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EXTENSIONS OF MATRIX MEAN INEQUALITIES TO SECTOR MATRICES

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Abstract. In this paper, extesions of some inequalities involving matrix means and sector matrices are considered. Among other results, we prove that if two sector matrices A and B satisfy $0 < mI_n \leqslant \Re(A), \Re(B) \leqslant MI_n$, then

$$\Phi^p\left(\Re(A\sigma B)\right) \leqslant \sec^{2p}(\theta)K^p(h)\Phi^p(\Re(B\sigma^{\perp}A)), \quad (0 \leqslant p \leqslant 2)$$

for every unital positive linear map Φ and arbitrary mean σ, where $K(h) := \frac{(M+m)^2}{4Mm}$ is the Kantorovich constant with $h := \frac{M}{m}$. In addition, we present some norm, numerical radius and determinantal inequalities for sector matrices.

1. Introduction and preliminaries

Let $\mathbb{M}_n$ be the space of all $n \times n$ complex matrices. For $A \in \mathbb{M}_n$, we let A^* denote the conjugate transpose of A. We say that $A \in \mathbb{M}_n$ is positive semi–definite and we write $A \geqslant 0$, if $\langle Ax, x\rangle \geqslant 0$ for all vectors $x \in \mathbb{C}^n$. In addition, if A is invertible, then we say that is positive definite. A matrix $A \in \mathbb{M}_n$ is accretive if its real part $\Re(A) = \dfrac{A+A^*}{2}$ is positive definite, that is, $\Re(A) > 0$. For $0 \leqslant \theta < \dfrac{\pi}{2}$, let S_θ is the sector in the complex plane defined as follows:

$$S_\theta = \{z \in \mathbb{C} : \Re(z) > 0, |\Im(z)| \leqslant (\Re(z))\tan\theta\}.$$

Let $A \in \mathbb{M}_n$ be whose numerical range $W(A)$ is contained in S_θ for some θ $\left(0 \leqslant \theta < \dfrac{\pi}{2}\right)$, where the numerical range $W(A)$ of a matrix $A \in \mathbb{M}_n$ is defined by

$$W(A) := \{\langle Ax, x\rangle : x \in \mathbb{C}^n, \ \|x\| = 1\}.$$

In this case, we say A is a sector matrix and we simply write $A \in S_\theta$ whenever $W(A) \subset S_\theta$. For more results on sector matrices see [1, 18, 22]. The numerical radius $w(A)$ of $A \in \mathbb{M}_n$ is also defned by

$$w(A) = \max\{|\langle Ax, x\rangle| : x \in \mathbb{C}^n, \ \|x\| = 1\}.$$

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It is clear that if $A \geqslant 0$, then $w(A) = \|A\|$ and in result $w(\Re(A)) = \|\Re(A)\|$. See [5, 7, 12] for recent results on numerical radius inequalities. For two positive definite matrices $A, B \in \mathbb{M}_n$, the arithmetic mean and harmonic mean are, respectively, denoted by

$$A\nabla B := \frac{A+B}{2} \quad \text{and} \quad A!B = \left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}.$$

The axiomatic theory for matrix means for pairs of positive semi–definite matrices has been developed by Kubo–Ando [15]. We briefly review a matrix mean. Let $\mathbb{M}_n^+ := \{A \in \mathbb{M}_n : A^* = A \geqslant 0\}$. A binary operation $\sigma : (A,B) \in \mathbb{M}_n^+ \times \mathbb{M}_n^+ \to A\sigma B \in \mathbb{M}_n^+$ is called a matrix mean if the following (i)–(iv) hold for any $A, B, C, D \in \mathbb{M}_n^+$.

(i) (Monotonicity) If $A \leqslant C$ and $B \leqslant D$, then $A\sigma B \leqslant C\sigma D$.

(ii) (Transfer inequality) $C(A\sigma B)C \leqslant (CAC)\sigma(CBC)$.

(iii) (Lower semicontinuity) If $A_m, B_m \in \mathbb{M}_n^+$, $A_m \downarrow A$ and $B_m \downarrow B$, then $A_m\sigma B_m \downarrow A\sigma B$, where $A_m \downarrow A$ means $A_1 \geqslant A_2 \geqslant \cdots$ and $\|A_m x - Ax\| \to 0$ for any $x \in \mathbb{C}^n$.

(iv) $I_n \sigma I_n = I_n$, where I_n is an identity matrix.

A matrix mean is connected to a matrix monotone function by the fundamental theory of Kubo–Ando [15]. We also review a matrix monotone function. Let $J \subset \mathbb{R}$ and let $f : J \to \mathbb{R}$ be a continuous function. Then f is called a matrix monotone function if $A \leqslant B \Longrightarrow f(A) \leqslant f(B)$ for any Hermitian matrices A and B with $\mathrm{Sp}(A), \mathrm{Sp}(B) \subset J$. Throughout this paper, we use the symbol

$$\mathbf{m} := \{f : (0,\infty) \to (0,\infty) : f \text{ is a matrix monotone function, } f(1) = 1\}.$$

For any matrix mean σ, there uniquely exsits a matrix monotone function $f \geqslant 0$ on $[0,\infty)$ such that $f(t)I_n = I_n\sigma(tI_n)$ for $t \geqslant 0$. And then we have the following (i) and (ii).

(i) $\sigma \to f$ is an affine one–to–one correspondence between a matrix mean σ and a matrix monotone function $f \in \mathbf{m}$. In addition, if $\sigma_1 \to f_1$ and $\sigma_2 \to f_2$, $A, B \in \mathbb{M}_n^+$ and $t \geqslant 0$, then we have

$$A\sigma_1 B \leqslant A\sigma_2 B \Longleftrightarrow f_1(t) \leqslant f_2(t).$$

(ii) For invertible A, we have the following form:

$$A\sigma B = A^{1/2} f\left(A^{-1/2} B A^{-1/2}\right) A^{1/2}.$$

Thus the function f is often called a representation function of a matrix mean σ.

