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SOME GENERALIZED INEQUALITIES FOR ACCRETIVE–DISSIPATIVE MATRICES

YONGHUI REN

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Abstract. In this paper, we present some generalized inequalities for accretive-dissipative matrices involving convex and concave functions which extend some results of Jabbarzadeh and Kaleibary. Among other results, we show that if $T_1, T_2, \cdots, T_n \in \mathbb{M}_n(\mathbb{C})$ are accretive-dissipative matrices, then for every non-negative increasing concave function f on $[0,\infty)$ and $p \geqslant 1$, we have

$$\left|\left|f\left(\sqrt{2}\left|\sum_{j=1}^{n} T_j\right|\right)\right|\right|_p^p \leqslant 2 \cdot n^{p-1} \sum_{j=1}^{n} \left|\left|f\left(|T_j|\right)\right|\right|_p^p.$$

Moreover, we also provide the generalized forms of Minkowski's determinant inequality and the Young type determinant inequality involving accretive-dissipative matrices.

1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ denote the set of $n \times n$ complex matrices and A^* denote the conjugate transpose of A. The matrix $A \in \mathbb{M}_n(\mathbb{C})$ is called accretive if $\Re A$ is positive semi-definite, and an accretive-dissipative matrix if both $\Re A$ and $\Im A$ are positive semi-definite, where $\Re A = \frac{1}{2}(A + A^*)$ and $\Im A = \frac{1}{2i}(A - A^*)$ are called the real part and imaginary part of A. For two Hermitian matrices $A, B \in \mathbb{M}_n(\mathbb{C})$, $A \geqslant B$ means that $A - B$ is positive semi-definite. In addition, a norm $\|\cdot\|$ on $\mathbb{M}_n(\mathbb{C})$ is unitarily invariant if $\|UAV\| = \|A\|$ for any $A \in \mathbb{M}_n(\mathbb{C})$ and all unitarily matrices $U, V \in \mathbb{M}_n(\mathbb{C})$. The singular values of A, that is, the eigenvalues of the positive semi-definite matrix $|A| = (A^*A)^{\frac{1}{2}}$, is denoted by $s_j(A)$, $j = 1, 2, \cdots, n,$ and arranged in a non-increasing order. For $A, B \in \mathbb{M}_n(\mathbb{C})$, the weak majorization relation $s(A) \prec_w s(B)$ means

$$\sum_{j=1}^{k} s_j(A) \leqslant \sum_{j=1}^{k} s_j(B)$$

for $k = 1, 2, \cdots, n$. It is well known that $||A|| \leqslant ||B||$ for all unitarily invariant norms with $A, B \in \mathbb{M}_n(\mathbb{C})$ if and only if $s(A) \prec_w s(B)$. For $A \in \mathbb{M}_n(\mathbb{C})$ and $1 \leqslant p < \infty$, the Schatten p-norms are defined as $||A||_p = \left(\sum_{j=1}^{n} s_j^p(A)\right)^{\frac{1}{p}}$. As we all konw that the Schatten p-norms are typical examples of unitarily invariant norms.

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The numerical range of $A \in \mathbb{M}_n(\mathbb{C})$ is defined by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n,\ x^*x = 1\}.$$

Moreover, a matrix $A \in \mathbb{M}_n(\mathbb{C})$ is said to be sectorial if, for some $\alpha \in [0, \frac{\pi}{2})$, we have

$$W(A) \subset S_\alpha := \left\{z \in \mathbb{C} : \Re z \geqslant 0, |\Im z| \leqslant (\Re z)\tan(\alpha)\right\}.$$

Note that the numerical range of an accretive-dissipative matrix A is located in the first quadrant, that is $W(e^{-\frac{i\pi}{4}}A) \subset S_{\frac{\pi}{4}}$.

In 2019, Kittaneh and Sakkijha [8] presented the following Schatten p-norm inequalities for accretive-dissipative matrices $T, S \in \mathbb{M}_n(\mathbb{C})$,

$$2^{-\frac{p}{2}}\left(||T||_p^p + ||S||_p^p\right) \leqslant ||T+S||_p^p \leqslant 2^{\frac{3p}{2}-1}\left(||T||_p^p + ||S||_p^p\right) \tag{1.1}$$

for $p \geqslant 1$.

