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# ON THE CONVERGENCE PROPERTIES OF DURRMEYER TYPE EXPONENTIAL SAMPLING SERIES IN (MELLIN) ORLICZ SPACES

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*Abstract.* In this study, by using the concept of modular convergence with the help of a suitable modular functional we obtain main theorem for the (Mellin) Orlicz spaces $X_0^{\eta} = L_{\mu}^{\phi}(\mathbb{R}^+)$ whose functions don't have to be bounded or continuous. Then we customize our theorems for $L_{\mu}^{p}(\mathbb{R}^+)$-space and $L_{\mu}^{\eta_{\alpha,\beta}}(\mathbb{R}^+)$ using these results. Finally, examples with graphical representations are given for some Durrmeyer type exponential sampling series with special kernels.

## 1. Introduction

How to reconstruct a function from another type of function, that is, how to find a useful function that approximates this function, is an important research topic. We can state that the first main result in this context is the Weierstrass approximation theorem, which has inspired studies that constitute the scope of the approximation theory. The Weierstrass approximation theory states that, for every continuous function on a compact interval, there is a polynomial approaching it (see [33]). Bernstein proved it algebraically in [14] by using the following polynomials sequence:

$$(B_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{(n-k)} f\left(\frac{k}{n}\right) \quad (x \in [0,1],\ n \in \mathbb{N})$$

where $f$ is a continuous function on $[0,1]$. Later, in order to get approximations for discontinuous functions (countable discontinuous), by replacing the values $f(\frac{k}{n})$ with Steklov mean values $(n+1)\int_{\frac{k}{n}}^{\frac{k+1}{n+1}} f(u)\,du$, the Kantorovich series in [24] are established as follows:

$$(K_n f)(x) = (n+1) \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{(n-k)} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(u)\,du.$$

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In 1967, Bernstein polynomials were modified by Durrmeyer to approximate Lebesgue integrable functions on $[0,1]$. The Berstein-Durrmeyer Operators in [22] were defined as follows

$$(\mathscr{D}_n f)(x) = n\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{(n-k)}\int_0^1\binom{n}{k}t^k(1-t)^{(n-k)}f(t)dt \quad (x\in[0,1]).$$

After all, when the interest was focused on functions defined on the whole real axis, the theory of sampling series offered useful and practical solutions. Initially, the sampling formula was given in [27], [31] and [34] independently of each other, but this formula worked for only Fourier band limited functions. To avoid this restriction, [16] has showed that a function $f$ that does not have to be band-limited can be reconstructed with the convolution series defined by

$$(S_w^{\phi}f)(x) = \sum_{k\in\mathbb{Z}} f(\frac{k}{w})\phi(wx-k), \quad w>0$$

where $\phi$ has some certain assumptions. It is called a generalization of the sampling series. Here, we see that the sampling values are equally spaced. This operator is so functional because it is the starting point of many studies (see [1], [4], [17], [19], [25], [26]). Also, inspired by the sampling formula and Mellin analysis, the exponential sampling formula, where sampling values were exponentially spaced, was introduced in [15], [23], and [30], also proved in [18] for investigation of concepts such as light scattering, Fraunhofer diffraction, and radio astronomy. It is remarked that the exponential sampling formula can be considered as the Mellin version of the classical sampling formula. Although such a connection is established between them, the exponential sampling formula is used for Mellin band-limited functions whose set is disjoint from the set of Fourier band-limited functions, when trivial functions are excluded. This distinction proved in [8] shows us the importance of studying the exponential sampling theory. In the next stage, to overcome the disadvantage of being band-limited in terms of Mellin setting on condition that the kernel function has some certain assumptions, the generalized exponential sampling series were defined in [9] as follows:

$$(E_w^{\phi}f)(x) = \sum_{k\in\mathbb{Z}} f(e^{\frac{k}{w}})\phi(e^{-k}x^w), \quad w>0,\, x\in\mathbb{R}^+ \tag{1}$$

for any function $f:\mathbb{R}^+\to\mathbb{R}$ where the series $E_w^{\phi}f$ are convergent for each $x$. Some researchers have studied these operators (see [7], [11]). We always can not have the exact sample values at the nodes $e^{\frac{k}{w}}$. To handle this problem, the so-called Kantorovich-type exponential sampling series was constructed in [28] as follows:

$$(I_w^{\phi})f(x) = \sum_{k\in\mathbb{Z}}\phi(e^{-k}x^w)w\int_{\frac{k}{w}}^{\frac{k+1}{w}} f(e^u)du \tag{2}$$

by replacing the value $f(e^{\frac{k}{w}})$ with the mean value of $f(e^x)$ in the interval $[\frac{k}{w},\frac{k+1}{w}]$ for $k\in\mathbb{Z}$, $w>0$. Recently, several modifications of it were studied in [2], [3], [5], [6], and [32]. Durrmeyer type exponential sampling series were presented in [10] in the following form

$$(D_w^{\phi,\psi} f)(x) := \sum_{k\in\mathbb{Z}} \phi(e^{-k}x^w) w \int_0^\infty \psi(e^{-k}u^w) f(u) \frac{du}{u}, \qquad w > 0. \tag{3}$$

It is important that the generalized exponential sampling series in (1) and Kantorovich-type exponential sampling series in (2) are included by the Durrmeyer-type exponential sampling series in (3).

This study aims to show that the Durrmeyer-type exponential sampling series offers us approximations for functions on $\mathbb{R}^+$ which are not only continuous or uniformly continuous, but also discontinuous such as elements of (Mellin) Orlicz spaces.

## 2. Preliminaries and notations

In our considerations, $\mathscr{C}_b(\mathbb{R}^+)$ denotes the space of log-uniformly continuous and bounded functions $f : \mathbb{R}^+ \to \mathbb{R}$ and $L^\infty(\mathbb{R}^+)$ represents the space of all essentially bounded functions $f : \mathbb{R}^+ \to \mathbb{R}$ equipped with the usual norm $\|f\|_\infty$. The symbol $f|_{\mathbb{R}^+} : \mathbb{R}^+ \to \mathbb{R}$ denotes the restriction of a function $f : \mathbb{R} \to \mathbb{R}$. The symbols $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}^+$, and $\mathbb{R}_0^+$ denote the set of positive natural numbers, the set of non-negative natural numbers, the set of integers, the set of positive real numbers and the set of non-negative real numbers respectively.

A version of the Haar measure is defined as follows:

$$\mu(A) = \int_A \frac{dt}{t}$$

for any measurable set $A \subset \mathbb{R}^+$, and it is also used in the construction of exponential Durrmeyer sampling series. We denote the space of all measurable functions over $\mathbb{R}^+$ with respect to the Haar measure $\mu$ by $M(\mathbb{R}^+, \mu)$.

Let $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a convex $\phi-$function, which means that the function $\phi$ has the following assumptions:

$(i)$ $\phi$ is convex in $\mathbb{R}_0^+$,
$(ii)$ $\phi(0) = 0$ and $\phi(u) > 0$ for all $u > 0$.