Let σ be a matrix mean with the representing function $f(t)$. The matrix mean with the representing function $t/f(t)$ is called the dual of σ and is denoted by $\sigma^\perp$. By above definition, for two positive definite matrices A and B, we can write

$$A\sigma^\perp B = (B^{-1}\sigma A^{-1})^{-1}. \tag{1.1}$$

A linear map Φ is positive on $\mathbb{M}_n$ if $\Phi(A) \geqslant 0$ whenever $A \geqslant 0$. Also, we say that Φ is unital whenever $\Phi(I_n) = I_n$. Throughout this paper, Φ is considered as a unital positive linear map, unless specified otherwise.

Recently, it has been proved in [17] that

$$\Phi^p(A\sigma B) \leqslant K^p(h)\Phi^p(B\sigma^\perp A) \tag{1.2}$$

where $0 < mI_n \leqslant A, B \leqslant MI_n$, σ is an arbitrary matrix mean, $\sigma^\perp$ is its dual, and the Kantorovich constant $K(h) := \dfrac{(M+m)^2}{4Mm}$ with $h := \dfrac{M}{m}$. $K(h)$ is used as the Kantorovich constant throughout this paper. Then, the inequality (1.2) for two arbitrary means σ_1 and σ_2 between σ and $\sigma^\perp$ has been generalized in [17, Theorem 2.7].

In this paper, one of our aims is to extend these inequalities to sectors matrices, and another purpose is to present some inequalities on determinant, unitarily invariant norm and numerical radius for sector matrices. A norm $\|\cdot\|_u$ on $\mathbb{M}_n$ is called a unitarily invariant norm if $\|U_1AU_2\|_u = \|A\|_u$ for any unitary matrices $U_1, U_2 \in \mathbb{M}_n$ and any $A \in \mathbb{M}_n$.

2. Inequalities for positive linear map

We first need to recall the following lemmas which are necessary for proving our main results.

LEMMA 2.1. [8] *Let $A, B \in S_\theta$ and σ be an arbitrary mean. Then,*

$$\Re(A)\sigma\Re(B) \leqslant \Re(A\sigma B) \leqslant \sec^2(\theta)(\Re(A)\sigma\Re(B)).$$

LEMMA 2.2. [19] (Choi's inequality) *Let $A \in \mathbb{M}_n$ be invertible. Then, for every unital positive linear map Φ,*

$$\Phi^{-1}(A) \leqslant \Phi\left(A^{-1}\right).$$

LEMMA 2.3. [19] *Suppose that $0 < mI_n \leqslant A \leqslant MI_n$. Then*

$$A + MmA^{-1} \leqslant (M+m)I_n.$$

Using these lemmas, our first main result is the following.

LEMMA 2.4. *Let $A, B \in S_\theta$ with $0 < mI_n \leqslant \Re(A), \Re(B) \leqslant MI_n$ for $0 < m < M$ and σ be an arbitrary mean. Then*

$$\cos^2(\theta)\Phi(\Re(A\sigma B)) + Mm\Phi^{-1}(\Re(B\sigma^\perp A)) \leqslant (M+m)I_n. \tag{2.1}$$

Proof. By Lemma 2.3, we have

$$\Re(A) + Mm\Re^{-1}(A) \leqslant (M+m)I_n \quad \text{and} \quad \Re(B) + Mm\Re^{-1}(B) \leqslant (M+m)I_n.$$

By the subadditivity and monotonicity properties of matrix means, it follows that

$$\begin{aligned}\Re(A)\sigma\Re(B)+Mm(\Re^{-1}(A)\sigma\Re^{-1}(B)) &\leqslant (\Re(A)+Mm\Re^{-1}(A))\sigma(\Re(B)+Mm\Re^{-1}(B))\\ &\leqslant (M+m)I_n\sigma(M+m)I_n=(M+m)I_n.\end{aligned}$$

By taking Φ for both sides of the above inequality, we get

$$\Phi(\Re(A)\sigma\Re(B))+Mm\Phi(\Re^{-1}(A)\sigma\Re^{-1}(B))\leqslant (M+m)I_n. \tag{2.2}$$

Applying Lemmas 2.1, 2.2 and the inequality (2.2), respectively, we obtain

$$\begin{aligned}&\cos^2(\theta)\Phi(\Re(A\sigma B))+Mm\Phi^{-1}(\Re(B\sigma^{\perp}A))\\ &\leqslant \Phi(\Re(A)\sigma\Re(B))+Mm\Phi^{-1}(\Re(B)\sigma^{\perp}\Re(A))\\ &\leqslant \Phi(\Re(A)\sigma\Re(B))+Mm\Phi\left((\Re(B)\sigma^{\perp}\Re(A))^{-1}\right)\\ &=\Phi(\Re(A)\sigma\Re(B))+Mm\Phi(\Re^{-1}(A)\sigma\Re^{-1}(B))\\ &\leqslant (M+m)I_n. \quad \square\end{aligned}$$

To extend the inequality (1.2) for sector matrices, we need to state the following lemma, in which $\|\cdot\|$ is the spectral norm:

LEMMA 2.5. [10, 2, 6] *Let $A,B>0$ and $1\leqslant\lambda<\infty$. Then*

(i) $\|AB\|\leqslant \frac{1}{4}\|A+B\|^2$.

(ii) $\|A^{\lambda}+B^{\lambda}\|\leqslant \|(A+B)^{\lambda}\|$.

(iii) $A\leqslant\lambda B$ *if and only if* $\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\|\leqslant\sqrt{\lambda}$.

