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ON COMPACTNESS OF OPERATORS FROM BANACH SPACES OF HOLOMORPHIC FUNCTIONS TO BANACH SPACES

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Abstract. We investigate a widely used application of compactness of bounded linear operators $T: X(\mathbb{B}) \to Y$, where $X(\mathbb{B})$ is a Banach space of holomorphic functions on the open unit ball $\mathbb{B} \subset \mathbb{C}^N$ and Y is a Banach space. In particular, we show that compactness of the operator when $X(\mathbb{B})$ is not reflexive, is not a sufficient condition for the property that every bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $X(\mathbb{B})$ such that $f_n \to 0$ with respect to the compact open topology as $n \to \infty$, implies that $T(f_n) \to 0$ with respect to the norm of Y as $n \to \infty$.

1. Introduction

Let $\mathbb{B} = \mathbb{B}^N$, $N \in \mathbb{N}$, be the open unit ball in the complex-vector space $\mathbb{C}^N$. For $N = 1$ the ball reduces to the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. As usual, by w we denote the weak topology on a Banach space Z, whereas by w^* we denote the weak-star topology on Z^*. With the norm $\|\cdot\|_Z$ of Z, we denote by B_Z the closed unit ball in Z, i.e., $B_Z = \{z \in Z : \|z\|_Z \leqslant 1\}$. We frequently omit the index Z in the notation for a norm and write $\|\cdot\|$.

If $\Omega \subset \mathbb{C}^N$ is an open set, then the space $H(\Omega)$ of all holomorphic functions on Ω endowed with the compact open topology (also called the topology of compact convergence) is a metrizable and complete locally convex space, i.e. a Fréchet space.

A bounded operator $T: Z \to Y$ between two normed spaces is called compact if the image of any bounded subset of Z is relatively compact in Y (see, e.g., [20]). To study compactness on functional spaces researchers need some applicable characterizations. For subspaces of $H(\Omega)$, one of the first such characterizations was given in [14], where the author studied compactness of the composition operator $C_\varphi(f)(z) = f(\varphi(z))$, $z \in \mathbb{D}$, on the Hardy spaces $H^p(\mathbb{D})$, $1 \leqslant p \leqslant \infty$ ([3]), where φ is a holomorphic self-map of the unit disk $\mathbb{D}$. Namely, the following theorem was proved:

THEOREM 1.1. *Let φ be a holomorphic self-map of $\mathbb{D}$ and $1 \leqslant p \leqslant \infty$. Then the operator C_φ is compact on the Hardy space $H^p(\mathbb{D})$ if and only if for any sequence $(f_n)_{n\in\mathbb{N}}$ bounded in $H^p(\mathbb{D})$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$ we have*

$$\lim_{n\to\infty} \|f_n \circ \varphi\|_{H^p} = 0.$$

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Many authors later noticed that similar arguments to those given in [14] can be used for characterizing the compactness of various concrete operators (see, e.g., [2, 8, 9, 15, 16, 17, 18, 19, 21] and the related references therein).

It is a typical situation that articles on compactness of concrete operators between subspaces of $H(\Omega)$ contain a result related to Theorem 1.1. It is also a frequent situation that authors only formulate such results without proving it, and only say that the results are proved similar to Theorem 1.1. In some articles, the authors study the operator $T\colon H^\infty(\Omega)\to Y$, where $H^\infty(\Omega)$ is the space of the bounded holomorphic functions on Ω with the norm $\|f\|_\infty=\sup_{z\in\Omega}|f(z)|$, and where Y is a Banach space, and use related, but not verified, claims (see, e.g., [10, 22, 23, 24]). The fact that $H^\infty(\Omega)$ has some characteristics not typical for the Hardy space for $0<p<\infty$, as well as for the Bergman, Dirichlet, Bloch, logarithmic Bloch, Zygmund, and weighted type spaces suggests studying the problem of characterizing compactness of the operator $T\colon H^\infty(\Omega)\to Y$ in detail.

