left(\frac{1}{2}+\left|t-\frac{1}{2}\right|\right)^{2}\|A\|^{2}+(1-t)t\omega\left(A^{2}\right)\right]^{\frac{1}{2}} \tag{27}$$

*for all* $t \in [0,1]$.

We close our paper with this important remark.

REMARK 9. If we take $t=\frac{1}{2}$ in (27), we derive (14), while from (26), we obtain

$$w(A) \leqslant \frac{\sqrt{2}}{2}\sqrt{\|A^{*}A+AA^{*}\|}. \tag{28}$$

Notice that (28) was first obtained by F. Kittaneh in [13] and provides an improvement of the second inequality in (1). We also mention here that the constant $\frac{\sqrt{2}}{2}$ in (28) is best possible.

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