THEOREM 2.6. *Let $A,B\in S_{\theta}$ with $0<mI_n\leqslant\Re(A),\Re(B)\leqslant MI_n$. If σ is an arbitrary mean, then*

$$\Phi^2\left(\Re(A\sigma B)\right)\leqslant \sec^4(\theta)K^2(h)\Phi^2\left(\Re(B\sigma^{\perp}A)\right). \tag{2.3}$$

Proof. Making use of Lemma 2.5 (i) and Lemma 2.4, we have

$$\begin{aligned}&\left\|\cos^2\theta Mm\Phi(\Re(A\sigma B))\Phi^{-1}(\Re(B\sigma^{\perp}A))\right\|\\ &\leqslant \frac{1}{4}\left\|\cos^2\Phi(\Re(A\sigma B))+Mm\Phi^{-1}(\Re(B\sigma^{\perp}A))\right\|^2\\ &\leqslant \frac{1}{4}(M+m)^2.\end{aligned}$$

This proves (2.3) by Lemma 2.5 (iii). $\square$

From the inequality (2.3) with the Löwner–Heinz inequality, we obtain the inequality

$$\Phi\left(\Re(A\sigma B)\right) \leqslant \sec^2(\theta)K(h)\Phi\left(\Re(B\sigma^{\perp}A)\right). \tag{2.4}$$

On the other hand, the inequality (2.4) does not imply the inequality (2.3) in general. That is, the inequality (2.3) is stronger than the inequality (2.4). The next result generalizes the inequality (2.3) for the higher powers.

THEOREM 2.7. *Let $A,B \in S_\theta$ with $0 < mI_n \leqslant \Re(A), \Re(B) \leqslant MI_n$ and σ be an arbitrary mean. For $0 \leqslant p \leqslant 2$, we have*

$$\Phi^p\left(\Re(A\sigma B)\right) \leqslant \sec^{2p}(\theta)K^p(h)\Phi^p\left(\Re(B\sigma^{\perp}A)\right). \tag{2.5}$$

For $p \geqslant 2$, we have

$$\Phi^p\left(\Re(A\sigma B)\right) \leqslant 4^{p-2}\sec^{2p}(\theta)K^p(h)\Phi^p\left(\Re(B\sigma^{\perp}A)\right). \tag{2.6}$$

Proof. For $0 \leqslant p \leqslant 2$ we have $0 \leqslant \frac{p}{2} \leqslant 1$. So, by the Löwner–Heinz inequality with (2.3), the inequality (2.5) is obvious. In the case that $p \geqslant 2$, we have

$$\begin{aligned}
&\left\|\cos^p(\theta)\Phi^{\frac{p}{2}}(\Re(A\sigma B))M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(\Re(B\sigma^{\perp}A))\right\| \\
&\leqslant \frac{1}{4}\left\|\cos^p(\theta)\Phi^{\frac{p}{2}}(\Re(A\sigma B)) + M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(\Re(B\sigma^{\perp}A))\right\|^2 \quad \text{(by Lemma 2.5 (i))} \\
&\leqslant \frac{1}{4}\left\|\cos^2(\theta)\Phi(\Re(A\sigma B)) + Mm\Phi^{-1}(\Re(B\sigma^{\perp}A))\right\|^p \quad \text{(by Lemma 2.5 (ii))} \\
&\leqslant \frac{1}{4}(M+m)^p \quad \text{(by the inequality (2.1))}.
\end{aligned}$$

By Lemma 2.5 (iii), we have

$$\cos^{2p}(\theta)\Phi^p\left(\Re(A\sigma B)\right) \leqslant \frac{(M+m)^{2p}}{16M^pm^p}\Phi^p\left(\Re(B\sigma^{\perp}A)\right)$$

which gives the desired inequality (2.6). □

Theorem 2.7 recovers Theorem 2.6 when $p = 2$. For the case $0 \leqslant p \leqslant 2$, the inequality (1.2) is recovered by taking $\theta = 0$ in the inequality (2.5). In order to try to find a sharper inequality than the obtained bound in Theorem 2.7, we need to state the following lemma.

LEMMA 2.8. *Let $A,B \in S_\theta$ with $0 < mI_n \leqslant \Re(A), \Re(B) \leqslant MI_n$. For an arbitrary mean σ, we have*

$$\Phi^2\left(\Re(A\sigma B)\right) + \sec^4(\theta)M^2m^2\Phi^{-2}\left(\Re(A\sigma B)\right) \leqslant (M^2\sec^4(\theta) + m^2)I_n.$$

Proof. By Lemma 2.1,

$$\Re(A)\sigma\Re(B) \leqslant \Re(A\sigma B) \leqslant \sec^2(\theta)(\Re(A)\sigma\Re(B)). \tag{2.7}$$

On the other hand, by the property of means, we have

$$mI_n \leqslant \Re(A)\sigma\Re(B) \leqslant MI_n. \tag{2.8}$$

From (2.7) and (2.8), it follows that

$$mI_n \leqslant \Re(A\sigma B) \leqslant (M\sec^2(\theta))I_n. \tag{2.9}$$

From (2.9), for a unital positive linear map Φ, we have

$$mI_n \leqslant \Phi(\Re(A\sigma B)) \leqslant (M\sec^2(\theta))I_n \Longrightarrow m^2 I_n \leqslant \Phi^2(\Re(A\sigma B)) \leqslant (M^2\sec^4(\theta))I_n.$$

Using Lemma 2.3, we obtain

$$\Phi^2(\Re(A\sigma B)) + \sec^4(\theta)M^2m^2\Phi^{-2}(\Re(A\sigma B)) \leqslant (M^2\sec^4(\theta)+m^2)I_n. \quad \square$$