Let $\eta$ be a convex $\phi-$function. Now, by using Haar measure, we define the following functional which is so important for us because we use it in our theorems:

$$I^\eta[f] := \int_0^\infty \eta(|f(x)|) \frac{dx}{x}, \qquad f \in M(\mathbb{R}^+, \mu).$$

From [13] and [29], $I^\eta$ is a convex modular functional on $M(\mathbb{R}^+, \mu)$.

Now we introduce Mellin-Orlicz space defined as follows:

$$X_c^\eta = \{f : \mathbb{R}^+ \to \mathbb{R} : f(\cdot)(\cdot)^c \in L_\mu^\eta(\mathbb{R}^+)\}$$

where

$$L_\mu^\eta(\mathbb{R}^+) = \{f \in M(\mathbb{R}^+, \mu) : \ \exists \Lambda > 0 : \ I^\eta[\Lambda f] < +\infty\}$$

is the Orlicz space with respect to the (invariant) Haar measure $\mu$. If we consider $\eta(u)=u^p$ $(1\leqslant p<\infty)$, the Mellin-Orlicz space $X_c^\eta$ turns into the Mellin-Lebesgue space $X_c^p=\{f:\mathbb{R}^+\to\mathbb{R}: f(\cdot)(\cdot)^c\in L_\mu^p(\mathbb{R}^+)\}$ where $L_\mu^p(\mathbb{R}^+)$ is the Lebesgue space with respect to invariant measure $\mu$.

In our study, we focus on the Mellin-Orlicz space $X_c^\eta$ for $c=0$. Frankly, the Mellin-Orlicz space $X_0^\eta$ is the Orlicz space $L_\mu^\eta(\mathbb{R}^+)$. Therefore, our theorems are related to the Orlicz space $L_\mu^\eta(\mathbb{R}^+)$ and from now on we use only the symbol $L_\mu^\eta(\mathbb{R}^+)$ instead of $X_0^\eta$. We also want to give

$$E_\mu^\eta(\mathbb{R}^+)=\{f\in M(\mathbb{R}^+,\mu):\ I^\eta[\Lambda f]<+\infty,\ \forall\Lambda>0\}$$

as a vector subspace consisting of all finite elements of the Orlicz space $L_\mu^\eta(\mathbb{R}^+)$.

For $f\in L_\mu^\eta(\mathbb{R}^+)$, we will use a notion which is called modular convergence. We can explain it in this way: for some $\Lambda>0$, if

$$\lim_{w\to\infty} I^\eta[\Lambda(f_w-f)]=0,$$

a net of functions $(f_w)_{w>0}\subset L_\mu^\eta(\mathbb{R}^+)$ is modularly convergent to $f$.

Let's recall Luxemburg norm given by

$$\|f\|_\eta:=\inf\{\Lambda>0: I^\eta[f/\Lambda]\leqslant 1\}$$

for $f\in L_\mu^\eta(\mathbb{R}^+)$. We denote the Luxemburg norm of a function $f\in L_\mu^p(\mathbb{R}^+)$ by $\|f\|_{p,\mu}$. Indeed, we have the following equality

$$\|f\|_{p,\mu}=\left(\int_{\mathbb{R}^+}|f(u)|^p\frac{du}{u}\right)^{\frac{1}{p}}$$

for $f\in L_\mu^p(\mathbb{R}^+)$. Here, for the $\phi$-function $\eta$, we mention an essential property called the $\Delta_2$-condition. To be more precise, we state that $\eta$ satisfies the $(\Delta_2)$-condition if there exists a constant $M>0$ such that

$$\eta(2u)\leqslant M\eta(u)\quad(\forall u\geqslant 0).$$

This condition helps us to get the relationship between norm (Luxemburg norm) convergence and modular convergence. To explain this relation, let's remind norm convergence in $L_\mu^\eta(\mathbb{R}^+)$:

$$\text{if } \|f_w-f\|_\eta\to 0 \text{ when } w\to+\infty, \text{ then } f_w\to f.$$

Now we can state that the $(\Delta_2)$-condition is necessary and sufficient in order that norm convergence and modular convergence are equivalent in $L_\mu^\eta(\mathbb{R}^+)$. Moreover, $L_\mu^\eta(\mathbb{R}^+)=E_\mu^\eta(\mathbb{R}^+)$ when $(\Delta_2)$-condition holds. For details on this subject, see, e.g., [13,29].

## 3. Exponential sampling Durrmeyer operator

In this section, we remind the Durrmeyer-type exponential sampling series defined in [10]. In this reference, point-wise and uniform convergence properties were investigated, and an asymptotic formula of Voronovskaja type was given for these series. In addition, by using the usual modulus of continuity for uniformly continuous functions, some results were obtained.

Below, the functions having some features and used in the construction of the exponential sampling Durrmeyer operator will be introduced.

Let $\phi : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function such that the following assumptions are satisfied

($\phi$.1) for $\forall u \in \mathbb{R}^+$,

$$\sum_{k\in\mathbb{Z}} \phi(e^{-k}u) = 1,$$

($\phi$.2)

$$M_0(\phi) := \sup_{u\in\mathbb{R}^+} \sum_{k\in\mathbb{Z}} \left|\phi(e^{-k}u)\right| < +\infty,$$

($\phi$.3) for some $r \in \mathbb{N}$,

$$\lim_{\gamma\to\infty} \sum_{|k-\log u|>\gamma} \left|\phi(e^{-k}u)\right| |k-\log u|^r = 0,$$

uniformly with respect to $u \in \mathbb{R}^+$;
we suppose that $\Phi$ denotes the class of all functions $\phi$ satisfying the above assumptions.

Let $\psi : \mathbb{R}^+ \to \mathbb{R}$ be a function satisfying the following conditions:
($\psi$.1)

$$\int_0^\infty \psi(u)\frac{du}{u} = 1,$$

($\psi$.2)

$$\tilde{M}_0(\psi) := \int_0^\infty |\psi(u)|\frac{du}{u} < +\infty.$$

Now, let's denote the space of all $\psi$ functions satisfying the above conditions by $\Psi$.