Let $X=X(\mathbb{B})$ be a Banach space of holomorphic functions on $\mathbb{B}$ and let $\|\cdot\|_X=\|\cdot\|$ be its norm. By τ_0 we denote the compact open topology on X. Our aim is to study the relation between compactness and a stronger compactness property (defined below) of a bounded operator $T\colon X(\mathbb{B})\to Y$, where Y is a Banach space. We show that compactness of T does not imply that the operator has this strong compactness property:

$$\forall (f_n)_{n\in\mathbb{N}}\subset B_X : f_n \xrightarrow{\tau_0} 0,\ \ n\to\infty \implies \|Tf_n\|_Y\to 0,\ \ n\to\infty. \tag{SC}$$

2. Some preliminaries and analyses

In this section we conduct some analyses and present some auxiliary results which are employed in the proofs of the main results in this note. See also [1] when $N=1$ for the general approach carried out next.

Recall that for each $z\in\mathbb{B}$, δ_z is the point evaluation functional defined as follows

$$\delta_z(f)=f(z), \quad \text{for } f\in X(\mathbb{B}).$$

First note that if the closed unit ball B_X of $X(\mathbb{B})$ is compact with respect to topology τ_0, then the identity map $Id\colon (X(\mathbb{B}),\|\cdot\|)\to(X(\mathbb{B}),\tau_0)$ is continuous, hence $\delta_z\in X(\mathbb{B})^*$. Indeed, suppose $(f_n)_{n\in\mathbb{N}}\subset X(\mathbb{B})$ and $\|f_n\|\to 0$ as $n\to\infty$. Then there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}\subset\mathbb{C}$ with $|\lambda_n|\to\infty$ and such that

$$\|\lambda_n f_n\|=|\lambda_n|\,\|f_n\|\to 0 \text{ as } n\to\infty.$$

The set $\{\lambda_n f_n : n\in\mathbb{N}\}$ is bounded in $(X(\mathbb{B}),\|\cdot\|)$ and hence also bounded in $(X(\mathbb{B}),\tau_0)$, since (B_X,τ_0) is compact.

For an arbitrary compact set $K\subset\mathbb{B}$ it holds that $\sup_{z\in K,n\in\mathbb{N}}|\lambda_n f_n(z)|<\infty$, so

$$\sup_{z\in K}|f_n(z)|=\frac{1}{|\lambda_n|}\sup_{z\in K}|\lambda_n f_n(z)|\to 0 \text{ as } n\to\infty.$$

Therefore $f_n\to 0$ with respect to τ_0 as $n\to\infty$.

Moreover, when (B_X, τ_0) is compact, we obtain by using the Dixmier-Ng theorem [12] that the space

$$^{*}X(\mathbb{B}) := \{ l \in X(\mathbb{B})^* : l_{|B_X} \text{ is } \tau_0\text{-continuous} \},$$

endowed with the norm induced by $X(\mathbb{B})^*$, is a Banach space. The evaluation map $e\colon X(\mathbb{B}) \to ({}^{*}X(\mathbb{B}))^*$, $f \mapsto \hat{f}$, where $\hat{f}(l) = l(f)$, is an onto isometric isomorphism. In particular, ${}^{*}X(\mathbb{B})$ is a predual of $X(\mathbb{B})$. From now on we regard $X(\mathbb{B}) = ({}^{*}X(\mathbb{B}))^*$ with isometric norms.

Note also that a standard application of Hahn-Banach's theorem shows that the linear span $Lin(\{\delta_z\})$ of the set $\{\delta_z : z \in \mathbb{B}\}$ is contained and norm dense in ${}^{*}X(\mathbb{B})$.

The mapping $\Delta\colon \mathbb{B} \to {}^{*}X(\mathbb{B})$, $z \mapsto \delta_z$, is weakly holomorphic, since

$$\hat{f} \circ \Delta(z) = \hat{f}(\delta_z) = f(z), \quad \text{for } z \in \mathbb{B},$$

and all $f \in X(\mathbb{B})$. Now Theorem 8.12 in [11] gives that $\Delta\colon \mathbb{B} \to {}^{*}X(\mathbb{B})$ is holomorphic and, in particular, continuous. This implies that $\overline{Lin(\{\delta_z\})} = \overline{Lin\{\delta_z : z \in S\}}$ for a dense subset S of $\mathbb{B}$.

So ${}^{*}X(\mathbb{B})$ is a separable Banach space. This means that the closed unit ball B_X is weak-star metrizable.