Throughout this paper, we assume that every function is continuous and all functions satisfy the conditions: J is a subinterval of $(0,\infty)$ and $f : J \to (0,\infty)$.

In 2022, Jabbarzadeh and Kaleibary [7] presented the following inequalities relevant to accretive-dissipative matrices involving convex and concave functions:

THEOREM 1.1. *Let $T, S \in \mathbb{M}_n(\mathbb{C})$ be accretive-dissipative and f be a nonnegative increasing concave function on $[0,\infty)$. Then for every $p \geqslant 1$,*

$$\frac{1}{4^p}\left(\left|\left|f(\sqrt{2}|T|)\right|\right|_p^p + \left|\left|f(\sqrt{2}|S|)\right|\right|_p^p\right) \leqslant \left|\left|f\left(\frac{|T+S|}{2}\right)\right|\right|_p^p.$$

THEOREM 1.2. *Let $T, S \in \mathbb{M}_n(\mathbb{C})$ be accretive-dissipative and f be an increasing convex function on $[0,\infty)$. Then for every $\alpha \in [0,1]$ and $p \geqslant 1$,*

$$\left|\left|f\left(|\alpha T + (1-\alpha)S|\right)\right|\right|_p^p \leqslant 2^{p-1}\left(\left|\left|\alpha f(\sqrt{2}|T|)\right|\right|_p^p + \left|\left|(1-\alpha)f(\sqrt{2}|S|)\right|\right|_p^p\right).$$

As the authors in [7] explained the left-hand side of inequality (1.1) comes from Theorem 1.1 when $f(t) = t$, and the right-hand side of inequality (1.1) follows as a special case of Theorem 1.2 with $f(t) = t$ and $\alpha = \frac{1}{2}$.

Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semi-definite. The famous Minkowski's determinant inequality reads

$$\left(\det(A+B)\right)^{\frac{1}{n}} \geqslant (\det A)^{\frac{1}{n}} + (\det B)^{\frac{1}{n}}, \tag{1.2}$$

and the Young type determinant inequality is:

$$(\det A)^{\alpha}(\det B)^{1-\alpha} \leqslant \det\left(\alpha A + (1-\alpha)B\right) \text{ for } 0 < \alpha < 1. \tag{1.3}$$

Kittaneh and Sakkijha [8] extended inequalities (1.2) and (1.3) to accretive-dissipative matrices: if $T, S \in \mathbb{M}_n(\mathbb{C})$ are accretive-dissipative, then

$$\sqrt{2}\left|\det(T+S)\right|^{\frac{1}{n}} \geqslant |\det T|^{\frac{1}{n}} + |\det S|^{\frac{1}{n}} \tag{1.4}$$

and

$$|\det T|^{\alpha}|\det S|^{1-\alpha} \leqslant 2^{\frac{n}{2}}\left|\det\left(\alpha T+(1-\alpha)S\right)\right| \text{ for } 0<\alpha<1. \tag{1.5}$$

In this paper, we will show some generalized inequalities for the accretive-dissipative matrices inequalities mentioned above.

2. Main results

We will list some auxiliary lemmas in front of our results. Firstly, we show generalized inequalities for Theorem 1.1.

LEMMA 2.1. ([7]) *Let A and B be positive and f be a non-negative increasing concave function on $[0,\infty)$. Then for every unitarily invariant norm $|||\cdot|||$,*

$$\frac{1}{2}|||f(2|A+iB|)||| \leqslant |||f(A+B)||| \leqslant |||f(\sqrt{2}\,|A+iB|)|||.$$

LEMMA 2.2. ([13]) *Let $A_1,A_2,\cdots,A_n \geqslant 0$ and $x_1,x_2,\cdots,x_n$ be positive real numbers with $\sum_{j=1}^{n} x_j=1$. Then for every unitarily invariant norm $|||\cdot|||$ on $M_n(\mathbb{C})$,*

$$\left|\left|\left|\sum_{j=1}^{n} x_j f(A_j)\right|\right|\right| \leqslant \left|\left|\left|f\left(\sum_{j=1}^{n} x_j A_j\right)\right|\right|\right|$$

for every non-negative concave function f on $[0,\infty)$.