Let $\nu \in \mathbb{N}_0$, for $\phi \in \Phi$ and $\psi \in \Psi$ discrete and continuous algebraic moments of order $\nu$ are defined as follows:

$$m_\nu(\phi,u) := \sum_{k\in\mathbb{Z}} \phi(e^{-k}u)\log^\nu(e^k u^{-1}) = \sum_{k\in\mathbb{Z}} \phi(e^{-k}u)(k-\log u)^\nu,\ u \in \mathbb{R}^+$$

and

$$\tilde{m}_\nu(\psi) := \int_0^\infty \psi(u)\log^\nu(u)\frac{du}{u}.$$

The absolute moment of order $v$ of $\phi \in \Phi$ is defined as follows:

$$M_v(\phi,u) := \sum_{k\in\mathbb{Z}} \left|\phi(e^{-k}u)\right| |\log^v(e^k u^{-1})| = \sum_{k\in\mathbb{Z}} \left|\phi(e^{-k}u)\right| |(k-\log u)|^v$$

and

$$M_v(\phi) := \sup_{u\in\mathbb{R}^+} \sum_{k\in\mathbb{Z}} \left|\phi(e^{-k}u)\right| |k-\log u|^v.$$

In addition to that, the absolute moment of order $v$ of $\psi \in \Psi$ is in the following form:

$$\tilde{M}_v(\psi) := \int_0^\infty |\psi(u)|\, |\log(u)|^v \frac{du}{u}.$$

Let $\phi \in \Phi$ and $\psi \in \Psi$. In [10], for any $w > 0$ and $f : \mathbb{R}^+ \to \mathbb{R}$, *exponential sampling Durrmeyer series* are defined by

$$(D_w^{\phi,\psi} f)(x) := \sum_{k\in\mathbb{Z}} \phi(e^{-k}x^w) w \int_0^\infty \psi(e^{-k}u^w) f(u) \frac{du}{u}, \qquad x \in \mathbb{R}^+$$

for any function $f \in dom(D_w^{\phi,\psi})$. This domain consists of all functions $f$ for which the series are absolutely convergent at each $x$.

REMARK 3.1. [10] Note that, for $\phi \in \Phi$, if $v_1, v_2 \in \mathbb{N}$ with $v_1 < v_2$, then $M_{v_2}(\phi) < \infty$ implies $M_{v_1}(\phi) < \infty$. This argument is valid for the absolute moments of $\psi \in \Psi$, i.e. when $v_1 < v_2$, if $\tilde{M}_{v_2}(\psi) < \infty$, then $\tilde{M}_{v_1}(\psi) < \infty$.

REMARK 3.2. From Remark 3.1 and the condition ($\phi$.3),

$$\lim_{\gamma\to\infty} \sum_{|k-\log u|>\gamma} \left|\phi(e^{-k}u)\right| = 0.$$

REMARK 3.3. [10] Using the conditions of the classes $\Phi$ and $\Psi$, the operators $D_w^{\phi,\psi}$ are well defined for any $f \in L^\infty(\mathbb{R}^+)$. Indeed,

$$|(D_w^{\phi,\psi} f)(x)| \leqslant M_0(\phi)\tilde{M}_0(\psi)\|f\|_\infty, \; x \in \mathbb{R}^+.$$

## 4. Convergence theorems

From now on, let kernels $\phi \in \Phi$ and $\psi \in \Psi$ for all remaining sections. In [10], the pointwise and uniform convergence theorem for the Durrmeyer type exponential sampling series $(D_w^{\phi,\psi})_{w>0}$ was obtained.

THEOREM 4.1. [10] *Let $f \in L^\infty(\mathbb{R}^+)$. We get*

$$\lim_{w\to+\infty} (D_w^{\phi,\psi} f)(x) = f(x)$$

*at any continuity point $x$ of $f$. Moreover, for $f \in \mathscr{C}_b(\mathbb{R}^+)$ we have*

$$\lim_{w\to+\infty} \|D_w^{\phi,\psi} f - f\|_\infty = 0.$$

For the remaining parts, let $\eta$ be a convex $\phi-$function. We will yield a modular convergence theorem in $L^{\eta}_{\mu}(\mathbb{R}^+)$ by using a modular continuity property for the Durrmeyer type exponential sampling series $(D^{\phi,\psi}_w)_{w>0}$.

THEOREM 4.2. *Let $\phi$ and $\psi$ be kernels with $\tilde{M}_0(\phi) < +\infty$ and $M_0(\psi) < +\infty$, also let $f \in L^{\eta}_{\mu}(\mathbb{R}^+)$ be fixed. Then there exists $\Lambda > 0$ such that*

$$I^{\eta}[\Lambda D^{\phi,\psi}_w f] \leqslant \frac{M_0(\psi)\tilde{M}_0(\phi)}{M_0(\phi)\tilde{M}_0(\psi)} I^{\eta}[\Lambda M_0(\phi)\tilde{M}_0(\psi) f], \; w > 0.$$

*Especially, $D^{\phi,\psi}_w$ is well-defined and $D^{\phi,\psi}_w(f)$ is an element of $L^{\eta}_{\mu}(\mathbb{R}^+)$ for every $w >$ 0.*

*Proof.* The assumption that $f \in L^{\eta}_{\mu}(\mathbb{R}^+)$ implies that there exists $\tilde{\Lambda} > 0$ such that $I^{\eta}[\tilde{\Lambda} f] < +\infty$. If $\Lambda > 0$ is chosen to be such that

$$\Lambda M_0(\phi)\tilde{M}_0(\psi) \leqslant \tilde{\Lambda},$$

we get $I^{\eta}[\Lambda M_0(\phi)\tilde{M}_0(\psi) f] < +\infty$.

We employ Jensen's inequality in our proofs; for further intriguing details on this, see [20]. Now, by taking advantage of the convexity of $\eta$, triangle inequality, Jensen inequality, and the change of variable $e^{-k}u^w = t$ respectively, we get

$$\begin{aligned}
&I^{\eta}[\Lambda(D^{\phi,\psi}_w f)] \\
&\leqslant \int_0^{\infty} \eta\left(\Lambda \sum_{k\in\mathbb{Z}} \left|\phi(e^{-k}x^w)\right| \left[w\int_0^{\infty} \left|\psi(e^{-k}u^w)\right|\left|f(u)\right|\frac{du}{u}\right]\right)\frac{dx}{x} \\
&\leqslant \frac{1}{M_0(\phi)} \int_0^{\infty} \left[\sum_{k\in\mathbb{Z}} \left|\phi(e^{-k}x^w)\right| \; \eta\left(\Lambda M_0(\phi) w \int_0^{\infty} \left|\psi(e^{-k}u^w)\right|\left|f(u)\right|\frac{du}{u}\right)\right]\frac{dx}{x} \\
&= \frac{1}{M_0(\phi)} \int_0^{\infty} \left[\sum_{k\in\mathbb{Z}} \left|\phi(e^{-k}x^w)\right| \; \eta\left(\Lambda M_0(\phi) \int_0^{\infty} \left|\psi(t)\right|\left|f((e^k t)^{\frac{1}{w}})\right|\frac{dt}{t}\right)\right]\frac{dx}{x} \\
&= \frac{1}{M_0(\phi)\tilde{M}_0(\psi)} \\
&\quad \times \int_0^{\infty} \left[\sum_{k\in\mathbb{Z}} \left|\phi(e^{-k}x^w)\right| \; \eta\left(\Lambda M_0(\phi)\tilde{M}_0(\psi) \int_0^{\infty} \frac{\left|\psi(t)\right|}{\tilde{M}_0(\psi)}\left|f((e^k t)^{\frac{1}{w}})\right|\frac{dt}{t}\right)\right]\frac{dx}{x} \\
&\leqslant \frac{1}{M_0(\phi)\tilde{M}_0(\psi)} \\
&\quad \times \int_0^{\infty} \left[\sum_{k\in\mathbb{Z}} \left|\phi(e^{-k}x^w)\right| w\int_0^{\infty} \left|\psi(e^{-k}u^w)\right| \; \eta\left(\Lambda M_0(\phi)\tilde{M}_0(\psi)\left|f(u)\right|\right)\frac{du}{u}\right]\frac{dx}{x} \\
&= \frac{1}{M_0(\phi)\tilde{M}_0(\psi)} \\
&\quad \times \int_0^{\infty} \sum_{k\in\mathbb{Z}} \left|\phi(e^{-k}x^w)\right|\frac{dx}{x}\, w\int_0^{\infty} \left|\psi(e^{-k}u^w)\right| \; \eta\left(\Lambda M_0(\phi)\tilde{M}_0(\psi)\left|f(u)\right|\right)\frac{du}{u},
\end{aligned}$$

