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THE CLAUSING INEQUALITY AND STRONG $\mathscr{F}$–CONCAVITY

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Abstract. In this paper, we introduce the class of strongly $\mathscr{F}$-concave functions as the class of functions $f\colon I\to\mathbb{R}$, where $I\subseteq\mathbb{R}$ is an interval, which satisfy

$$tf(x)+(1-t)f(y)-f(tx+(1-t)y)\leqslant F(tx+(1-t)y)-tF(x)-(1-t)F(y)$$

for $x,y\in I$ and $t\in[0,1]$ and some convex function F on I called control function. This class contains the class of strongly concave functions. Analogous generalization of strongly convex functions is also given.

We investigate possibilities to use this class to refine the Clausing inequality. The refinement of the left-hand side of the Clausing inequality has the same form as refinements of any Jensen type inequality (for example, the Hermite-Hadamard inequality), but we introduce a suitable class of control functions F such that these refinements are applicable to much broader class of $\mathscr{F}$-concave functions than it is possible for strongly concave functions. The refinements for the right-hand side of the inequality are more subtle to obtain, but flexibility of choosing control functions enables us to refine this side also.

1. Introduction

Let $I\subseteq\mathbb{R}$ be an interval and c a positive real number. A function $f\colon I\to\mathbb{R}$ is called strongly convex with modulus c if

$$f(tx+(1-t)y)\leqslant tf(x)+(1-t)f(y)-ct(1-t)(x-y)^2$$

holds for every $x,y\in I$ and $t\in[0,1]$. If $-f$ is strongly convex (with modulus c) then we say that f is strongly concave (with modulus c). Strongly convex functions were introduced by B. T. Polyak in [18]. There is a vast literature on these notions, although strong concavity is rarely mentioned (see for example [6], [13], [14], [19] and the references therein).

It is a surprise that not much is done in generalizing strong convexity and strong concavity. See [9] as one example. We introduce in a natural way the class of strongly $\mathscr{F}$-convex ($\mathscr{F}$-concave) functions which is wider than the class of strongly convex (concave) functions.

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DEFINITION 1. Let $I \subseteq \mathbb{R}$ be an interval and $F: I \to \mathbb{R}$ be a convex function. We say that a function $f: I \to \mathbb{R}$ is strongly $\mathscr{F}$-convex with control function F if

$$tf(x)+(1-t)f(y)-f(tx+(1-t)y) \geqslant tF(x)+(1-t)F(y)-F(tx+(1-t)y) \quad (1)$$

holds for all $x, y \in I$ and $t \in [0,1]$. If $-f$ is strongly $\mathscr{F}$-convex, then we say that f is strongly $\mathscr{F}$-concave function.

We will also use the term strongly convex (concave) with a control function F. For related notions see [3], [4], [10], [20], and references therein.

It is easy to verify

$$tF(x)+(1-t)F(y)-F(tx+(1-t)y) = ct(1-t)(x-y)^2$$

for $F(x) = cx^2$. This shows that the class of strongly $\mathscr{F}$-convex (strongly $\mathscr{F}$-concave) functions contains the class of strongly convex (strongly concave) functions.

It is obvious from (1) that f is strongly $\mathscr{F}$-convex (strongly $\mathscr{F}$-concave) iff $f-F$ ($f+F$) is convex (concave) on I.

EXAMPLE 1. Let $f(x) = \sqrt{x}$, $x \in [0,1]$ and $F(x) = cx^{\alpha}$, $c > 0$, $\alpha > 1$, $x \in [0,1]$. Then $f+F$ is concave iff

$$c \leqslant \frac{1}{4\alpha(\alpha-1)} x^{\frac{1}{2}-\alpha}, \quad x \in (0,1],$$

which gives that f is strongly $\mathscr{F}$-concave for $c \leqslant \frac{1}{4\alpha(\alpha-1)}$.

Many papers are written on refinements of classical inequalities for strongly convex functions. These inequalities are mainly of the Jensen type (the Hermite-Hadamard inequality, the Hölder inequality, the Popoviciu inequality, the converse Jensen inequality, the Lah-Ribarič inequality, and similar). As an example, we give a short proof of the improvement of the Lah-Ribarič inequality (the converse Jensen inequality) for strongly $\mathscr{F}$-convex functions, which was given in [6] for strongly convex functions.

THEOREM 1. [6, Theorem 1; see also Theorem 5] *Let $I \subseteq \mathbb{R}$ be an interval and $m, M \in I$, $m < M$. If $f: I \to \mathbb{R}$ is strongly convex with modulus c, then*

$$\begin{aligned} f(\bar{x}) &\leqslant \sum_{i=1}^{n} t_i f(x_i) - c\sum_{i=1}^{n} t_i (x_i-\bar{x})^2 \\ &\leqslant \frac{M-\bar{x}}{M-m} f(m) + \frac{\bar{x}-m}{M-m} f(M) - c(M-\bar{x})(\bar{x}-m), \end{aligned}$$

where $x_1, \ldots, x_n \in [m, M]$, $t_1, \ldots, t_n \geqslant 0$ with $\sum_{i=1}^{n} t_i = 1$, and $\bar{x} = \sum_{i=1}^{n} t_i x_i$.