LEMMA 2.3. ([2]) *Let $A_1,A_2,\cdots,A_n$ be positive and $p\geqslant 1$. Then*

$$\sum_{j=1}^{n}||A_j||_p^p \leqslant \left|\left|\sum_{j=1}^{n} A_j\right|\right|_p^p \leqslant n^{p-1}\sum_{j=1}^{n}||A_j||_p^p.$$

THEOREM 2.4. *Let $T_1,T_2,\cdots,T_n \in \mathbb{M}_n(\mathbb{C})$ be accretive-dissipative and $x_1,x_2,\cdots,x_n$ be positive real numbers with $\sum_{j=1}^{n} x_j=1$. Then for $p\geqslant 1$ and every increasing concave function f on $[0,\infty)$, we have*

$$\left|\left|f\left(\sqrt{2}\left|\sum_{j=1}^{n} x_j T_j\right|\right)\right|\right|_p^p \geqslant \sum_{j=1}^{n}\left(\frac{x_j}{2}\right)^p||f(2|T_j|)||_p^p.$$

Proof. Let $T_j = A_j + iB_j$, $j = 1,2,\cdots,n$, be the Cartesian decompositions of T_j. Then we have

$$\begin{aligned}\left|\left|f\left(\sqrt{2}\left|\sum_{j=1}^{n} x_j T_j\right|\right)\right|\right|_p^p &= \left|\left|f\left(\sqrt{2}\left|\sum_{j=1}^{n} x_j A_j + i\sum_{j=1}^{n} x_j B_j\right|\right)\right|\right|_p^p \\ &\geqslant \left|\left|f\left(\sum_{j=1}^{n} x_j A_j + \sum_{j=1}^{n} x_j B_j\right)\right|\right|_p^p \quad \text{(by Lemma 2.1)} \\ &= \left|\left|f\left(\sum_{j=1}^{n} x_j (A_j + B_j)\right)\right|\right|_p^p \\ &\geqslant \left|\left|\sum_{j=1}^{n} x_j f(A_j + B_j)\right|\right|_p^p \quad \text{(by Lemma 2.2)} \\ &\geqslant \sum_{j=1}^{n} x_j^p \left|\left|f(A_j + B_j)\right|\right|_p^p \quad \text{(by Lemma 2.3)} \\ &\geqslant \sum_{j=1}^{n} x_j^p \frac{1}{2^p}\left|\left|f(2|A_j + iB_j|)\right|\right|_p^p \quad \text{(by Lemma 2.1)} \\ &= \sum_{j=1}^{n} \left(\frac{x_j}{2}\right)^p \left|\left|f(2|T_j|)\right|\right|_p^p. \quad \square\end{aligned}$$

REMARK 2.5. We can obtain Theorem 1.1 by Theorem 2.4 with $x_1 = x_2 = \frac{1}{2}$ and $n = 2$.

The authors ([12] Theorem 10) presented a reverse of Theorem 2.4. However, there is a minor flaw. In fact, it should be

$$\frac{1}{2^p}\left|\left|f(2|A+iB|)\right|\right|_p^p \leqslant \left|\left|f(A+B)\right|\right|_p^p$$

instead of $\frac{1}{2}\left|\left|f(2|A+iB|)\right|\right|_p^p \leqslant \left|\left|f(A+B)\right|\right|_p^p$ under the conditions as in Lemma 2.1. We now correct it as follows.

LEMMA 2.6. ([4]) *Let* $A_1, A_2, \cdots, A_n \geqslant 0$. *Then for every non-negative concave function* f *on* $[0,\infty)$ *and for every unitarily invariant norm* $|||\cdot|||$,

$$\left|\left|\left|f\left(\sum_{j=1}^{n} A_j\right)\right|\right|\right| \leqslant \left|\left|\left|\sum_{j=1}^{n} f(A_j)\right|\right|\right|.$$