finally, if we apply the Fubini-Tonelli theorem to the inequality above. After this, by the change of variable $e^{-k}x^w = y$, we deduce

$$
\begin{aligned}
& I^\eta[\Lambda(D_w^{\phi,\psi}f)] \\
&= \frac{1}{M_0(\phi)\tilde{M}_0(\psi)} \int_0^\infty |\phi(y)| \frac{dy}{y} \int_0^\infty \sum_{k\in\mathbb{Z}} \left|\psi(e^{-k}u^w)\right| \eta\left(\Lambda M_0(\phi)\tilde{M}_0(\psi)\left|f(u)\right|\right) \frac{du}{u} \\
&= \frac{\tilde{M}_0(\phi)M_0(\psi)}{M_0(\phi)\tilde{M}_0(\psi)} \int_0^\infty \eta\left(\Lambda M_0(\phi)\tilde{M}_0(\psi)|f(u)|\right) \frac{du}{u} \\
&= \frac{\tilde{M}_0(\phi)M_0(\psi)}{M_0(\phi)\tilde{M}_0(\psi)} I^\eta[\Lambda M_0(\phi)\tilde{M}_0(\psi)] < +\infty.
\end{aligned}
$$

Theorem 4.2 is proved. □

Due to the previous theorem, we can obtain that the operators $D_w^{\phi,\psi}$ are well-defined mappings from $L_\mu^\eta(\mathbb{R}^+)$ to itself. Furthermore, we assert that $D_w^{\phi,\psi}$ is modularly continuous. Firstly, we explain this notion: for any sequence $(f_k) \subset L_\mu^\eta(\mathbb{R}^+)$ which is modularly convergent to $f \in L_\mu^\eta(\mathbb{R}^+)$, if $I^\eta[\Lambda(D_w^{\phi,\psi}f - D_w^{\phi,\psi}f_k)] \to 0$ $(k \to +\infty)$ for some $\Lambda > 0$, $D_w^{\phi,\psi}$ is modularly continuous. Let's show the validity of our claim; for a modularly convergent sequence $(f_k)$, there exists $\tilde{\Lambda} > 0$ such that $I^\eta[\tilde{\Lambda}(f - f_k)] \to 0$, as $k \to \infty$. Now let's choose $\Lambda > 0$ such that $\Lambda M_0(\phi)\tilde{M}_0(\psi) \leqslant \tilde{\Lambda}$, then we get

$$
\begin{aligned}
I^\eta[\Lambda\big(D_w^{\phi,\psi}f - D_w^{\phi,\psi}f_k\big)] &= I^\eta[\Lambda D_w^{\phi,\psi}(f - f_k)] \\
&\leqslant \frac{\tilde{M}_0(\phi)M_0(\psi)}{M_0(\phi)\tilde{M}_0(\psi)} I^\eta[\Lambda M_0(\phi)\tilde{M}_0(\psi)(f - f_k)] \\
&\leqslant \frac{\tilde{M}_0(\phi)M_0(\psi)}{M_0(\phi)\tilde{M}_0(\psi)} I^\eta[\tilde{\Lambda}(f - f_k)].
\end{aligned}
\tag{4}
$$

$D_w^{\phi,\psi}$ is modularly continuous because the right part of inequality in (4) goes to 0 when $k \to +\infty$.

After all these explanations, we give the main theorem of this section in the following theorem.

THEOREM 4.3. *Let $\phi$ and $\psi$ be kernels with $\tilde{M}_0(\phi) < +\infty$ and $M_0(\psi) < +\infty$, also let $f \in L_\mu^\eta(\mathbb{R}^+)$ be fixed. Then there exists $\Lambda > 0$ such that*

$$
\lim_{w\to\infty} I^\eta[\Lambda\left(D_w^{\phi,\psi}f - f\right)] = 0.
$$

*Proof.* First of all, since $f \in L_\mu^\eta(\mathbb{R}^+)$ there exists $\Lambda_1, \Lambda_2 > 0$ such that $I^\eta\,[\Lambda_1 f] < \infty$, and

$$
I^\eta[\Lambda_2(f(\cdot) - f(\cdot\, t^{\frac{1}{w}}))] \to 0, \text{ as } w \to \infty,
$$

i.e., for every fixed $\varepsilon$ there exists $\delta > 0$ such that

$$\int_0^\infty \eta\left(\Lambda_2(f(x) - f(xt^{\frac{1}{w}})|\right)\frac{dx}{x} < \varepsilon \tag{5}$$

for every $t \in \mathbb{R}^+$ such that $|\log t| < w\delta$. This explanation is concluded from p. 178 and Example 1 in [12] and also from Theorem 2.4 in [13].

Now, we choose $\Lambda > 0$ such that

$$\Lambda \leqslant \min\left\{\frac{\Lambda_1}{4M_0(\phi)\tilde{M}_0(\psi)}, \frac{\Lambda_2}{2M_0(\phi)\tilde{M}_0(\psi)}\right\}.$$

Applying the properties of the convex modular functional $I^\eta$ (in fact, the properties of convex $\phi$-function $\eta$) and the Fubini-Tonelli theorem, we obtain the following inequality.