We give a proof of Theorem 1 in the strongly $\mathscr{F}$-convex setting. Suppose that f is strongly $\mathscr{F}$-convex on I (with control function F). Using the discrete Jensen inequality and the Lah-Ribarič inequality for $f-F$ we get:

$$\begin{aligned} f(\bar{x}) &\leqslant \sum_{i=1}^{n} t_i f(x_i) - \left(\sum_{i=1}^{n} t_i F(x_i) - F(\bar{x}) \right) = \sum_{i=1}^{n} t_i (f-F)(x_i) + F(\bar{x}) \\ &\leqslant \frac{M-\bar{x}}{M-m}(f-F)(m) + \frac{\bar{x}-m}{M-m}(f-F)(M) + F(\bar{x}) \\ &= \frac{M-\bar{x}}{M-m} f(m) + \frac{\bar{x}-m}{M-m} f(M) - \left(\frac{M-\bar{x}}{M-m} F(m) + \frac{\bar{x}-m}{M-m} F(M) - F(\bar{x}) \right). \end{aligned}$$

It is straightforward to see that

$$\frac{M-\bar{x}}{M-m} F(m) + \frac{\bar{x}-m}{M-m} F(M) - F(\bar{x}) = c(M-\bar{x})(\bar{x}-m)$$

for $F(x) = cx^2$.

In [6] the supporting functions $S(x) = c(x-\bar{x})^2 + a(x-\bar{x}) + b$ and the generalized Beckenbach convexity with respect to the family $\mathscr{F}_c = \{cx^2 + ax + b;\ a,b \in \mathbb{R}\}$ for strongly convex functions (with modulus c) are used. It could be of interest to develop an analogous theory for strongly $\mathscr{F}$-convex (concave) functions.

In this paper we consider the problem of improving the Clausing inequality for strongly $\mathscr{F}$-concave ($\mathscr{F}$-convex) functions and by that for strongly concave (convex) functions. It turns out that for this inequality (for its one side) the answer is not as simple as in the case of Jensen type inequalities (see Theorem 1 as a typical case).

The following theorem is given in [2] (see also [1], [11]).

THEOREM 2. (The Clausing inequality) *Let w be continuous on $[0,1]$ and increasing on $[0,1/2]$, with $w(x) = w(1-x)$ for $x \in [0,1]$. If f is concave and positive on $(0,1)$, then*

$$\int_0^1 f(x)dx \int_0^1 w(x)dx \leqslant \int_0^1 f(x)w(x)dx \leqslant \int_0^1 f(x)dx \int_0^1 \widehat{w}(x)dx, \tag{2}$$

where $\widehat{w}(x) = 4\min\{x, 1-x\}\, w(x)$.

Both sides of (2) are sharp. It is easy to see that equality is achieved on the left-hand side for $f(x) = kx + l$, $k,l \in \mathbb{R}$ and on the right-hand side for $f(x) = \min\{x, 1-x\}$.

Although this is the standard form of the Clausing inequality, a more careful inspection shows that there is a significant difference between the left-hand side and the right-hand side of (2). The left-hand side remains unchanged if f is replaced with $f + C$ for any $C \in \mathbb{R}$. This implies that this side holds for any concave f and the reverse inequality holds for convex f on $(0,1)$. This inequality is better known as the Levin-Stečkin inequality (see [8] or for example [14]). There are various more general forms of the Levin-Stečkin inequality. Since we are interested in how strong concavity and convexity influence on inequalities, we present here the following version (see [15], and compare to [12, Chapter 4] or [13]) as probably the most simple to express.

THEOREM 3. *For $i=1,2$, let $\lambda_i\colon [a,b]\to\mathbb{R}$ be non-constant continuous functions of bounded variation and $\lambda_i(a+b-x)=-\lambda_i(x)$ on $[a,b]$. Then the following two statements are equivalent:*

1. *For every concave function $f\colon [a,b]\to\mathbb{R}$ holds*

$$\frac{\int_a^b f(x)d\lambda_2(x)}{\int_a^b d\lambda_2(x)} \leqslant \frac{\int_a^b f(x)d\lambda_1(x)}{\int_a^b d\lambda_1(x)}. \tag{3}$$

2. *For every $s\in[a,b]$ holds*

$$\frac{\int_a^s \lambda_2(x)dx}{\lambda_2(b)} \geqslant \frac{\int_a^s \lambda_1(x)dx}{\lambda_1(b)}. \tag{4}$$

The reverse inequality in (3) for convex f is equivalent to (4).

Since obviously $\int_a^s \lambda_2(x)dx=\int_a^{a+b-s}\lambda_2(x)dx$, it is enough to check the condition (4) for $s\in[a,(a+b)/2]$.

In the case $\lambda=\lambda_2$ and $d\lambda_1(x)=w(x)d\lambda(x)$, where w is increasing on $[a,(a+b)/2]$, the condition (4) easily follows.