THEOREM 2.7. *Let* $T_1, T_2, \cdots, T_n \in \mathbb{M}_n(\mathbb{C})$ *be accretive-dissipative. Then for every non-negative increasing concave function* f *on* $[0,\infty)$ *and* $p \geqslant 1$, *we have*

$$\left|\left|f\left(\sqrt{2}\left|\sum_{j=1}^{n} T_j\right|\right)\right|\right|_p^p \leqslant 2^p \cdot n^{p-1} \sum_{j=1}^{n} \left|\left|f\left(|T_j|\right)\right|\right|_p^p.$$

Proof. Let $T_j = A_j + iB_j$, $j = 1,2,\cdots,n$, be the cartesian decompositions of T_j. Then we have

$$
\begin{aligned}
\left\|f\left(\sqrt{2}\left|\sum_{j=1}^{n} T_j\right|\right)\right\|_p^p &= 2^p\left(\frac{1}{2}\left\|f\left(\sqrt{2}\left|\sum_{j=1}^{n} A_j + i\sum_{j=1}^{n} B_j\right|\right)\right\|_p\right)^p \\
&\leqslant 2^p\left\|f\left(\frac{\sum_{j=1}^{n} A_j + \sum_{j=1}^{n} B_j}{\sqrt{2}}\right)\right\|_p^p \quad \text{(by Lemma 2.1)} \\
&= 2^p\left\|f\left(\sum_{j=1}^{n}\left(\frac{A_j + B_j}{\sqrt{2}}\right)\right)\right\|_p^p \\
&\leqslant 2^p\left\|\sum_{j=1}^{n} f\left(\frac{A_j + B_j}{\sqrt{2}}\right)\right\|_p^p \quad \text{(by Lemma 2.6)} \\
&\leqslant 2^p n^{p-1}\sum_{j=1}^{n}\left\|f\left(\frac{A_j + B_j}{\sqrt{2}}\right)\right\|_p^p \quad \text{(by Lemma 2.3)} \\
&\leqslant 2^p n^{p-1}\sum_{j=1}^{n}\left\|f\left(|A_j + iB_j|\right)\right\|_p^p \quad \text{(by Lemma 2.1)} \\
&= 2^p n^{p-1}\sum_{j=1}^{n}\left\|f\left(|T_j|\right)\right\|_p^p. \quad \square
\end{aligned}
$$

We now show generalized inequalities for Theorem 1.2.

LEMMA 2.8. ([3]) *Let $A_1, A_2, \cdots, A_n \geqslant 0$ and $x_1, x_2, \cdots, x_n$ be positive real numbers with $\sum_{j=1}^{n} x_j = 1$. Then for every unitarily invariant norm $|||\cdot|||$ on $M_n(\mathbb{C})$,*

$$\left|\left|\left|\sum_{j=1}^{n} x_j f(A_j)\right|\right|\right| \geqslant \left|\left|\left|f\left(\sum_{j=1}^{n} x_j A_j\right)\right|\right|\right|$$

for every non-negative convex function f on $[0,\infty)$.

LEMMA 2.9. ([7]) *Let A, B be positive and f be an increasing convex function on $[0,\infty)$. Then for every unitarily invariant norm $|||\cdot|||$,*

$$|||f(|A+iB|)||| \leqslant |||f(A+B)||| \leqslant |||f(\sqrt{2}\,|A+iB|)|||.$$

THEOREM 2.10. *Let $T_1, T_2, \cdots, T_n \in \mathbb{M}_n(\mathbb{C})$ be accretive-dissipative and $x_1, x_2, \cdots, x_n$ be positive real numbers with $\sum_{j=1}^{n} x_j = 1$. Then for $p \geqslant 1$ and every increasing convex function f on $[0,\infty)$, we have*

$$\left\|f\left(\left|\sum_{j=1}^{n} x_j T_j\right|\right)\right\|_p^p \leqslant n^{p-1}\sum_{j=1}^{n} x_j^p \|f(\sqrt{2}|T_j|)\|_p^p.$$