$$\begin{aligned}
I^\eta\left[\Lambda\left(D_w^{\phi,\psi}f - f\right)\right] &= \int_0^\infty \eta\left(\Lambda\left|(D_w^{\phi,\psi}f)(x) - f(x)\right|\right)\frac{dx}{x} \\
&= \int_0^\infty \eta\left(\Lambda\left|(D_w^{\phi,\psi}f)(x) - \sum_{k\in\mathbb{Z}}\phi(e^{-k}x^w)\, w\int_0^\infty \psi(e^{-k}u^w)f(uxe^{\frac{-k}{w}})\frac{du}{u}\right.\right. \\
&\qquad \left.\left. + \sum_{k\in\mathbb{Z}}\phi(e^{-k}x^w)\, w\int_0^\infty \psi(e^{-k}u^w)f(uxe^{\frac{-k}{w}})\frac{du}{u} - f(x)\right|\right)\frac{dx}{x} \\
&\leqslant \frac{1}{2}\left\{\int_0^\infty \eta\left(2\Lambda\left|(D_w^{\phi,\psi}f)(x) - \sum_{k\in\mathbb{Z}}\phi(e^{-k}x^w)\, w\int_0^\infty \psi(e^{-k}u^w)f(uxe^{\frac{-k}{w}})\frac{du}{u}\right|\right)\frac{dx}{x}\right. \\
&\qquad \left. + \int_0^\infty \eta\left(2\Lambda\left|\sum_{k\in\mathbb{Z}}\phi(e^{-k}x^w)\, w\int_0^\infty \psi(e^{-k}u^w)f(uxe^{\frac{-k}{w}})\frac{du}{u} - f(x)\right|\right)\frac{dx}{x}\right\} \\
&=: \frac{1}{2}\{J_1 + J_2\}.
\end{aligned}$$

At first, we examine $J_1$. Applying Jensen inequality twice similarly to the proof of Theorem 4.2 and change of variable $e^{-k}x^w = t$, we obtain

$$\begin{aligned}
|J_1| &\leqslant \int_0^\infty \eta\left(2\Lambda\sum_{k\in\mathbb{Z}}|\phi(e^{-k}x^w)|\, w\int_0^\infty |\psi(e^{-k}u^w)|\left|f(u) - f(uxe^{\frac{-k}{w}})\right|\frac{du}{u}\right)\frac{dx}{x} \\
&\leqslant \frac{1}{M_0(\phi)}\int_0^\infty \sum_{k\in\mathbb{Z}}|\phi(e^{-k}x^w)|\ \eta \\
&\quad \times \left(2\Lambda M_0(\phi)\, w\int_0^\infty |\psi(e^{-k}u^w)|\left|f(u) - f(uxe^{\frac{-k}{w}})\right|\frac{du}{u}\right)\frac{dx}{x} \\
&\leqslant \frac{1}{M_0(\phi)\tilde{M}_0(\psi)}\int_0^\infty \sum_{k\in\mathbb{Z}}|\phi(e^{-k}x^w)|\, w \\
&\quad \times \left[\int_0^\infty |\psi(e^{-k}u^w)|\ \eta\left(2\Lambda M_0(\phi)\tilde{M}_0(\psi)\ \left|f(u) - f(uxe^{\frac{-k}{w}})\right|\right)\frac{du}{u}\right]\frac{dx}{x} \\
&= \frac{M_0(\psi)}{M_0(\phi)\tilde{M}_0(\psi)}\int_0^\infty |\phi(t)|\left[\int_0^\infty \eta\left(2\Lambda M_0(\phi)\tilde{M}_0(\psi)\ \left|f(u) - f(ut^{\frac{1}{w}})\right|\right)\frac{du}{u}\right]\frac{dt}{t}.
\end{aligned}$$

Now, using $\delta$ given in (5), we can rewrite the above integral as follows

$$
\begin{aligned}
|J_1| &\leqslant \frac{M_0(\psi)}{M_0(\phi)\tilde{M}_0(\psi)}\left\{\int_{|\log t|\leqslant w\delta}+\int_{|\log t|>w\delta}\right\}|\phi(t)| \\
&\quad\times\left[\int_0^{\infty}\eta\left(2\Lambda M_0(\phi)\tilde{M}_0(\psi)\left|f(u)-f(ut^{\frac{1}{w}})\right|\right)\frac{du}{u}\right]\frac{dt}{t} \\
&=: J_{1,1}+J_{1,2}.
\end{aligned}
$$

Now, let's use the inequality in (5), we have

$$
\begin{aligned}
|J_{1,1}| &= \frac{M_0(\psi)}{M_0(\phi)\tilde{M}_0(\psi)} \\
&\quad\times\int_{|\log t|\leqslant w\delta}|\phi(t)|\left[\int_0^{\infty}\eta\left(2\Lambda M_0(\phi)\tilde{M}_0(\psi)\left|f(u)-f(ut^{\frac{1}{w}})\right|\right)\frac{du}{u}\right]\frac{dt}{t} \\
&\leqslant \frac{M_0(\psi)}{M_0(\phi)\tilde{M}_0(\psi)}\int_{|\log t|\leqslant w\delta}|\phi(t)|\left[\int_0^{\infty}\eta\left(\Lambda_2\left|f(u)-f(ut^{\frac{1}{w}})\right|\right)\frac{du}{u}\right]\frac{dt}{t} \\
&\leqslant \frac{M_0(\psi)\tilde{M}_0(\phi)}{M_0(\phi)\tilde{M}_0(\psi)}\varepsilon \leqslant \frac{M_0(\psi)\tilde{M}_0(\phi)}{M_0(\phi)\tilde{M}_0(\psi)}\varepsilon,
\end{aligned}
$$

for every $w>0$.

From the convexity of $\eta$, we obtain

$$
\begin{aligned}
|J_{1,2}| \leqslant \frac{M_0(\psi)}{M_0(\phi)\tilde{M}_0(\psi)}\int_{|\log t|>w\delta}|\phi(t)|\frac{1}{2}&\left[\int_0^{\infty}\eta\left(4\Lambda M_0(\phi)\tilde{M}_0(\psi)\,|f(u)|\right)\frac{du}{u}\right. \\
&\left.+\int_0^{\infty}\eta\left(4\Lambda M_0(\phi)\tilde{M}_0(\psi)\left|f(ut^{\frac{1}{w}})\right|\right)\frac{du}{u}\right]\frac{dt}{t}.
\end{aligned}
$$

In order to get a result for $J_{1,2}$, we must note that

$$
\int_0^{\infty}\eta\left(4\Lambda M_0(\phi)\tilde{M}_0(\psi)\left|f(ut^{\frac{1}{w}})\right|\right)\frac{du}{u}=\int_0^{\infty}\eta\left(4\Lambda M_0(\phi)\tilde{M}_0(\psi)\,|f(u)|\right)\frac{du}{u}
$$

for every $t\in\mathbb{R}^+$ and $w>0$. Moreover, there exists $\overline{w_1}>0$ such that

$$
\int_{|\log t|>w\delta}|\phi(t)|\frac{dt}{t}<\varepsilon
$$

for every $w\geqslant\overline{w_1}$ because of the assumption that $\tilde{M}_0(\phi)<\infty$. Using these explanations and Fubini Tonelli theorem, for every $w\geqslant\overline{w_1}$ we get

$$
\begin{aligned}
J_{1,2} &\leqslant \frac{M_0(\psi)}{M_0(\phi)\tilde{M}_0(\psi)}\int_{|\log t|>w\delta}|\phi(t)|\frac{dt}{t}\int_0^{\infty}\eta\left(4\Lambda M_0(\phi)\tilde{M}_0(\psi)\,|f(u)|\right)\frac{du}{u} \\
&\leqslant \frac{M_0(\psi)}{M_0(\phi)\tilde{M}_0(\psi)}I^{\eta}[\Lambda_1 f]\int_{|\log t|>w\delta}|\phi(t)|\frac{dt}{t}<\frac{M_0(\psi)}{M_0(\phi)\tilde{M}_0(\psi)}I^{\eta}[\Lambda_1 f]\,\varepsilon.
\end{aligned}
$$