The right-hand side of (2) is less investigated. Set:

$$\begin{aligned}\widehat{c_3} &= 2\lambda(b)\frac{\int_a^{(a+b)/2}\int_{(a+b)/2}^x w(t)d\lambda(t)dx}{\int_a^{(a+b)/2}\lambda(x)dx} \\ &= 2\lambda(b)\frac{\int_{(a+b)/2}^b (b-x)w(x)d\lambda(x)}{\int_{(a+b)/2}^b \lambda(x)dx} = 2\lambda(b)\frac{\int_a^{(a+b)/2}(x-a)w(x)d\lambda(x)}{\int_{(a+b)/2}^b \lambda(x)dx}.\end{aligned} \tag{5}$$

It is straightforward to see that $\widehat{c_3}$ for $\lambda(x)=x-(a+b)/2$ reduces to $\displaystyle\int_a^b \widehat{w}(x)d\lambda(x)$, where $\widehat{w}(x)=4\min\left\{\dfrac{x-a}{b-a},\dfrac{b-x}{b-a}\right\}w(x)$ (see (2)).

The following theorem from [16] gives a generalization of the right-hand side of (2).

THEOREM 4. *Let $\lambda\colon [a,b]\to\mathbb{R}$ be a non-constant continuous increasing function such that $\lambda(a+b-x)=-\lambda(x)$ for any $x\in[a,b]$. Let w be a non-negative continuous function on $[a,b]$ increasing on $[a,\frac{a+b}{2}]$, with $w(x)=w(a+b-x)$ for any $x\in[a,b]$. Let $\widehat{c_3}$ be as in (5). Then*

$$\frac{\int_a^b f(x)w(x)d\lambda(x)}{\widehat{c_3}} \leqslant \frac{\int_a^b f(x)d\lambda(x)}{2\lambda(b)}, \tag{6}$$

holds for every concave function f on $[a,b]$ assuming $f(a)+f(b)\geqslant 0$. Equality in (6) is attained for $f(x)=\min\{x-a,b-x\}$. If f is convex on $[a,b]$ assuming $f(a)+f(b)\leqslant 0$, then the reverse inequality holds in (6).

Another proof can be given by using the integral representation involving the Green function (see for example [5]), or by following the method given by Mercer in [11]. We notice that the setting of the Mercer's paper is very similar to the setting of Theorem 4, which is evident from the equivalent form of (6) written for concave f as:

$$\frac{\int_a^b f(x)w(x)d\lambda(x)}{\int_a^b f(x)d\lambda(x)} \leqslant \frac{\widehat{c_3}}{2\lambda(b)} = \frac{\int_a^b \min\{x-a,b-x\}w(x)d\lambda(x)}{\int_a^b \min\{x-a,b-x\}d\lambda(x)}.$$

In Section 2 we refine the left-hand side of the Clausing inequality (in the generalized form given in Theorem 3). A suitable class of control functions F is introduced in such a way that the refinements are applicable to much broader class of functions than refinements obtained for strongly concave (or strongly convex) functions. In Section 3 some refinements are given for the righ-hand side of the Clausing inequality (in the generalized form given in Theorem 4). However, since the convex case of this theorem is more restrictive than the case of Jensen type inequalities, some additional effort is required to refine this side more generally. The last section (Section 4) has twofold purpose. The first one is to give a synthesis of the refinements obtained in the previous sections. The second one is to discuss more closely the convex case of the right-hand side of the Clausing inequality, which is closely related to a refinement of this side of the inequality in the case of strongly concave functions.

2. The Levin-Stečkin inequality and strong $\mathscr{F}$-concavity

Improvements of the Levin-Stečkin inequality (3) follow the same lines of arguments as improvements of Jensen type inequalities (the Jensen inequality, the converse Jensen inequality, the Hermite-Hadamard inequality and similar). In this section, we concentrate more on how the choice of functions F impacts on the improvements.

The following result was obtained in a similar form for strongly convex functions in [13]. It gives a refinement of the Levin-Stečkin inequality (3) for strongly $\mathscr{F}$-concave (convex) functions.

THEOREM 5. *For $i=1,2$, let $\lambda_i\colon [a,b]\to\mathbb{R}$ be non-constant continuous functions of bounded variation and $\lambda_i(a+b-x)=-\lambda_i(x)$ on $[a,b]$. Suppose that*

$$\frac{\int_a^s \lambda_2(x)dx}{\lambda_2(b)} \geqslant \frac{\int_a^s \lambda_1(x)dx}{\lambda_1(b)} \tag{7}$$

holds for every $s\in[a,(a+b)/2]$. If f is strongly $\mathscr{F}$-concave on $[a,b]$ with control function F, then

$$\begin{aligned}
&\frac{\int_a^b f(x)d\lambda_2(x)}{\int_a^b d\lambda_2(x)} \\
&\leqslant \frac{\int_a^b f(x)d\lambda_1(x)}{\int_a^b d\lambda_1(x)} - \left(\frac{\int_a^b F(x)d\lambda_2(x)}{\int_a^b d\lambda_2(x)} - \frac{\int_a^b F(x)d\lambda_1(x)}{\int_a^b d\lambda_1(x)}\right) \\
&\leqslant \frac{\int_a^b f(x)d\lambda_1(x)}{\int_a^b d\lambda_1(x)}.
\end{aligned} \tag{8}$$

If f is strongly $\mathscr{F}$-convex with control function F, then (8) holds with the reverse inequalities and plus sign in front of the brackets in the middle term.