Proof. Let $T_j = A_j + iB_j$, $j = 1, 2, \cdots, n$, be the Cartesian decompositions of T_j. Then we have

$$\begin{aligned}
\left|\left|f\left(\left|\sum_{j=1}^{n} x_j T_j\right|\right)\right|\right|_p^p &= \left|\left|f\left(\left|\sum_{j=1}^{n} x_j A_j + i\sum_{j=1}^{n} x_j B_j\right|\right)\right|\right|_p^p \\
&\leqslant \left|\left|f\left(\sum_{j=1}^{n} x_j A_j + \sum_{j=1}^{n} x_j B_j\right)\right|\right|_p^p \quad \text{(by Lemma 2.9)} \\
&= \left|\left|f\left(\sum_{j=1}^{n} x_j (A_j + B_j)\right)\right|\right|_p^p \\
&\leqslant \left|\left|\sum_{j=1}^{n} x_j f(A_j + B_j)\right|\right|_p^p \quad \text{(by Lemma 2.8)} \\
&\leqslant n^{p-1} \sum_{j=1}^{n} ||x_j f(A_j + B_j)||_p^p \quad \text{(by Lemma 2.3)} \\
&= n^{p-1} \sum_{j=1}^{n} x_j^p ||f(A_j + B_j)||_p^p \\
&\leqslant n^{p-1} \sum_{j=1}^{n} x_j^p ||f(\sqrt{2}|A_j + iB_j|)||_p^p \quad \text{(by Lemma 2.9)} \\
&= n^{p-1} \sum_{j=1}^{n} x_j^p ||f(\sqrt{2}|T_j|)||_p^p. \quad \square
\end{aligned}$$

REMARK 2.11. The Theorem 1.2 is a special case of Theorem 2.10 with $x_1 = \alpha$, $x_2 = 1 - \alpha$ and $n = 2$.

Next, we extend Theorem 2.10 to sectorial matrices.

LEMMA 2.12. ([15]) *Let $A \in \mathbb{M}_n(\mathbb{C})$ with $W(A) \subset S_\alpha$ for $\alpha \in [0, \frac{\pi}{2})$. Then*

$$s(A) \prec_w \sec(\alpha) s(\Re A).$$

Equivalently, for all unitarily invariant norms $||\cdot||$ on $\mathbb{M}_n(\mathbb{C})$

$$||A|| \leqslant \sec(\alpha) ||\Re A||.$$

LEMMA 2.13. ([5]) *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ and $f : (0, \infty) \to (0, \infty)$ be continuous increasing convex function. If $||A|| \leqslant ||B||$, then*

$$||f(|A|)|| \leqslant ||f(|B|)||$$

for every unitarily invariant norm $||\cdot||$.

LEMMA 2.14. ([1]) *Let $A, B \in \mathbb{M}_n(\mathbb{C})$. If A is positive semi-definite and B is Hermitian, then*

$$s_j(A) \leqslant s_j(A + iB)$$

for $j = 1, 2, \cdots, n$.

THEOREM 2.15. *Let $T_j \in \mathbb{M}_n(\mathbb{C})$ be such that $W(T_j) \subset S_\alpha$ for $\alpha \in [0, \frac{\pi}{2})$ and $x_1, x_2, \cdots, x_n$ be positive real numbers with $\sum_{j=1}^{n} x_j = 1$. Then for $p \geqslant 1$ and every increasing convex function f on $[0,\infty)$, we have*

$$\left|\left|f\left(\left|\sum_{j=1}^{n} x_j T_j\right|\right)\right|\right|_p^p \leqslant n^{p-1} \sum_{j=1}^{n} \left|\left|x_j f\big(\sec(\alpha)|T_j|\big)\right|\right|_p^p.$$

Proof. Let $T_j = A_j + iB_j$, $j = 1, 2, \cdots, n$, be the Cartesian decompositions of T_j. By Lemma 2.12, we have

$$\left|\left|\sum_{j=1}^{n} x_j T_j\right|\right|_p^p \leqslant \left|\left|\sum_{j=1}^{n} x_j \sec(\alpha) A_j\right|\right|_p^p. \tag{2.1}$$