We continue for $J_2$, using the change of variable $e^{-k}u^w = t$, applying Jensen inequality twice and the Fubini-Tonelli theorem, we have

$$\begin{aligned}
J_2 &= \int_0^\infty \eta\left(2\Lambda\left|\sum_{k\in\mathbb{Z}} \phi(e^{-k}x^w)\int_0^\infty \psi(e^{-k}u^w)\left[f(uxe^{\frac{-k}{w}})-f(x)\right]\frac{du}{u}\right|\right)\frac{dx}{x}\\
&= \int_0^\infty \eta\left(2\Lambda\left|\sum_{k\in\mathbb{Z}} \phi(e^{-k}x^w)\int_0^\infty \psi(t)\left[f(xt^{\frac{1}{w}})-f(x)\right]\frac{dt}{t}\right|\right)\frac{dx}{x}\\
&\leqslant \int_0^\infty \eta\left(2\Lambda\sum_{k\in\mathbb{Z}} \left|\phi(e^{-k}x^w)\right|\int_0^\infty |\psi(t)|\left|f(xt^{\frac{1}{w}})-f(x)\right|\frac{dt}{t}\right)\frac{dx}{x}\\
&\leqslant \frac{1}{M_0(\phi)}\int_0^\infty \sum_{k\in\mathbb{Z}} \left|\phi(e^{-k}x^w)\right|\eta\left(2\Lambda M_0(\phi)\int_0^\infty |\psi(t)|\left|f(xt^{\frac{1}{w}})-f(x)\right|\frac{dt}{t}\right)\frac{dx}{x}\\
&= \frac{1}{\tilde{M}_0(\psi)}\int_0^\infty \left[\int_0^\infty |\psi(t)|\,\eta\left(2\Lambda M_0(\phi)\tilde{M}_0(\psi)\left|f(xt^{\frac{1}{w}})-f(x)\right|\frac{dt}{t}\right)\right]\frac{dx}{x}\\
&= \frac{1}{\tilde{M}_0(\psi)}\int_0^\infty |\psi(t)|\left[\int_0^\infty \eta\left(2\Lambda M_0(\phi)\tilde{M}_0(\psi)\left|f(xt^{\frac{1}{w}})-f(x)\right|\frac{dx}{x}\right)\right]\frac{dt}{t}.
\end{aligned}$$

Then using again $\delta$ in (5), we can convert above integral to the following form:

$$\begin{aligned}
J_2 &\leqslant \frac{1}{\tilde{M}_0(\psi)}\left\{\int_{|\log t|\leqslant w\delta}+\int_{|\log t|>w\delta}\right\}|\psi(t)|\\
&\quad\times\left[\int_0^\infty \eta\left(2\Lambda M_0(\phi)\tilde{M}_0(\psi)\left|f(xt^{\frac{1}{w}})-f(x)\right|\frac{dx}{x}\right)\right]\frac{dt}{t}\\
&= J_{2,1}+J_{2,2}.
\end{aligned}$$

Again using the inequality in (5), we obtain

$$\begin{aligned}
J_{2,1} &= \frac{1}{\tilde{M}_0(\psi)}\int_{|\log t|\leqslant w\delta}|\psi(t)|\left[\int_0^\infty \eta\left(2\Lambda M_0(\phi)\tilde{M}_0(\psi)\left|f(xt^{\frac{1}{w}})-f(x)\right|\frac{dx}{x}\right)\right]\frac{dt}{t}\\
&\leqslant \frac{1}{\tilde{M}_0(\psi)}\int_{|\log t|\leqslant w\delta}|\psi(t)|\left[\int_0^\infty \eta\left(\Lambda_2\left|f(xt^{\frac{1}{w}})-f(x)\right|\frac{dx}{x}\right)\right]\frac{dt}{t} < \varepsilon
\end{aligned}$$

for every $w>0$.

Thanks to the convexity of $\eta$, we get

$$\begin{aligned}
|J_{2,2}| &\leqslant \frac{1}{\tilde{M}_0(\psi)}\int_{|\log t|>w\delta}|\psi(t)|\frac{1}{2}\left[\int_0^\infty \eta\left(4\Lambda M_0(\phi)\tilde{M}_0(\psi)\left|f(xt^{\frac{1}{w}})\right|\right)\frac{dx}{x}\right.\\
&\quad\left.+\int_0^\infty \eta\left(4\Lambda M_0(\phi)\tilde{M}_0(\psi)|f(x)|\right)\frac{dx}{x}\right]\frac{dt}{t}\\
&= \frac{1}{\tilde{M}_0(\psi)}\int_{|\log t|>w\delta}|\psi(t)|\left[\int_0^\infty \eta\left(4\Lambda M_0(\phi)\tilde{M}_0(\psi)|f(x)|\right)\frac{dx}{x}\right]\frac{dt}{t}.
\end{aligned}$$

By the property $(\psi.2)$, i.e $\tilde{M}_0(\psi)<\infty$, so there exists $\overline{w_2}>0$ such that

$$\int_{|\log t|>w\delta}|\psi(t)|\frac{dt}{t}<\varepsilon$$

for every $w > \overline{w_2}$ and similarly to before, we have

$$|J_{2,2}| \leqslant \frac{1}{\tilde{M}_0(\psi)} \int_{|\log t|>w\delta} |\psi(t)| \frac{dt}{t} I^\eta[\Lambda_1 f] < \frac{\varepsilon}{\tilde{M}_0(\psi)} I^\eta[\Lambda_1 f]$$

for every $w \geqslant \overline{w_2}$.

Eventually, let $\overline{w} := \max\{\overline{w_1}, \overline{w_2}\}$ and

$$C := \frac{1}{2}\left\{\frac{M_0(\psi)\tilde{M}_0(\phi)}{M_0(\phi)\tilde{M}_0(\psi)} + \frac{M_0(\psi)}{M_0(\phi)\tilde{M}_0(\psi)} I^\eta[\Lambda_1 f] + 1 + \frac{1}{\tilde{M}_0(\psi)} I^\eta[\Lambda_1 f]\right\},$$

so we can write

$$I^\eta\left[\Lambda(D_w^{\phi,\psi} f - f)\right] \leqslant C\varepsilon,$$

for every $w \geqslant \overline{w}$. □

In the last part of this section, we will obtain convergence results for some special cases of Orlicz spaces. Firstly, we examine the situation where $\phi$-function $\eta(u) = u^p$ $(1 \leqslant p < \infty)$. Due to the fact that $\eta(u) = u^p$ satisfies $\Delta_2$-condition, we get $L_\mu^\eta(\mathbb{R}^+) = E_\mu^\eta(\mathbb{R}^+ = L_\mu^p(\mathbb{R}^+)$ endowed with the Luxemburg norm $\|f\|_{p,\mu}$. As mentioned before, modular convergence is equivalent to norm convergence. By making inferences from previous theorems, we yield the following corollaries.