Proof. The first inequality follows from (3) for the concave function $f+F$. The second inequality follows from the convexity part of Theorem 3. □

Although many papers are written on improvements of some classical inequalities for strongly convex functions, there are remarkably few examples of these functions. The following notions are motivated by the following simple observation. Suppose that $f \in C^2([a,b])$. Then $f(x)+cx^2$ is concave iff $f''(x) \leqslant -2c$. This means that if there is an $x_0 \in [a,b]$ such that $f''(x_0)=0$, such $c>0$ cannot exist. It follows that the method of strongly concave (and similarly strongly convex) functions cannot be used to improve Jensen type inequalities (and the Levin-Steckin inequality), for example for such a simple function as $f(x)=\sin(\pi x/2)$, $x \in [0,1]$. To remedy this, at least partially, it is natural to consider suitable classes of convex functions F for this purpose. The simplest one is the class:

$$F(x)=c\,|x-x_0|^{\alpha}, \quad c>0, \quad \alpha>1, \quad x_0 \in \mathbb{R}. \tag{9}$$

This class naturally generalizes the class of functions $x \mapsto cx^2$, $c>0$, which generates strongly convex and strongly concave functions.

We give some examples. These examples are illustrations of some general problems naturally imposed by considering the family of functions given in (9) mainly used to find an optimal refinement in (8). These examples basically contain necessary apparatus to solve these problems.

EXAMPLE 2. Suppose that f'' exists and $f''(x) \leqslant -m < 0$ on $[a,b] \subset \mathbb{R}$. Then $f(x)+c\,|x-x_0|^{\alpha}$ is concave iff

$$c \leqslant -\frac{f''(x)}{\alpha(\alpha-1)}\frac{1}{|x-x_0|^{\alpha-2}}.$$

Hence f is strongly $\mathscr{F}$-concave with control function $F(x)=c\,|x-x_0|^{\alpha}$ for $\alpha \geqslant 2$ at least (see Example 1),

$$0<c \leqslant \frac{m}{\alpha(\alpha-1)} \min_{x\in[a,b]} \frac{1}{|x-x_0|^{\alpha-2}},$$

and any $x_0 \in \mathbb{R}$.

EXAMPLE 3. Left $f(x)=\sin^p(\pi x/2)$, $x \in [0,1]$, $0<p \leqslant 1$. Let $F(x)=cx^{\alpha}$, $x \in [0,1]$, $c>0$, $\alpha>1$. Then $f(x)+F(x)$ is concave iff

$$c \leqslant \frac{\pi^2}{4}\frac{p}{\alpha(\alpha-1)}\left((1-p)\sin^{p-2}\left(\frac{\pi}{2}x\right)\cos^2\left(\frac{\pi}{2}x\right)+\sin^p\left(\frac{\pi}{2}x\right)\right)\frac{1}{x^{\alpha-2}}.$$

Obviously

$$(1-p)\sin^{p-2}\left(\frac{\pi}{2}x\right)\cos^2\left(\frac{\pi}{2}x\right)+\sin^p\left(\frac{\pi}{2}x\right) \geqslant \sin^p\left(\frac{\pi}{2}x\right) \geqslant x^p$$

for $x \in [0,1]$. Thus we infer that for

$$\alpha \geqslant p+2, \ \ 0 < c \leqslant \frac{\pi^2}{4}\frac{p}{\alpha(\alpha-1)}$$

the function f is strongly $\mathscr{F}$-concave with control function $F(x) = cx^{\alpha}$.

In the following examples, we give a comparison of improvements of the Levin-Stečkin inequality given in Theorem 5 with respect to either α or x_0 in choosing functions F given by (9).

EXAMPLE 4. Similar conclusions (as in the previous example) hold for $f(x) = \cos^p\left(\frac{\pi}{2}x\right)$, $F(x) = c(1-x)^{\alpha}$, $x \in [0,1]$, $0 < p \leqslant 1$, $\alpha > 1$.