Then we can get the follow inequalities

$$\begin{aligned}
\left|\left|f\left(\left|\sum_{j=1}^{n} x_j T_j\right|\right)\right|\right|_p^p &\leqslant \left|\left|f\left(\left|\sum_{j=1}^{n} x_j \sec(\alpha) A_j\right|\right)\right|\right|_p^p \quad \text{(by (2.1) and Lemma 2.13)}\\
&= \left|\left|f\left(\sum_{j=1}^{n} x_j \sec(\alpha) A_j\right)\right|\right|_p^p\\
&\leqslant \left|\left|\sum_{j=1}^{n} x_j f\big(\sec(\alpha) A_j\big)\right|\right|_p^p \quad \text{(by Lemma 2.8)}\\
&\leqslant n^{p-1} \sum_{j=1}^{n} \left|\left|x_j f\big(\sec(\alpha) A_j\big)\right|\right|_p^p \quad \text{(by Lemma 2.3)}\\
&= n^{p-1} \sum_{j=1}^{n} \left|\left|x_j f\big(\sec(\alpha) |A_j|\big)\right|\right|_p^p\\
&\leqslant n^{p-1} \sum_{j=1}^{n} \left|\left|x_j f\big(\sec(\alpha) |T_j|\big)\right|\right|_p^p \quad \text{(by Lemma 2.14 and 2.13).} \qquad \square
\end{aligned}$$

REMARK 2.16. Putting $\alpha = \frac{\pi}{4}$ in Theorem 2.15, then we can obtain Theorem 2.10.

Next, we present a reverse of Theorem 2.15.

LEMMA 2.17. ([9]) *Let $A_1, A_2, \cdots A_n \geqslant 0$. Then for every non-negative convex function f on $[0,\infty)$ with $f(0) = 0$ and for every unitarily invariant norm $|||\cdot|||$,*

$$\left|\left|\left|\sum_{j=1}^{n} f(A_j)\right|\right|\right| \leqslant \left|\left|\left|f\Big(\sum_{j=1}^{n} A_j\Big)\right|\right|\right|.$$

THEOREM 2.18. *Let $T_j \in \mathbb{M}_n(\mathbb{C})$ be such that $W(T_j) \subset S_\alpha$ for $\alpha \in [0, \frac{\pi}{2})$. Then for every increasing convex function f on $[0,\infty)$ with $f(0) = 0$ and $p \geqslant 1$, we have*

$$\left|\left|f\left(\sec(\alpha)\left|\sum_{j=1}^{n} T_j\right|\right)\right|\right|_p^p \geqslant \sum_{j=1}^{n} ||f(|T_j|)||_p^p.$$

Proof. Let $T_j = A_j + iB_j$, $j = 1,2,\cdots,n$, be the Cartesian decompositions of T_j. By Lemma 2.14, we have

$$\left|\left|\left|\sum_{j=1}^{n} T_j\right|\right|\right| \geqslant \left|\left|\left|\sum_{j=1}^{n} A_j\right|\right|\right|. \tag{2.2}$$

So we have

$$\begin{aligned}
\sum_{j=1}^{n} ||f(|T_j|)||_p^p &\leqslant \sum_{j=1}^{n} ||f(\sec(\alpha)|A_j|)||_p^p \quad \text{(by Lemma 2.12 and 2.13)}\\
&= \sum_{j=1}^{n} ||f(\sec(\alpha)A_j)||_p^p\\
&\leqslant \left|\left|\sum_{j=1}^{n} f(\sec(\alpha)A_j)\right|\right|_p^p \quad \text{(by Lemma 2.3)}\\
&\leqslant \left|\left|f\left(\sum_{j=1}^{n} \sec(\alpha)A_j\right)\right|\right|_p^p \quad \text{(by Lemma 2.17)}\\
&= \left|\left|f\left(\sec(\alpha)\left|\sum_{j=1}^{n} A_j\right|\right)\right|\right|_p^p\\
&\leqslant \left|\left|f\left(\sec(\alpha)\left|\sum_{j=1}^{n} T_j\right|\right)\right|\right|_p^p \quad \text{(by (2.2) and Lemma 2.13).} \qquad \square
\end{aligned}$$

Putting $\alpha = \frac{\pi}{4}$ in Theorem 2.18, we the following Corollary 2.19.

COROLLARY 2.19. *Let $T_1, T_2, \cdots T_n \in \mathbb{M}_n(\mathbb{C})$ be accretive-dissipative. Then for every increasing convex function f on $[0,\infty)$ with $f(0) = 0$ and $p \geqslant 1$, we have*

$$\sum_{j=1}^{n} ||f(|T_j|)||_p^p \leqslant \left|\left|f\left(\sqrt{2}\left|\sum_{j=1}^{n} T_j\right|\right)\right|\right|_p^p.$$

Let $f(t) = t$ in Theorem 2.4, Theorem 2.7, Theorem 2.10 and Corollary 2.19, respectively, we get the following corollary, which also obtained by Yang and Lu [14].