THEOREM 2.9. *Let $A,B \in S_\theta$ be such that $0 < mI_n \leqslant \Re(A), \Re(B) \leqslant MI_n$ and σ be an arbitrary mean. For $p \geqslant 4$, we have*

$$\Phi^p(\Re(A\sigma B)) \leqslant \left(\frac{K(h)\left(M^2\sec^4(\theta)+m^2\right)}{2^{\frac{4}{p}}Mm}\right)^p \Phi^p\left(\Re(B\sigma^\perp A)\right).$$

For $0 \leqslant p \leqslant 4$, we have

$$\Phi^p(\Re(A\sigma B)) \leqslant \left(\frac{K(h)\left(M^2\sec^4(\theta)+m^2\right)}{2Mm}\right)^p \Phi^p\left(\Re(B\sigma^\perp A)\right).$$

Proof. From Theorem 2.6, we have

$$\Phi^{-2}\left(\Re(B\sigma^\perp A)\right) \leqslant \sec^4(\theta)K^2(h)\Phi^{-2}(\Re(A\sigma B)). \tag{2.10}$$

A simple computation shows that for $p \geqslant 4$,

$$\begin{aligned}
&\left\|\Phi^{\frac{p}{2}}(\Re(A\sigma B))M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{-\frac{p}{2}}\left(\Re(B\sigma^\perp A)\right)\right\| \\
&\leqslant \frac{1}{4}\left\|K^{\frac{p}{4}}(h)\Phi^{\frac{p}{2}}(\Re(A\sigma B)) + \left(\frac{M^2m^2}{K(h)}\right)^{\frac{p}{4}}\Phi^{-\frac{p}{2}}\left(\Re(B\sigma^\perp A)\right)\right\|^2 \quad \text{(by Lemmas 2.5 (i))} \\
&\leqslant \frac{1}{4}\left\|K(h)\Phi^2(\Re(A\sigma B)) + \frac{M^2m^2}{K(h)}\Phi^{-2}\left(\Re(B\sigma^\perp A)\right)\right\|^{\frac{p}{2}} \quad \text{(by Lemmas 2.5 (ii))} \\
&\leqslant \frac{1}{4}\left\|K(h)\Phi^2(\Re(A\sigma B)) + M^2m^2\sec^4(\theta)K(h)\Phi^{-2}(\Re(A\sigma B))\right\|^{\frac{p}{2}} \\
&\qquad \text{(by the inequality (2.10))} \\
&= \frac{1}{4}K^{\frac{p}{2}}(h)\left\|\Phi^2(\Re(A\sigma B)) + M^2m^2\sec^4(\theta)\Phi^{-2}(\Re(A\sigma B))\right\|^{\frac{p}{2}} \\
&\leqslant \frac{1}{4}\left(K(h)\left(M^2\sec^4(\theta)+m^2\right)\right)^{\frac{p}{2}} \quad \text{(by Lemma 2.8).}
\end{aligned}$$

If $0 \leqslant p \leqslant 4$, then $0 \leqslant \frac{p}{4} \leqslant 1$. By considering the case of $p=4$ and using the Löwner–Heinz inequality, we conclude the inequality for $0 \leqslant p \leqslant 4$. □

If we take $\theta=0$ in Theorem 2.9, then we have [17, Theorem 2.9] and [17, Corollary 2.10].

REMARK 2.10. We compare two ratios

$$\alpha(\theta,m,M) := \begin{cases} \sec^{2p}(\theta)K^p(h) = \left(\sec^2(\theta)K(h)\right)^p, & (0 \leqslant p \leqslant 2) \\ 4^{p-2}\sec^{2p}(\theta)K^p(h) = \left(\dfrac{4\sec^2(\theta)K(h)}{2^{4/p}}\right)^p, & (p \geqslant 2) \end{cases}$$

and

$$\beta(\theta,m,M) := \begin{cases} \left(\dfrac{K(h)\left(M^2\sec^4(\theta)+m^2\right)}{2Mm}\right)^p, & (0 \leqslant p \leqslant 4) \\ \left(\dfrac{K(h)\left(M^2\sec^4(\theta)+m^2\right)}{2^{4/p}Mm}\right)^p, & (p \geqslant 4) \end{cases}$$

respectively given in Theorem 2.7 and Theorem 2.9.

(i) For the case $0 \leqslant p \leqslant 2$, it is sufficient to compare $a_1(\theta) := \sec^2(\theta)$ and $b_1(\theta,m,M) := \dfrac{M^2\sec^4(\theta)+m^2}{2Mm}$ for $0<m\leqslant M$ and $0\leqslant\theta<\frac{\pi}{2}$. Since $(M\sec^2(\theta) - m)^2 \geqslant 0$, we have $a_1(\theta) \leqslant b_1(\theta,m,M)$ so that $\alpha(\theta,m,M) \leqslant \beta(\theta,m,M)$.

(ii) For the case $2 \leqslant p \leqslant 4$, it is sufficient to compare $a_2(\theta,p) := \dfrac{4\sec^2(\theta)}{2^{2/p}}$ and $b_1(\theta,m,M) := \dfrac{M^2\sec^4(\theta)+m^2}{2Mm}$ for $0<m\leqslant M$ and $0\leqslant\theta<\frac{\pi}{2}$. Then we have the following examples:

$$a_2\left(\frac{\pi}{3},3\right) = 2^{8/3} < \frac{65}{4} = b_1\left(\frac{\pi}{3},1,2\right)$$

and

$$a_2\left(\frac{\pi}{6},3\right) = \frac{2^{8/3}}{3} > \frac{73}{36} = b_1\left(\frac{\pi}{6},1,2\right).$$

There is no ordering between $\alpha(\theta,m,M)$ and $\beta(\theta,m,M)$ for the case $2\leqslant p\leqslant 4$.