COROLLARY 4.4. *If $\phi$ and $\psi$ are kernels with the assumptions $\tilde{M}_0(\phi) < +\infty$ and $M_0(\psi) < +\infty$, then we obtain*

$$\|D_w^{\phi,\psi} f\|_{p,\mu} \leqslant M_0(\psi)^{\frac{1}{p}} \tilde{M}_0(\phi)^{\frac{1}{p}} M_0(\phi)^{\frac{p-1}{p}} \tilde{M}_0(\psi)^{\frac{p-1}{p}} \|f\|_{p,\mu} \ \ (w > 0)$$

*for every $f \in L_\mu^p(\mathbb{R}^+)$ $(1 \leqslant p < \infty)$. We conclude that $D_w^{\phi,\psi} : L_\mu^p(\mathbb{R}^+) \to L_\mu^p(\mathbb{R}^+)$ is well-defined.*

*Proof.* Let $\eta(u) = u^p$, so the convex $\phi$-function $\eta$ satisfies $\Delta_2$-condition. Therefore, when we consider a function $f \in L_\mu^p(\mathbb{R}^+)$, $\int_0^\infty |\tilde{\Lambda} f(x)|^p \frac{dx}{x} < \infty$ for all $\tilde{\Lambda} > 0$. So, when we choose $\tilde{\Lambda} = 1$, it is satisfied that $\int_0^\infty |f(x)|^p \frac{dx}{x} < \infty$, i.e. $\|f\|_{p,\mu}^p < \infty$. Using Theorem 4.2, we get

$$\begin{aligned}
\left[\|D_w^{\phi,\psi} f\|_{p,\mu}\right]^p &= I^\eta[D_w^{\phi\psi} f] \\
&\leqslant \frac{M_0(\psi)\tilde{M}_0(\phi)}{M_0(\phi)\tilde{M}_0(\psi)} I^\eta[M_0(\phi)\tilde{M}_0(\psi) f] \\
&= M_0(\psi)\tilde{M}_0(\phi) M_0(\phi)^{p-1} \tilde{M}_0(\psi)^{p-1} \|f\|_{p,\mu}^p.
\end{aligned}$$

By this result, we conclude that

$$\|D_w^{\phi,\psi} f\|_{p,\mu} \leqslant M_0(\psi)^{\frac{1}{p}} \tilde{M}_0(\phi)^{\frac{1}{p}} M_0(\phi)^{\frac{p-1}{p}} p\tilde{M}_0(\psi)^{\frac{p-1}{p}} \|f\|_{p,\mu}. \quad \square$$

Additionally, we directly reach the following consequence.

COROLLARY 4.5. *Let $\phi$ and $\psi$ be kernels with $\tilde{M}_0(\phi) < +\infty$ and $M_0(\psi) < +\infty$. We obtain*

$$\lim_{w\to+\infty} \|D_w^{\phi,\psi} f - f\|_{p,\mu} = 0$$

*for every $f \in L^p_\mu(\mathbb{R}^+)$ $(1 \leqslant p < \infty)$.*

Secondly, let us consider the function $\eta_{\alpha,\beta}(u) = u^\alpha(\log(e+u))^\beta$, $u \geqslant 0$ for $\alpha \geqslant 1$ and $\beta > 0$ which satisfies the $\Delta_2$-property. The Orlicz spaces corresponding to $\eta_{\alpha,\beta}$ comprise measurable functions $f \in M(\mathbb{R}^+,\mu)$ where

$$I^{\eta_{\alpha,\beta}}[\Lambda f] = \int_{\mathbb{R}^+} \Lambda^\alpha |f(x)|^\alpha (\log(e+\Lambda|f(x)|))^\beta \frac{dx}{x} < +\infty$$

for $\exists\Lambda$. We denote it by $L^{\eta_{\alpha,\beta}}_\mu(\mathbb{R}^+)$. If we apply Theorem 4.2 for this space when $\alpha = \beta = 1$, we deduce the following corollary.

COROLLARY 4.6. *Let $\phi$ and $\psi$ be kernels such that $M_0(\psi) < \infty$ and $\tilde{M}_0(\phi)$. For every $f \in L^{\eta_{1,1}}_\mu(\mathbb{R}^+)$, we have*

$$\begin{aligned}
&\int_0^\infty \left|D_w^{\phi,\psi} f(x)\right| \log\left(e + \Lambda\left|D_w^{\phi,\psi} f(x)\right|\right) \frac{dx}{x} \\
&\leqslant M_0(\psi)\tilde{M}_0(\phi) \int_0^\infty |f(u)| \log\left(e + \Lambda M_0(\phi)\tilde{M}_0(\psi)|f(u)|\right) \frac{du}{u}
\end{aligned}$$

*for $\Lambda > 0$. Moreover, $D_w^{\phi,\psi}$ is a well-defined operator from $L^{\eta_{1,1}}_\mu(\mathbb{R}^+)$ to $L^{\eta_{1,1}}_\mu(\mathbb{R}^+)$.*

*Proof.*

$$\begin{aligned}
&\int_0^\infty \Lambda\left|D_w^{\phi,\psi} f\right| \log(e + \Lambda\left|D_w^{\phi,\psi} f(x)\right|) \frac{dx}{x} \\
&= I^{\eta_{1,1}}[\Lambda D_w^{\phi,\psi} f] \\
&\leqslant \frac{M_0(\psi)\tilde{M}_0(\phi)}{M_0(\phi)\tilde{M}_0(\psi)} I^{\eta_{1,1}}[\Lambda M_0(\phi)\tilde{M}_0(\psi) f] \\
&= \Lambda M_0(\psi)\tilde{M}_0(\phi) \int_0^\infty |f(u)| \log\left(e + \Lambda M_0(\phi)\tilde{M}_0(\psi)|f(u)|\right) \frac{du}{u}.
\end{aligned}$$

If we eliminate $\Lambda$'s, the desired result is obtained. □

As $\eta_{1,1}$ has the $\Delta_2$-property, modular convergence and norm convergence are equivalent in $L^{\eta_{1,1}}_\mu(\mathbb{R}^+)$. Thanks to this equivalence, we have the following corollary.