COROLLARY 2.20. *Let $T_1, T_2, \cdots, T_n \in \mathbb{M}_n(\mathbb{C})$ be accretive-dissipative. Then for $p \geqslant 1$, we have*

$$2^{-\frac{p}{2}} \sum_{j=1}^{n} ||T_j||_p^p \leqslant \left|\left|\sum_{j=1}^{n} T_j\right|\right|_p^p \leqslant 2^{\frac{p}{2}} n^{p-1} \sum_{j=1}^{n} ||T_j||_p^p.$$

At the end of this paper, we will give generalized forms of Minkowski's determinant inequality (1.4) and Young type determinant inequality (1.5) involving accretive-dissipative matrices as promised.

LEMMA 2.21. *Let k be a positive integer and let $A_1, A_2, \cdots, A_k \in M_n(\mathbb{C})$ be positive definite matrices. Then we have*

$$\left(\det\left(\sum_{j=1}^{k} A_j\right)\right)^{\frac{1}{n}} \geqslant \sum_{j=1}^{k} (\det A_j)^{\frac{1}{n}}.$$

Proof. It can be proved by mathematical induction with (1.2) easily, so we omit the details. □

LEMMA 2.22. ([10]) *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive definite matrices. Then*

$$|\det(A+iB)| \leqslant \det(A+B) \leqslant 2^{\frac{n}{2}} |\det(A+iB)|.$$

LEMMA 2.23. ([11]) *Let $A \in \mathbb{M}_n(\mathbb{C})$ with $W(A) \subset S_\alpha$. Then*

$$|\det A| \leqslant \sec^n(\alpha) \det(\Re A).$$

LEMMA 2.24. ([6]) *If $A \in \mathbb{M}_n(\mathbb{C})$ has positive definite real part, then*

$$\det(\Re A) \leqslant |\det A|.$$

THEOREM 2.25. *Let $T_1, T_2, \cdots, T_k \in \mathbb{M}_n(\mathbb{C})$ be accretive-dissipative. Then*

$$\sqrt{2}\left|\det\left(\sum_{j=1}^{k} T_j\right)\right|^{\frac{1}{n}} \geqslant \sum_{j=1}^{k} |\det T_j|^{\frac{1}{n}}.$$

Proof. Let $T_j = A_j + iB_j$, $j = 1, 2, \cdots, k$, be the Cartesian decompositions of T_j. Then we have

$$\begin{aligned}
\left|\det\left(\sum_{j=1}^{k} T_j\right)\right|^{\frac{1}{n}} &= \left|\det\left(\sum_{j=1}^{k} A_j + i\sum_{j=1}^{k} B_j\right)\right|^{\frac{1}{n}} \\
&\geqslant \left(2^{-\frac{n}{2}} \det\left(\sum_{j=1}^{k} A_j + \sum_{j=1}^{k} B_j\right)\right)^{\frac{1}{n}} \quad \text{(by Lemma 2.22)} \\
&= 2^{-\frac{1}{2}} \left(\det\left(\sum_{j=1}^{k} (A_j + B_j)\right)\right)^{\frac{1}{n}} \\
&\geqslant 2^{-\frac{1}{2}} \sum_{j=1}^{k} \left(\det(A_j + B_j)\right)^{\frac{1}{n}} \quad \text{(by Lemma 2.21)} \\
&\geqslant 2^{-\frac{1}{2}} \sum_{j=1}^{k} \left|\det(A_j + iB_j)\right|^{\frac{1}{n}} \quad \text{(by Lemma 2.22)} \\
&= 2^{-\frac{1}{2}} \sum_{j=1}^{k} |\det T_j|^{\frac{1}{n}}.
\end{aligned}$$

In fact, there is another proof of this result.