(iii) For the case $p \geqslant 4$, it is sufficient to compare $a_3(\theta) := 4\sec^2(\theta)$ and $b_2(\theta,m,M) := \dfrac{M^2\sec^4(\theta)+m^2}{Mm}$ for $0<m\leqslant M$ and $0\leqslant\theta<\frac{\pi}{2}$. Then we have the following examples:

$$a_3\left(\frac{\pi}{3}\right) = 16 < \frac{65}{2} = b_2\left(\frac{\pi}{3},1,2\right) \quad \text{and} \quad a_3\left(\frac{\pi}{6}\right) = \frac{16}{3} > \frac{73}{18} = b_2\left(\frac{\pi}{6},1,2\right).$$

There is no ordering between $\alpha(\theta,m,M)$ and $\beta(\theta,m,M)$ for the case $p \geqslant 4$.

Consequently, we have for $0 \leqslant p \leqslant 2$,

$$\Phi^p\left(\Re(A\sigma B)\right) \leqslant \alpha(\theta,m,M)\Phi^p\left(\Re(B\sigma^{\perp}A)\right) \leqslant \beta(\theta,m,M)\Phi^p\left(\Re(B\sigma^{\perp}A)\right)$$

and for $p \geqslant 2$

$$\Phi^p\left(\Re(A\sigma B)\right) \leqslant \min\{\alpha(\theta,m,M),\beta(\theta,m,M)\}\Phi^p\left(\Re(B\sigma^{\perp}A)\right).$$

from Theorem 2.7 and Theorem 2.9.

THEOREM 2.11. *Let $A,B \in S_\theta$ with $0 < mI_n \leqslant \Re(A),\Re(B) \leqslant MI_n$ and σ_1,σ_2 be two arbitrary means between σ and $\sigma^{\perp}$ for a certain matrix mean σ. Then for $p \geqslant 2$ we have*

$$\Phi^p\left(\Re(A\sigma_2 B)\right) \leqslant 4^{p-2}\sec^{2p}(\theta)K^p(h)\Phi^p\left(\Re(B\sigma_1 A)\right), \tag{2.11}$$

and for $0 \leqslant p \leqslant 2$ we have

$$\Phi^p\left(\Re(A\sigma_2 B)\right) \leqslant \sec^{2p}(\theta)K^p(h)\Phi^p\left(\Re(B\sigma_1 A)\right). \tag{2.12}$$

Proof. Let $\sigma_1 \geqslant \sigma^{\perp}$ and $\sigma_2 \leqslant \sigma$. Using Lemma 2.1, Lemma 2.2, the equality (1.1) and the inequality (2.2), we get the following chain of inequalities:

$$\begin{aligned}
&\cos^2(\theta)\Phi(\Re(A\sigma_2 B)) + Mm\Phi^{-1}(\Re(B\sigma_1 A)) \\
&\leqslant \Phi(\Re(A)\sigma_2\Re(B)) + Mm\Phi^{-1}(\Re(B)\sigma_1\Re(A)) \quad \text{(by Lemma 2.1)} \\
&\leqslant \Phi(\Re(A)\sigma_2\Re(B)) + Mm\Phi^{-1}(\Re(B)\sigma^{\perp}\Re(A)) \quad \text{(by } \sigma_1 \geqslant \sigma^{\perp}\text{)} \\
&\leqslant \Phi(\Re(A)\sigma_2\Re(B)) + Mm\Phi(\Re^{-1}(A)\sigma\Re^{-1}B) \quad \text{(by Lemma 2.2 with (1.1))} \\
&\leqslant \Phi(\Re(A)\sigma\Re(B)) + Mm\Phi(\Re^{-1}(A)\sigma\Re^{-1}B) \quad \text{(by } \sigma_2 \leqslant \sigma\text{)} \\
&\leqslant (M+m)I_n \quad \text{(by the inequality (2.2))}.
\end{aligned}$$

Thus we have the inequality (2.11) by the similar way to the proof of Theorem 2.7. If we set $p = 2$ in the inequality (2.11) and use the Löwner–Heinz inequality, then we obtain the inequality (2.12). □

3. Numerical radius, norm and determinantal inequalities

The next lemma is necessary to obtain more results.

LEMMA 3.1. [8] *Let $A,B \in S_\theta$ and $f \in \mathbf{m}$. Then*

$$1)\ f(\Re(A)) \leqslant \Re(f(A)) \leqslant \sec^2(\theta)f(\Re(A)), \tag{3.1}$$

$$2)\ w(\Re(A)) \leqslant w(A) \leqslant \sec(\theta)w(\Re(A)), \tag{3.2}$$

$$3)\ f(\|\Re(A)\|_u) \leqslant \|\Re f(A)\|_u \leqslant \sec^2(\theta)f(\|\Re(A)\|_u). \tag{3.3}$$

LEMMA 3.2. [21] *Let $A \in S_\theta$ for $0 \leqslant \theta < \frac{\pi}{2}$. Then*

$$\cos(\theta)\|A\|_u \leqslant \|\Re(A)\|_u \leqslant \|A\|_u.$$

COROLLARY 3.3. *Let $A,B \in S_\theta$ with $0 < mI_n \leqslant \Re(A), \Re(B) \leqslant MI_n$ and σ_1, σ_2 be two arbitrary means between σ and $\sigma^\perp$ for a certain matrix mean σ and $f \in \mathbf{m}$. Then*

$$\Re f(\Phi(A\sigma_2 B)) \leqslant \sec^2(\theta)\Re f(\sec^2(\theta)K(h)\Phi(B\sigma_1 A)) \leqslant \sec^4(\theta)K(h)\Re f(\Phi(B\sigma_1 A)).$$