EXAMPLE 5. Let $f(x) = \log(x+1)$, $F_{\alpha}(x) = c(x+1)^{\alpha}$, $x \in [0,1]$, $\alpha > 1$, $c > 0$. Then f is strongly $\mathscr{F}$-concave with control function F_{α} iff

$$c \leqslant \frac{1}{\alpha(\alpha-1)}\frac{1}{(1+x)^{\alpha}}, \ \ x \in [0,1],$$

which gives $c \leqslant \frac{1}{\alpha(\alpha-1)}\frac{1}{2^{\alpha}}$. Since the expression $\frac{\int_a^b F(x)d\lambda_2(x)}{\int_a^b d\lambda_2(x)} - \frac{\int_a^b F(x)d\lambda_1(x)}{\int_a^b d\lambda_1(x)}$ in (8) with $F \equiv F_{\alpha}$ is increasing in $c > 0$, it is optimal to take $c = c(\alpha) = \frac{1}{\alpha(\alpha-1)}\frac{1}{2^{\alpha}}$. The claim is that this expression (in (8) containing F_{α}) is decreasing in α, or for $1 < \alpha_1 < \alpha_2$ it holds:

$$\begin{aligned} &\frac{\int_0^1 F_{\alpha_1}(x)d\lambda_2(x)}{2\lambda_2(1)} - \frac{\int_0^1 F_{\alpha_1}(x)d\lambda_1(x)}{2\lambda_1(1)} \\ &\quad \geqslant \frac{\int_0^1 F_{\alpha_2}(x)d\lambda_2(x)}{2\lambda_2(1)} - \frac{\int_0^1 F_{\alpha_2}(x)d\lambda_1(x)}{2\lambda_1(1)} \end{aligned}$$

if (7) holds (on $[0,1/2]$). Using Theorem 3 (the convex part of this theorem), it is enough to prove that

$$\phi(x) = F_{\alpha_1}(x) - F_{\alpha_2}(x) = \frac{1}{\alpha_1(\alpha_1-1)}\frac{1}{2^{\alpha_1}}(x+1)^{\alpha_1} - \frac{1}{\alpha_2(\alpha_2-1)}\frac{1}{2^{\alpha_2}}(x+1)^{\alpha_2}$$

is a convex function on $[0,1]$, which is immediate from

$$\phi''(x) = \frac{1}{4}\left(\frac{x+1}{2}\right)^{\alpha_1-2}\left(1 - \left(\frac{x+1}{2}\right)^{\alpha_2-\alpha_1}\right) \geqslant 0.$$

It follows that the best possible improvements of the Levin-Stečkin inequality in this case is equal to:

$$\begin{aligned} &\lim_{\alpha\to 1}\left(\frac{\int_0^1 F_{\alpha}(x)d\lambda_2(x)}{2\lambda_2(1)} - \frac{\int_0^1 F_{\alpha}(x)d\lambda_1(x)}{2\lambda_1(1)}\right) \qquad (10) \\ &\quad = \frac{1}{2}\left(\frac{\int_0^1 (x+1)\log(x+1)d\lambda_2(x)}{2\lambda_2(1)} - \frac{\int_0^1 (x+1)\log(x+1)d\lambda_1(x)}{2\lambda_1(1)}\right). \end{aligned}$$

EXAMPLE 6. Let $f(x) = \log(x+1)$, $G_\alpha(x) = cx^\alpha$, $x \in [0,1]$, $\alpha > 1$, $c > 0$. Then f is strongly $\mathscr{F}$-concave with control function G_α iff

$$c \leqslant \frac{1}{\alpha(\alpha-1)} \frac{1}{(1+x)^2} \frac{1}{x^{\alpha-2}}, \quad x \in (0,1],$$

which gives $c \leqslant \frac{1}{\alpha(\alpha-1)} \frac{1}{2^2}$ for $\alpha \geqslant 2$. Since the expression $\frac{\int_a^b F(x)d\lambda_2(x)}{\int_a^b d\lambda_2(x)} - \frac{\int_a^b F(x)d\lambda_1(x)}{\int_a^b d\lambda_1(x)}$ in (8) with, in this case, $F \equiv G_\alpha$ is increasing in $c > 0$, it is optimal to take $c = c(\alpha) = \frac{1}{\alpha(\alpha-1)} \frac{1}{2^2}$. As in the previous example, it is easy to see that this expression (in (8) containing, in this case, G_α) is decreasing in α. The best possible improvements of the Levin-Stečkin inequality in this case is obtained for $\alpha = 2$ and is equal to

$$\frac{1}{2^3} \left(\frac{\int_0^1 x^2 d\lambda_2(x)}{2\lambda_2(1)} - \frac{\int_0^1 x^2 d\lambda_1(x)}{2\lambda_1(1)} \right). \tag{11}$$

To show that the improvement given by (10) is better that the improvement given by (11), it is enough (again using the convexity part on Theorem 3) to check that

$$\psi(x) = 4(x+1)\log(x+1) - x^2$$

is convex on $[0,1]$, which is trivial since $\psi''(x) = 2\dfrac{1-x}{1+x} \geqslant 0$ on $[0,1]$.

3. The right-hand side of the Clausing inequality and strong $\mathscr{F}$-concavity

The both terms of the right-hand side of the Clausing inequality (2) (see also (6)) are linear in f. This means that the method of obtaining inequalities of the same type for strongly $\mathscr{F}$-concave functions is the same as in the case of Levin-Stečkin's inequality (or the Jensen type inequalities). Whether this new inequality is an improvement on the original inequality is more subtle than in the case of Jensen type inequalities. It can be said that the properties of the inequalities generated in this way reveal some not so obvious properties of the original inequality.