REMARK 1. For the Hardy spaces $H^p(\mathbb{B})$, the mixed norm spaces $H(p,q,\gamma)$ (the Bergman-type spaces are obtained for $p = q$), the weighted-type spaces $H^\infty_\alpha(\mathbb{B})$, the Bloch type spaces $\mathscr{B}^\alpha(\mathbb{B})$, the logarithmic Bloch-type spaces $\mathscr{B}^\alpha_{\log^\beta}(\mathbb{B})$, and the weighted Dirichlet spaces $\mathscr{D}_\beta(\mathbb{B})$, among many others, the closed unit balls endowed with the compact open topology τ_0 are compact. This can be proved by using the point evaluation estimates (see, e.g., [4, 8, 9, 13, 19]), Montel's theorem, and Fatou's lemma for the spaces defined by integrals.

Now, we prove a useful auxiliary result which we apply in the proof of the main result in the next section.

LEMMA 2.1. *Assume that (B_X, τ_0) is compact and let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $X(\mathbb{B})$. Then the following conditions are equivalent:*

(a) $f_n \to f \in X(\mathbb{B})$ *with respect to the* w^* *topology as* $n \to \infty$;

(b) $\sup_{n\in\mathbb{N}} \|f_n\| < \infty$, $f \in X(\mathbb{B})$ *and* $\lim_{n\to\infty} f_n(z) = f(z)$ *for all* $z \in \mathbb{B}$;

(c) $\sup_{n\in\mathbb{N}} \|f_n\| < \infty$, $f \in X(\mathbb{B})$ *and* $f_n \to f$ *uniformly on compact subsets of* $\mathbb{B}$ *as* $n \to \infty$, *that is,* $f_n \to f$ *with respect to* τ_0.

Proof. Since the sequence $(f_n)_{n\in\mathbb{N}}$ is bounded with respect to τ_0, it is well known that (b) $\Leftrightarrow$ (c) (see, e.g., [5]).

If (c) holds (by scaling we may assume that $\sup_{n\in\mathbb{N}} \|f_n\| \leqslant 1$ and $\|f\| \leqslant 1$), and from the definition of ${}^{*}X(\mathbb{B})$ we have that for each $l \in {}^{*}X(\mathbb{B})$, $l(f_n) \to l(f)$ as $n \to \infty$, so (a) holds.

If $\hat{f}_n \to \hat{f}$ in the weak-star topology of $({}^*X(\mathbb{B}))^*$ as $n \to \infty$, then by the principle of uniform boundedness we have $\sup_{n\in\mathbb{N}} \|f_n\| = \sup_{n\in\mathbb{N}} \|\hat{f}_n\| < \infty$. Since (c) implies (a) and (B_X, τ_0) is metrizable, it follows that the identity map $Id\colon (B_X, \tau_0) \to (B_X, w^*)$ is continuous. Now by the compactness of (B_X, τ_0) we get $(B_X, w^*) = (B_X, \tau_0)$. This means that (a) implies (c). □

3. Main results

In this section we formulate and prove our main results and present a versatile corollary. The next theorem when $N = 1$ is closely related to a claim by Tjani in [21]. See also Lemma 3 in [17].

THEOREM 3.1. *Assume that (B_X, τ_0) is compact and let $T\colon X(\mathbb{B}) \to Y$ be a bounded linear operator, where Y is a Banach space. Let τ be any locally convex topology coarser than the norm topology of Y. Then $T\colon X(\mathbb{B}) \to Y$ is compact and $T_{|B_X}\colon (B_X, w^*) \to (Y, \tau)$ is continuous if and only if T has the (SC)-property.*

Proof. $\Rightarrow$) Let $(f_n)_{n\in\mathbb{N}} \subset B_X$ such that $f_n \to 0$ with respect to τ_0. By Lemma 2.1, $f_n \to 0$ with respect to w^*, as $n \to \infty$, and so by the continuity assumption we have that $T(f_n) \to T(0) = 0$ with respect to topology τ as $n \to \infty$. Since $K := \overline{T(B_X)}^{\|\cdot\|}$ is compact with respect to the norm $\|\cdot\|$ of Y and $Id\colon (K, \|\cdot\|) \to (K, \tau)$ is continuous, we conclude that $(K, \|\cdot\|) = (K, \tau)$. Hence $T(f_n) \to 0$ in the norm of Y, as $n \to \infty$.