Proof. From Theorem 2.11 with $p = 1$ and the inequality (3.1), we get

$$\begin{aligned}\cos^2(\theta)\Re f(\Phi(A\sigma_2 B)) &\leqslant f(\Re\Phi(A\sigma_2 B)) \quad \text{(by the inequality (3.1))}\\ &\leqslant f(\sec^2(\theta)K(h)\Re\Phi(B\sigma_1 A)) \quad \text{(by Theorem 2.11 with } p=1)\\ &\leqslant \Re f(\sec^2(\theta)K(h)\Phi(B\sigma_1 A)). \quad \text{(by the inequality (3.1))}\end{aligned}$$

It is known that [13, Lemma 2.2] if $\lambda \geqslant 1$, then $f(\lambda x) \leqslant \lambda f(x)$ for every $x > 0$ and $f \in \mathbf{m}$. This proves the second inequality. □

COROLLARY 3.4. *Let $A,B \in S_\theta$ with $0 < mI_n \leqslant \Re(A), \Re(B) \leqslant MI_n$ and σ_1, σ_2 be two arbitrary means between σ and $\sigma^\perp$ for a certain matrix mean σ. Then*

$$\|\Phi(A\sigma_2 B))\|_u \leqslant \sec^3(\theta)K(h)\|\Phi(B\sigma_1 A))\|_u.$$

Proof. By Lemma 3.2 and Theorem 2.11 with $p = 1$, we coclude that

$$\begin{aligned}\|\Phi(A\sigma_2 B))\|_u &\leqslant \sec(\theta)\|\Re\Phi(A\sigma_2 B)\|_u \quad \text{(by Lemma 3.2)}\\ &\leqslant \sec^3(\theta)K(h)\|\Re\Phi(B\sigma_1 A))\|_u \quad \text{(by Theorem 2.11 with } p=1)\\ &\leqslant \sec^3(\theta)K(h)\|\Phi(B\sigma_1 A))\|_u \quad \text{(by Lemma 3.2).}\end{aligned}$$ □

THEOREM 3.5. *Let $A,B \in S_\theta$ with $0 < mI_n \leqslant \Re(A), \Re(B) \leqslant MI_n$ and σ_1, σ_2 be two arbitrary means between σ and $\sigma^\perp$ for a certain matrix mean σ and $f \in \mathbf{m}$. Then*

$$f(w(A\sigma_2 B)) \leqslant \sec^3(\theta)K(h)w(f(B\sigma_1 A)).$$

Proof. By the second and the third inequalities of Lemma 3.1 and Theorem 2.11 with $p = 1$, we get

$$\begin{aligned}f(w(A\sigma_2 B)) &\leqslant f(\sec(\theta)w(\Re(A\sigma_2 B))) \quad \text{(by the inequality (3.2))}\\ &\leqslant f(\sec(\theta)w(\sec^2(\theta)K(h)\Re(B\sigma_1 A))) \quad \text{(by Theorem 2.11 for } p=1)\\ &= f(\sec^3(\theta)K(h)w(\Re(B\sigma_1 A)))\\ &\leqslant \sec^3(\theta)K(h)f(w\Re(B\sigma_1 A))\\ &= \sec^3(\theta)K(h)f(\|\Re(B\sigma_1 A)\|)\\ &\leqslant \sec^3(\theta)K(h)\|\Re f(B\sigma_1 A)\| \quad \text{(by the inequality (3.3))}\\ &= \sec^3(\theta)K(h)w(\Re f(B\sigma_1 A))\\ &\leqslant \sec^3(\theta)K(h)w(f(B\sigma_1 A)). \quad \text{(by the inequality (3.2))}\end{aligned}$$

In the third inequality above, we used the fact $f(\lambda x) \leqslant \lambda f(x)$ if $\lambda \geqslant 1$ and $x > 0$. $\square$

To obtain a determinantal inequality, we use the following lemmas.

LEMMA 3.6. [14, 16] *Let $A \in S_\theta$. Then*

$$\det(\Re(A)) \leqslant |\det(A)| \leqslant \sec^n(\theta)\det(\Re(A)).$$

LEMMA 3.7. [20] *Let $A, B \in S_\theta$. Then*

$$|\det(A+B)| \leqslant \sec^{2n}(\theta)|\det(I_n+A)| \cdot |\det(I_n+B)|$$

and

$$\|A+B\|_u \leqslant \sec(\theta)\|I_n+A\|_u \cdot \|I_n+B\|_u.$$

The authors showed in [8, Theorem 8.1] that if $A, B \in S_\theta$ and $f \in \mathbf{m}$, then

$$\|f(A+B)\|_u \leqslant \sec^3(\theta)\|f(A)+f(B)\|_u.$$

On the other hand, we have

$$\|\Re(f(A+B))\|_u \leqslant \|f(A+B)\|_u.$$

Thus,

$$\|\Re(f(A+B))\|_u \leqslant \sec^3(\theta)\|f(A)+f(B)\|_u. \tag{3.4}$$

If we put $f(x) := x$ in (3.4), we get

$$\|\Re(A+B)\|_u \leqslant \sec^3(\theta)\|A+B\|_u.$$

From the above inequality and the second relations of Lemma 3.7, we have

$$\|\Re(A+B)\|_u \leqslant \sec^4(\theta)\|I_n+A\|_u \cdot \|I_n+B\|_u. \tag{3.5}$$

COROLLARY 3.8. *Let $A, B \in S_\theta$ with $0 < mI_n \leqslant \Re(A), \Re(B) \leqslant MI_n$. Then*

$$|\det(A!B)| \leqslant \frac{\sec^{3n+2}(\theta)K(h)}{2^n}|\det(I_n+A)| \cdot |\det(I_n+B)|$$

and

$$\|A!B\|_u \leqslant \frac{\sec^7(\theta)K(h)}{2}\|I_n+A\|_u \cdot \|I_n+B\|_u.$$