THEOREM 6. *Let $\lambda : [a,b] \to \mathbb{R}$ be a non-constant continuous increasing function such that $\lambda(a+b-x) = -\lambda(x)$ for any $x \in [a,b]$. Let w be a non-negative continuous function on $[a,b]$ increasing on $[a, \frac{a+b}{2}]$, with $w(x) = w(a+b-x)$ for any $x \in [a,b]$. Let $\widehat{c_3}$ be as in (5). Then*

$$\frac{\int_a^b f(x)w(x)d\lambda(x)}{\widehat{c_3}} \leqslant \frac{\int_a^b f(x)d\lambda(x)}{2\lambda(b)} - \left(\frac{\int_a^b F(x)w(x)d\lambda(x)}{\widehat{c_3}} - \frac{\int_a^b F(x)d\lambda(x)}{2\lambda(b)} \right), \tag{12}$$

holds for every strongly $\mathscr{F}$-concave function f on $[a,b]$ (with control function F) assuming $f(a)+f(b)+F(a)+F(b) \geqslant 0$. If $F(a)+F(b) \leqslant 0$, then

$$\frac{\int_a^b f(x)d\lambda(x)}{2\lambda(b)} - \left(\frac{\int_a^b F(x)w(x)d\lambda(x)}{\widehat{c_3}} - \frac{\int_a^b F(x)d\lambda(x)}{2\lambda(b)} \right) \leqslant \frac{\int_a^b f(x)d\lambda(x)}{2\lambda(b)}. \tag{13}$$

Proof. The inequality (12) follows from Theorem 4 for the concave function $f+F$. The inequality (13) follows from the convexity part of Theorem 4. □

Note that all examples from the previous section satisfy the condition $f(a)+f(b)+F(a)+F(b)\geqslant 0$, but none satisfies $F(a)+F(b)\leqslant 0$. Of course, this is a sufficient condition. It remains to investigate is it possible to have an improvement of (6) for strongly $\mathscr{F}$-concave functions with control function F with $F(a)+F(b)>0$, or equivalently, is it possible that

$$\frac{\int_a^b F(x)w(x)d\lambda(x)}{\widehat{c_3}}-\frac{\int_a^b F(x)d\lambda(x)}{2\lambda(b)}\geqslant 0 \tag{14}$$

holds for convex F with $F(a)+F(b)>0$. In the following example we show that (14) does not hold in the case of strongly concave functions. This implies that Theorem 6 does not give an improvement of the right-side of the Clausing inequality (see also (6)) for strongly concave functions.

EXAMPLE 7. Let $F(x)=cx^2$, $c>0$, $x\in[0,1]$ and $d\lambda(x)=dx$. To show that the reverse inequality holds in (14), it is enough to check

$$3\int_0^1 x^2 w(x)dx\leqslant \widehat{c_3}=\int_0^1 \widehat{w}(x)dx. \tag{15}$$

The right-hand side of (2) gives

$$\int_0^1 (2x)w(x)dx\leqslant \int_0^1 \widehat{w}(x)dx.$$

Inequality (15) will follow if

$$\int_0^1 \left(2x-3x^2\right)w(x)dx\geqslant 0$$

holds. Using simple substitution and integration by parts (if w is not smooth enough some uniform approximation is assumed; the Bernstein polynomials for example), we get:

$$\begin{aligned}\int_0^1 \left(2x-3x^2\right)w(x)dx &= \int_0^{1/2}\left(2x-3x^2\right)w(x)dx+\int_{1/2}^1\left(2x-3x^2\right)w(x)dx\\ &= \int_0^{1/2}\left(-1+6x-6x^2\right)w(x)dx\\ &= \int_0^{1/2} w'(x)\int_0^x\left(6t^2-6t+1\right)dt\,dx\\ &= \int_0^{1/2} x(1-x)(1-2x)w'(x)dx\geqslant 0,\end{aligned}$$

where $\int_0^{1/2}\left(-1+6x-6x^2\right)dx=0$ is also used.

This discussion will be given generally in the last section.

Notice that $F(a)+F(b) \leqslant 0$ implies

$$\int_a^b F(x)w(x)d\lambda(x) \leqslant \frac{F(a)+F(b)}{2}\int_a^b w(x)d\lambda(x) \leqslant 0,$$

and similarly for the second term in (14). This follows from the Fejér's variant of the Hermite-Hadamard inequality (see for example [7], [17]).

Our primary concern in this section is to get improvements of (6) for strongly $\mathscr{F}$-concave functions removing the above mentioned obstacles, mainly $F(a)+F(b) \leqslant 0$. The idea is based on the following simple observation:

f is strongly $\mathscr{F}$-concave with control function F if and only if f is strongly $\mathscr{F}$-concave with control function $F(x)+kx+l$ for any $k,l \in \mathbb{R}$.