$$\begin{aligned}
\left|\det\left(\sum_{j=1}^{k} T_j\right)\right|^{\frac{1}{n}} &\geqslant \left(\det\left(\sum_{j=1}^{k} A_j\right)\right)^{\frac{1}{n}} \quad \text{(by Lemma 2.24)}\\
&\geqslant \sum_{j=1}^{k} (\det A_j)^{\frac{1}{n}} \quad \text{(by Lemma 2.21)}\\
&\geqslant \sum_{j=1}^{k} \left(\cos^n\left(\frac{\pi}{4}\right)|\det T_j|\right)^{\frac{1}{n}} \quad \text{(by Lemma 2.23)}\\
&= 2^{-\frac{1}{2}} \sum_{j=1}^{k} |\det T_j|^{\frac{1}{n}}. \quad \square
\end{aligned}$$

LEMMA 2.26. *Let k be a positive integer and $x_1, x_2, \cdots, x_k$ be positive real numbers with $\sum_{j=1}^{k} x_j = 1$. If $A_1, A_2, \cdots, A_k \in M_n(\mathbb{C})$ are positive definite matrices, then we have*

$$\det\left(\sum_{j=1}^{k} x_j A_j\right) \geqslant \prod_{j=1}^{k} (\det A_j)^{x_j}.$$

Proof. It can be proved by mathematical induction with (1.3). $\square$

THEOREM 2.27. *Let $T_1, T_2, \cdots, T_k \in \mathbb{M}_n(\mathbb{C})$ be accretive-dissipative and $x_1, x_2, \cdots, x_k$ be positive real numbers with $\sum_{j=1}^{k} x_j = 1$. Then*

$$\prod_{j=1}^{k} |\det T_j|^{x_j} \leqslant 2^{\frac{n}{2}} \left|\det\left(\sum_{j=1}^{k} x_j T_j\right)\right|.$$

Proof. Let $T_j = A_j + iB_j$, $j = 1, 2, \cdots, k$, be the Cartesian decompositions of T_j. Then

$$\begin{aligned}
\prod_{j=1}^{k} |\det T_j|^{x_j} &= \prod_{j=1}^{k} \left|\det(A_j + iB_j)\right|^{x_j}\\
&\leqslant \prod_{j=1}^{k} \left(\det(A_j + B_j)\right)^{x_j} \quad \text{(by Lemma 2.22)}\\
&\leqslant \det\left(\sum_{j=1}^{k} x_j (A_j + B_j)\right) \quad \text{(by Lemma 2.26)}\\
&= \det\left(\sum_{j=1}^{k} x_j A_j + \sum_{j=1}^{k} x_j B_j\right)
\end{aligned}$$

$$\leqslant 2^{\frac{n}{2}} \left| \det \left(\sum_{j=1}^{k} x_j A_j + i \sum_{j=1}^{k} x_j B_j \right) \right| \quad \text{(by Lemma 2.22)}$$
$$= 2^{\frac{n}{2}} \left| \det \left(\sum_{j=1}^{k} x_j T_j \right) \right|.$$

Similar to the proof of Theorem 2.25, we now show another proof of this result.

$$\begin{aligned} \left| \det \left(\sum_{j=1}^{k} x_j T_j \right) \right| &\geqslant \det \left(\Re \left(\sum_{j=1}^{k} x_j T_j \right) \right) \quad \text{(by Lemma 2.24)} \\ &= \det \left(\sum_{j=1}^{k} x_j \Re(T_j) \right) \\ &\geqslant \prod_{j=1}^{k} \left(\det(\Re(T_j)) \right)^{x_j} \quad \text{(by Lemma 2.26)} \\ &\geqslant \prod_{j=1}^{k} \left(\cos^n\left(\frac{\pi}{4}\right) |\det T_j| \right)^{x_j} \quad \text{(by Lemma 2.23)} \\ &= 2^{-\frac{n}{2}} \prod_{j=1}^{k} |\det T_j|^{x_j}. \quad \square \end{aligned}$$

REMARK 2.28. We can get (1.5) by Theorem 2.27 when $x_1 = \alpha$, $x_2 = 1 - \alpha$ and $n = 2$.

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Yonghui Ren
School of Mathematics and Statistics
Zhoukou Normal University
Zhoukou 466001, China
e-mail: yonghuiren1992@163.com