COROLLARY 4.7. *Let $\phi$ and $\psi$ be kernels such that $M_0(\psi) < \infty$ and $\tilde{M}_0(\phi)$. We have*

$$\lim_{w\to\infty} \|D_w^{\phi,\psi} f - f\|_{\eta_{1,1}} = 0$$

*for $\forall f \in L^{\eta_{1,1}}_\mu(\mathbb{R}^+)$.*

## 5. Examples

In this section, we will show our theory is valid for some examples. Therefore, firstly, we want to introduce Mellin splines which are analogues of the classical central B-splines in the Mellin setting. The Mellin B-splines of order $n$ are defined as

$$B_n(x) = \frac{1}{(n-1)!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\left(\frac{n}{2}+\log x - j\right)_+^{n-1},$$

where $r_+$ denotes the positive part of the number $r \in \mathbb{R}$, i.e. $r_+ := \max\{r, 0\}$. Since $B_n(x)$ has compact supports $[e^{-\frac{n}{2}}, e^{\frac{n}{2}}]$, the functions $B_n(x)$ are bounded on $\mathbb{R}^+$ and the moment condition $M_\nu(\phi) < +\infty$ is held for all $\nu > 0$. If we reduce (or simplify) $B_n(x)$ for $n = 2$, we get

$$B_2(x) := (1 - |\log x|)_+ = \begin{cases} 1 - \log x, & 1 < x < e \\ 1 + \log x, & e^{-1} < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

which is the second order of Mellin spline whose graphic is given in Figure 1.

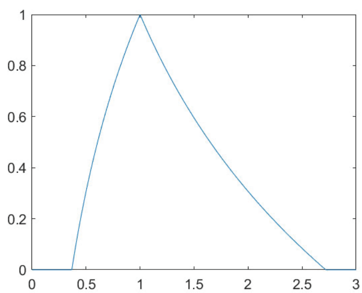


Figure 1: *The graphic of $B_2(x)$.*

If we put $\phi(x) = B_2(x)$ and $\psi(t) = \chi_{[1,e]}(t)$, so the Durrmeyer-type exponential sampling series turn into

$$\begin{aligned}\left(D_w^{B_2;\chi_{[1,e]}} f\right)(x) &= \sum_{k\in\mathbb{Z}} B_2(e^{-k}x^w)w\int_0^\infty \chi_{[1,e]}(e^{-k}u^w)f(u)\frac{du}{u} \\ &= \sum_{k\in\mathbb{Z}} B_2(e^{-k}x^w)\,w\int_{e^{k/w}}^{e^{(k+1)/w}} f(u)\frac{du}{u} \\ &= \sum_{k\in\mathbb{Z}} B_2(e^{-k}x^w)\,w\int_{\frac{k}{w}}^{\frac{k+1}{w}} f(e^u)du = (K_w^{B_2}f)(x), \qquad (x \in \mathbb{R}^+)\end{aligned}$$

which is in the form of a Kantorovich-type exponential sampling series.

We recall the Orlicz space $L^{\eta}(\mathbb{R})$ studied in [21]. If there exists $\exists\Lambda$ such that $\int_{\mathbb{R}} \eta(\Lambda|f(x)|)dx < \infty$ for a function $f$, then $f \in L^{\eta}(\mathbb{R})$. On one hand, it is trivially non sense to investigate whether the space $L^{\eta}(\mathbb{R})$ is equal to the space $L^{\eta}_{\mu}(\mathbb{R}^+)$ as the domains of their functions are different. On the other hand, we may think of the question of whether there is any function $f$ such that $f \notin L^{\eta}(\mathbb{R})$ but $f|_{\mathbb{R}^+} \in L^{\eta}_{\mu}(\mathbb{R}^+)$. The answer to this question is so important, because if we can answer it positively, although we can not find a series approximating a function $f \in L^{\eta}(\mathbb{R})$, the Durrmeyer-type exponential sampling series can be used to approximate this function $f$ on the positive part of its domain. Now, in this sense let's examine the following function defined by

$$f(x) = \begin{cases} \frac{1}{\sqrt[3]{x}}, & x \in [1\,,\infty) \\ 0 & x \in \mathbb{R} - [1\,,\infty) \end{cases}. \tag{6}$$

Let us consider $\eta_1(u) = u^3$ as a convex $\phi$-function. Because there does not exist any $\Lambda > 0$ such that $\int_{\mathbb{R}} \eta_1(\Lambda|f(x)|)\,dx < \infty$, $f \notin L^{\eta_1}(\mathbb{R})$. However, let's denote $f|_{\mathbb{R}^+}$ by $g$, since there exists $\exists\Lambda$ such that

$$\int_{\mathbb{R}^+} \eta_1\left(\Lambda\,|g(x)|\right)\,\frac{dx}{x} < \infty,$$

we have $f|_{\mathbb{R}^+} \in L^{\eta_1}_{\mu}(\mathbb{R}^+)$. Here, we give the plot of $g(x)$ as follows:

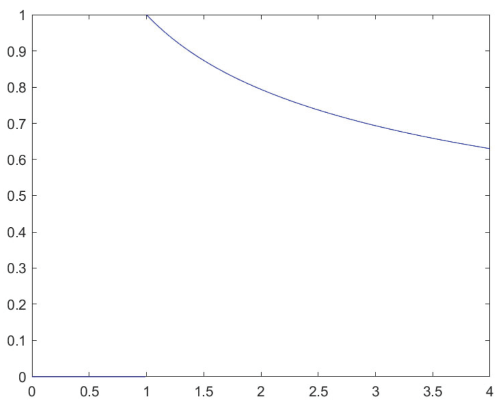


Figure 2: *The graphic of* $g(x) = f|_{\mathbb{R}^+}(x)$.

Now, we apply the Durrmeyer-type exponential sampling operator $D_w^{B_2,\chi_{[1,e]}}$ to $g = f|_{\mathbb{R}^+}$ which is a restriction of $f(x)$ given in (6). Additionally, we present the comparison of the function $g(x)$ with the series $(D_w^{B_2,\chi_{[1,e]}}g)(x)$ for $w = 5$ and $w = 10$ in Figure 3.

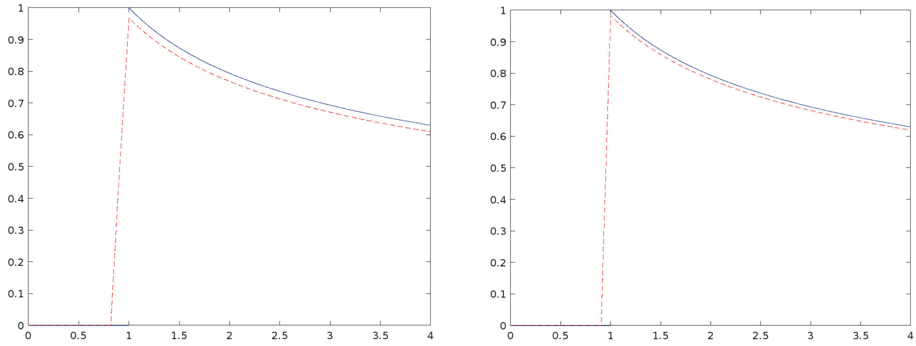


Figure 3: *Comparison of the function* $g(x)$ *with the series* $(D_w^{B_2,\chi_{[1,e]}}g)(x)$ *with respect to* $w=5$ *and* $w=10$.

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*Conflicts of Interest.* The authors declare that they have no conflicts of interest.

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