LEMMA 1. *Let $\lambda : [a,b] \to \mathbb{R}$ be a non-constant continuous increasing function such that $\lambda(a+b-x) = -\lambda(x)$ for any $x \in [a,b]$. Let w be a non-negative continuous function on $[a,b]$ increasing on $[a,\frac{a+b}{2}]$, with $w(x) = w(a+b-x)$ for any $x \in [a,b]$. Suppose $F(a), F(b) \in \mathbb{R}$ and $0 \neq c_3 \in \mathbb{R}$. If*

$$L(x) = \frac{F(b)-F(a)}{b-a}(x-a)+F(a), \tag{16}$$

then

$$\frac{\int_a^b L(x)w(x)d\lambda(x)}{c_3} - \frac{\int_a^b L(x)d\lambda(x)}{2\lambda(b)} = \frac{F(a)+F(b)}{2}\left(\frac{1}{c_3}\int_a^b w(x)d\lambda(x) - 1\right). \tag{17}$$

Proof. Using the substitution $t = a+b-x$, we get

$$\begin{aligned}\int_a^b (x-a)w(x)d\lambda(x) &= \int_a^{(a+b)/2}(x-a)w(x)d\lambda(x) + \int_{(a+b)/2}^b (x-a)w(x)d\lambda(x)\\ &= \int_a^{(a+b)/2}(x-a)w(x)d\lambda(x) + \int_a^{(a+b)/2}(b-x)w(x)d\lambda(x)\\ &= (b-a)\int_a^{(a+b)/2} w(x)d\lambda(x),\end{aligned}$$

and similarly

$$\int_a^b (x-a)d\lambda(x) = (b-a)\int_a^{(a+b)/2} d\lambda(x) = \lambda(b)(b-a).$$

It follows:

$$\begin{aligned}&\frac{\int_a^b L(x)w(x)d\lambda(x)}{c_3} - \frac{\int_a^b L(x)d\lambda(x)}{2\lambda(b)}\\ &= \frac{1}{c_3}\left((F(b)-F(a))\int_a^{(a+b)/2} w(x)d\lambda(x) + 2F(a)\int_a^{(a+b)/2} w(x)d\lambda(x)\right)\\ &\quad -\frac{1}{2}(F(a)+F(b)) = \frac{F(a)+F(b)}{2}\left(\frac{1}{c_3}\int_a^b w(x)d\lambda(x) - 1\right). \qquad \square\end{aligned}$$

The following corollary gives an improvement of the convex case of Theorem 4 if $f(a)+f(b) \leqslant 0$ and the new estimation of the convex variant of (6) in the case $f(a)+f(b) \geqslant 0$. The important remark here is that (under assumptions of Theorem 4)

$$\widehat{c_3} \geqslant \int_a^b w(x)d\lambda(x), \tag{18}$$

which is a simple consequence of the same theorem applied on $f(x)=1$. This estimation has interesting feature. It is equivalent to the inequality

$$\frac{\Lambda(b)}{\lambda(b)} \leqslant \frac{\int_{(a+b)/2}^b \Lambda(x)dx}{\int_{(a+b)/2}^b \lambda(x)dx},$$

where $d\Lambda(x) = w(x)d\lambda(x)$ on $[(a+b)/2, b]$. It could be instructive to give an independent (of Theorem 4) proof.

COROLLARY 1. *Let λ, w and $\widehat{c_3}$ be as in Theorem 4. If F is a continuous convex function on $[a,b]$, then*

$$\frac{\int_a^b F(x)w(x)d\lambda(x)}{\widehat{c_3}} \geqslant \frac{\int_a^b F(x)d\lambda(x)}{2\lambda(b)} - \frac{F(a)+F(b)}{2}\left(1-\frac{1}{\widehat{c_3}}\int_a^b w(x)d\lambda(x)\right)$$

Proof. Set $F_1(x) = F(x) - L(x)$, where $L(x)$ is given by (16). Using the convex case of Theorem 4, rearranging (6) for F_1 instead of f and applying Lemma 1 for $c_3 = \widehat{c_3}$ the claim easily follows. □

THEOREM 7. *Let $\lambda : [a,b] \to \mathbb{R}$ be a non-constant continuous increasing function such that $\lambda(a+b-x) = -\lambda(x)$ for any $x \in [a,b]$. Let w be a non-negative continuous function on $[a,b]$ increasing on $[a, \frac{a+b}{2}]$, with $w(x) = w(a+b-x)$ for any $x \in [a,b]$. Let $\widehat{c_3}$ be as in (5). If f is strongly $\mathscr{F}$-concave on $[a,b]$ with control function F, then*

$$\begin{aligned}\frac{\int_a^b f(x)w(x)d\lambda(x)}{\widehat{c_3}} &\leqslant \frac{\int_a^b f(x)d\lambda(x)}{2\lambda(b)} - \left(\frac{\int_a^b F(x)w(x)d\lambda(x)}{\widehat{c_3}} - \frac{\int_a^b F(x)d\lambda(x)}{2\lambda(b)}\right. \\ &\qquad \left. + \frac{F(a)+F(b)}{2}\left(1-\frac{1}{\widehat{c_3}}\int_a^b w(x)d\lambda(x)\right)\right) \\ &\leqslant \frac{\int_a^b f(x)d\lambda(x)}{2\lambda(b)},\end{aligned} \tag{19}$$

where in the first inequality $f(a)+f(b) \geqslant 0$ is assumed.