$\Leftarrow$) Let $(f_n)_{n\in\mathbb{N}} \subset B_X$ and $f_n \to f \in B_X$ with respect to w^* as $n \to \infty$. Then $\sup_{n\in\mathbb{N}} \|f_n\| \leqslant 1$. By Lemma 2.1, $f_n \to f$ with respect to τ_0 in $X(\mathbb{B})$ as $n \to \infty$. Now the assumption gives that $T(f_n - f) \to 0$ with respect to the norm of Y as $n \to \infty$. Since (B_X, w^*) is metrizable we conclude that $T_{|B_X}\colon (B_X, w^*) \to (Y, \tau)$ is continuous.

Further let $(g_n)_{n\in\mathbb{N}} \subset B_X$. Since (B_X, τ_0) is compact, there is a subsequence $(g_{n_k})_{k\in\mathbb{N}}$ of $(g_n)_{n\in\mathbb{N}}$ with $g_{n_k} \to g$ with respect to τ_0 as $k \to \infty$ for some $g \in B_X$. Then, by the strong compactness assumption, $T(g_{n_k}) \to T(g)$ with respect to the norm in Y, as $k \to \infty$, that is, $T\colon X(\mathbb{B}) \to Y$ is compact. □

REMARK 2. Note that the claim in Theorem 3.1 is valid if $X(\mathbb{B})$ is replaced by a dual space Z^*, τ_0 by w^* and the sequence $(f_n)_{n\in\mathbb{N}} \subset B_X$ by a sequence $(l_n)_{n\in\mathbb{N}} \subset B_{Z^*}$. A similar characterization also holds for weakly compact operators $T\colon X(\mathbb{B}) \to Y$ with the norm topology changed to the w-topology.

If $X(\mathbb{B})$ is reflexive, then $(B_X, w) = (B_X, w^*)$. Hence $T_{|B_X}\colon (B_X, w^*) \to (Y, w)$ is continuous, and we get the following result.

COROLLARY 3.2. *Assume that (B_X, τ_0) is compact, $X(\mathbb{B})$ is reflexive and $T\colon X(\mathbb{B}) \to Y$ is a bounded linear operator, where Y is a Banach space. Then $T\colon X(\mathbb{B}) \to Y$ is compact if and only if T has the (SC)-property.*

Now, we show that compactness of $T\colon X(\mathbb{B}) \to Y$, when $X(\mathbb{B})$ is not reflexive, is not a sufficient condition for T to have the (SC)-property. Non-reflexive spaces are, e.g., $H^\infty(\mathbb{B})$ and $H^\infty_\alpha(\mathbb{B})$.

THEOREM 3.3. *Assume that (B_X, τ_0) is compact but that $X(\mathbb{B})$ is not reflexive and Y is a Banach space. Then there is a finite-rank operator $T\colon X(\mathbb{B}) \to Y$, which does not have the (SC)-property.*

Proof. By James' theorem [6, 7] we have that there exists a bounded linear functional $l \in X(\mathbb{B})^*$ that is not norm-attaining. Take a sequence $(f_n)_{n\in\mathbb{N}} \subset B_X$ such that $|l(f_n)| \to \|l\|$ as $n \to \infty$ and a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ with $f_{n_k} \to f$ with respect to τ_0 as $k \to \infty$ for some $f \in B_X$. Without loss of generality we can now assume that

$$\lim_{k\to\infty} |l(f_{n_k})| = \|l\| \quad \text{and} \quad l(f_{n_k}) \to \alpha \in \mathbb{C}$$

as $k \to \infty$. Therefore $\|l\| = |\alpha|$ but since l is not norm-attaining we have $\alpha \notin l(B_X)$.

Let

$$g_k(z) := \frac{f_{n_k}(z) - f(z)}{2}, \quad z \in \mathbb{B},\ k \in \mathbb{N}.$$

Then $g_k \to 0$ with respect to τ_0 as $k \to \infty$ and $(g_k)_{k\in\mathbb{N}} \subset B_X$. Moreover, with fixed $0 \neq y_0 \in Y$, we have that $T := l \otimes y_0$ is a finite-rank bounded linear operator mapping the space $X(\mathbb{B})$ to Y such that

$$T(g_k) = l(g_k)y_0 = \frac{1}{2}\Big(l(f_{n_k}) - l(f)\Big)y_0 \xrightarrow{\|\cdot\|} \frac{1}{2}\Big(\alpha - l(f)\Big)y_0 \neq 0$$

as $k \to \infty$. $\square$

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