Proof. We set $\Phi(X) := X$ for any $X \in \mathbb{M}_n$. If we put $p := 1$ and $\sigma = !$ in Theorem 2.7, then we get

$$\Re(A!B) \leqslant \sec^2(\theta)K(h)\Re(A\nabla B). \tag{3.6}$$

Applying Lemma 3.6, the inequality (3.6) and the first inequality of Lemma 3.7, we have

$$\begin{aligned}|\det(A!B)| &\leqslant \sec^n(\theta)\det\Re(A!B)\\ &\leqslant \frac{\sec^{n+2}(\theta)K(h)}{2^n}\det(\Re(A+B))\\ &\leqslant \frac{\sec^{3n+2}(\theta)K(h)}{2^n}|\det(I_n+A)|\cdot|\det(I_n+B)|.\end{aligned}$$

From Lemma 3.2, the inequality(3.6) and the inequality (3.5), respectively, we have

$$\begin{aligned}\|A!B\|_u &\leqslant \sec(\theta)\|\Re(A!B)\|_u\\ &\leqslant \frac{\sec^3(\theta)K(h)}{2}\|\Re(A+B)\|_u\\ &\leqslant \frac{\sec^7(\theta)K(h)}{2}\|I_n+A\|_u\cdot\|I_n+B\|_u. \quad \square\end{aligned}$$

4. Some applications with matrix means

We notice that [3] if A,B are positive definite, then for any unitarily invariant norm $\|\cdot\|_u$

$$\|A\sigma B\|_u \leqslant \|A\|_u\sigma\|B\|_u. \tag{4.1}$$

Recently, Y. Bedrani et al. presented the accretive version of the inequality (4.1) in [8, Theorem 8.2]:

$$\|A\sigma B\|_u \leqslant \sec^3(\theta)\left(\|A\|_u\sigma\|B\|_u\right), \tag{4.2}$$

where $A,B\in\mathbb{M}_n$ are accretive matrices such that $W(A),W(B)\subset S_\theta$. Also we have the following inequality for a positive linear map in connection with the matrix means, shown by T. Ando [4]. We use the following known facts with proofs for the convenience to the readers.

LEMMA 4.1. *Let $X\in S_\theta$. Then*

(i) $w(\Re(X)) = \|\Re(X)\|$.

(ii) $\Re(\Phi(X)) = \Phi(\Re(X))$ *for any positive linear map* Φ.

Proof.

(i) The result follows from $\Re(X)>0$.

(ii) It follows that

$$\Phi(\Re(X)) = \Phi\left(\frac{X+X^*}{2}\right) = \frac{\Phi(X)+\Phi(X^*)}{2} = \frac{\Phi(X)+\Phi(X)^*}{2} = \Re(\Phi(X)),$$

since $\Phi(X^*)=\Phi(X)^*$ which can be shown by the Cartesian decomposition $X = Y+iZ$ where $Y^*=Y$ and $Z^*=Z$. $\square$

LEMMA 4.2. [4] *If $A,B \in \mathbb{M}_n$ be positive definite. Then*

$$\Phi(A\sigma B) \leqslant \Phi(A)\sigma\Phi(B). \tag{4.3}$$

To get our results, we need to prove the next lemma.

LEMMA 4.3. *Let $A,B \in S_\theta$ and σ_1,σ_2 be two arbitrary means such that $\sigma_1 \leqslant \sigma_2$. Then*

$$\Phi\left(\Re(A\sigma_1 B)\right) \leqslant \sec^2(\theta)\Re(\Phi(A))\sigma_2\Re(\Phi(B)) \leqslant \sec^2(\theta)\Re(\Phi(A)\sigma_2\Phi(B)).$$

Proof. The following chain of the ineualities is computable.

$$\begin{aligned}
\Phi\left(\Re(A\sigma_1 B)\right) &\leqslant \sec^2(\theta)\Phi(\Re(A)\sigma_1\Re(B)) \quad \text{(by Lemma 2.1)}\\
&\leqslant \sec^2(\theta)\Phi(\Re(A))\sigma_1\Phi(\Re(B)) \quad \text{(by the inequality (4.3))}\\
&\leqslant \sec^2(\theta)\Phi(\Re(A))\sigma_2\Phi(\Re(B))\\
&= \sec^2(\theta)\Re(\Phi(A))\sigma_2\Re(\Phi(B)) \quad \text{(by Lemma 4.1 (ii))}\\
&\leqslant \sec^2(\theta)\Re(\Phi(A)\sigma_2\Phi(B)) \quad \text{(by Lemma 2.1)}. \quad \square
\end{aligned}$$

COROLLARY 4.4. *Let $A,B \in S_\theta$ and σ_1,σ_2 be two arbitrary means such that $\sigma_1 \leqslant \sigma_2$. Then*

$$\Re\langle(A\sigma_1 B)x,x\rangle \leqslant \sec^2(\theta)\Re(\langle Ax,x\rangle\sigma_2\langle Bx,x\rangle).$$

Proof. By considering $\Phi(A) = \langle Ax,x\rangle$ in Lemma 4.3, we obtain the desired result. $\square$

Now we are ready to prove our theorem which is an extension of the inequality (4.2).

THEOREM 4.5. *Let $A,B \in S_\theta$ and σ_1,σ_2 be two arbitrary means such that $\sigma_1 \leqslant \sigma_2$. Then for any unitarily invariant norm*

$$\|\Phi(A\sigma_1 B)\|_u \leqslant \sec^3\theta\|\Phi(A)\sigma_2\Phi(B)\|_u. \tag{4.4}$$

and

$$\|\Phi(A\sigma_1 B)\|_u \leqslant \sec^3(\theta)\left(\|\Phi(A)\|_u\sigma_2\|\Phi(B)\|_u\right). \tag{4.5}$$

Proof. We calculate as follows

$$\begin{aligned}
\cos(\theta)\|\Phi(A\sigma_1 B)\|_u &\leqslant \|\Re(\Phi(A\sigma_1 B))\|_u \quad \text{(by Lemma 3.2)}\\
&= \left\|\Phi\left(\Re(A\sigma_1 B)\right)\right\|_u \quad \text{(by Lemma 4.1 (ii))}\\
&\leqslant \sec^2(\theta)\|\Re(\Phi(A))\sigma_2\Re(\Phi(B))\|_u \quad \text{(by Lemma 4.3)}\\
&\leqslant \sec^2(\theta)\left\|\Re\left(\Phi(A)\sigma_2\Phi(B)\right)\right\|_u \quad \text{(by Lemma 2.1)}\\
&\leqslant \sec^2(\theta)\|\Phi(A)\sigma_2\Phi(B)\|_u. \quad \text{(by Lemma 3.2)}
\end{aligned}$$

Thus, we obtain the inequality (4.4).

We also have

$$\begin{aligned}
\|\Phi(A\sigma_1 B)\|_u &\leqslant \sec(\theta)\|\Re(\Phi(A\sigma_1 B))\|_u \quad \text{(by Lemma 3.2)}\\
&= \sec(\theta)\|\Phi(\Re(A\sigma_1 B))\|_u \quad \text{(by Lemma 4.1 (ii))}\\
&\leqslant \sec^3(\theta)\|\Re(\Phi(A))\sigma_2\Re(\Phi(B))\|_u \quad \text{(by Lemma 4.3)}\\
&= \sec^3(\theta)\|\Phi(\Re(A))\sigma_2\Phi(\Re(B))\|_u \quad \text{(by Lemma 4.1 (ii))}\\
&\leqslant \sec^3(\theta)\left(\|\Phi(\Re(A))\|_u\sigma_2\|\Phi(\Re(B))\|_u\right) \quad \text{(by the inequality (4.1))}\\
&= \sec^3(\theta)\left(\|\Re(\Phi(A))\|_u\sigma_2\|\Re(\Phi(B))\|_u\right) \quad \text{(by Lemma 4.1 (ii))}\\
&\leqslant \sec^3(\theta)\left(\|\Phi(A)\|_u\sigma_2\|\Phi(B)\|_u\right) \quad \text{(by Lemma 3.2).} \quad \square
\end{aligned}$$

Taking an account for the relation (4.2), it may be interesting to compare the inequality (4.4) and the inequality (4.5) as the bound of $\|\Phi(A\sigma_1 B)\|_u$. The authors in [9, Theorem 3.7] proved that

$$w(A\sigma B) \leqslant \sec^3(\theta)w(A)\sigma w(B). \tag{4.6}$$

The next result extends the inequality (4.6) for two arbitrary means.

THEOREM 4.6. *Let $A,B \in S_\theta$ and σ_1,σ_2 be two arbitrary means such that $\sigma_1 \leqslant \sigma_2$. Then*

$$w(\Phi(A\sigma_1 B)) \leqslant \sec^3(\theta)w(\Phi(A))\sigma_2 w(\Phi(B))$$

and

$$w(\Phi(A\sigma_1 B)) \leqslant \sec^3(\theta)w(\Phi(A)\sigma_2\Phi(B)).$$

Proof. We obtain the first inequality as follows

$$\begin{aligned}
w(\Phi(A\sigma_1 B)) &\leqslant \|\Phi(A\sigma_1 B)\|\\
&\leqslant \sec(\theta)\|\Re(\Phi(A\sigma_1 B))\| \quad \text{(by Lemma 3.2)}\\
&= \sec(\theta)\|\Phi(\Re(A\sigma_1 B))\| \quad \text{(by Lemma 4.1 (ii))}\\
&\leqslant \sec^3(\theta)\|\Re(\Phi(A))\sigma_2\Re(\Phi(B))\| \quad \text{(by Lemma 4.3)}\\
&\leqslant \sec^3(\theta)\|\Re(\Phi(A))\|\sigma_2\|\Re(\Phi(B))\|\\
&\qquad\qquad \text{(by the inequality (4.1) and Lemma 4.1 (ii))}\\
&= \sec^3(\theta)w(\Re(\Phi(A))\sigma_2 w(\Re(\Phi(B))) \quad \text{(by Lemma 4.1 (i))}\\
&\leqslant \sec^3(\theta)w(\Phi(A))\sigma_2 w(\Phi(B)). \quad \text{(by Lemma 3.1)}
\end{aligned}$$

We also obtain the second inequality as follows

$$\begin{aligned} w(\Phi(A\sigma_1 B)) &\leqslant \|\Phi(A\sigma_1 B)\| \\ &\leqslant \sec(\theta)\|\Re(\Phi(A\sigma_1 B))\| \quad \text{(by Lemma 3.2)} \\ &\leqslant \sec^3(\theta)\|\Re(\Phi(A)\sigma_2\Phi(B))\| \quad \text{(by Lemma 4.3 and Lemma 4.1 (ii))} \\ &= \sec^3(\theta) w(\Re(\Phi(A)\sigma_2\Phi(B))) \quad \text{(by Lemma 4.1 (i))} \\ &\leqslant \sec^3(\theta) w(\Phi(A)\sigma_2\Phi(B)) \quad \text{(by Lemma 3.1).} \quad \square \end{aligned}$$

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Availability of data and materials. Not applicable.

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