Proof. It is obvious that if f is strongly $\mathscr{F}$-concave with control function F, then f is strongly $\mathscr{F}$-concave with control function $F_1(x) = F(x) - L(x)$, where $L(x)$ is given by (16). The first inequality easily follows from Theorem 6 (using F_1 instead of F) and Lemma 1. The second inequality in (19) follows from Corollary 1. □

4. Final conclusions and the right-hand side of the Clausing inequality for convex functions

As a kind of synthesis of the results given in previous sections, in this section we first give the compound improvement for strongly $\mathscr{F}$-concave functions of the generalized Clausing inequality

$$\frac{\int_a^b f(x)w(x)d\lambda(x)}{\widehat{c_3}} \leqslant \frac{\int_a^b f(x)d\lambda(x)}{\int_a^b d\lambda(x)} \leqslant \frac{\int_a^b f(x)w(x)d\lambda(x)}{\int_a^b w(x)d\lambda(x)},$$

where $\lambda, w, \widehat{c_3}$ are as in Theorem 4, and f is a concave function on $[a,b]$ such that $f(a)+f(b) \geqslant 0$ is assumed in the first inequality. See Theorem 3 (and the first remark below that) and Theorem 4.

COROLLARY 2. *Let $\lambda : [a,b] \to \mathbb{R}$ be a non-constant continuous increasing function such that $\lambda(a+b-x) = -\lambda(x)$ for any $x \in [a,b]$. Let w be a non-negative continuous function on $[a,b]$ increasing on $[a, \frac{a+b}{2}]$, with $w(x) = w(a+b-x)$ for any $x \in [a,b]$. Let $\widehat{c_3}$ be as in (5). If f is strongly $\mathscr{F}$-concave function on $[a,b]$ with control function F, then*

$$\begin{aligned}
&\frac{\int_a^b f(x)w(x)d\lambda(x)}{\widehat{c_3}} \\
&\leqslant \frac{\int_a^b f(x)w(x)d\lambda(x)}{\widehat{c_3}} + \left(\frac{\int_a^b F(x)w(x)d\lambda(x)}{\widehat{c_3}} - \frac{\int_a^b F(x)d\lambda(x)}{2\lambda(b)} \right. \\
&\quad \left. + \frac{F(a)+F(b)}{2} \left(1 - \frac{1}{\widehat{c_3}} \int_a^b w(x)d\lambda(x) \right) \right) \\
&\leqslant \frac{\int_a^b f(x)d\lambda(x)}{2\lambda(b)} \\
&\leqslant \frac{\int_a^b f(x)w(x)d\lambda(x)}{\int_a^b w(x)d\lambda(x)} - \left(\frac{\int_a^b F(x)d\lambda(x)}{\int_a^b d\lambda(x)} - \frac{\int_a^b F(x)w(x)d\lambda(x)}{\int_a^b w(x)\lambda(x)} \right) \\
&\leqslant \frac{\int_a^b f(x)w(x)d\lambda(x)}{\int_a^b w(x)d\lambda(x)},
\end{aligned}$$

where $f(a)+f(b) \geqslant 0$ is assumed in the second inequality.

Proof. The last two inequalities are as in Theorem 5 for $\lambda_2 = \lambda$ and $d\lambda_1(x) = w(x)d\lambda(x)$. The first two inequalities are rearranged inequalities in (19). □

The final part of this paper has twofold purpose. To complete Theorem 4 for convex functions F for which $\int_a^b F(x)w(x)d\lambda(x) \geqslant 0$ (we slightly changed notations because of strongly $\mathscr{F}$-concave context), and secondly to continue with the discussion below Theorem 6 on improving the right-hand side of the Clausing inequality (in a general form given in (6)) for strongly $\mathscr{F}$-concave functions with control function F with formerly mentioned property.

Suppose that λ, w and $\widehat{c_3}$ are as in Theorem 4 (and Theorem 6). Let F be a convex function on $[a,b]$ such that

$$\int_a^b F(x)w(x)d\lambda(x) \geqslant 0.$$

Note that if $\int_a^b F(x)w(x)d\lambda(x) = 0$, then from the convex case of (3) (for $\lambda = \lambda_2$ and $d\lambda_1(x) = w(x)d\lambda(x)$) follows $\int_a^b F(x)d\lambda(x) \geqslant 0$. We get that (6) trivially holds (with the same inequality) and that the improvement of the form (13) does not hold.

Suppose that $\int_a^b F(x)w(x)d\lambda(x) > 0$. Then

$$\frac{\int_a^b F(x)d\lambda(x)}{\int_a^b d\lambda(x)} \geqslant \frac{\int_a^b F(x)w(x)d\lambda(x)}{\int_a^b w(x)d\lambda(x)} \geqslant \frac{\int_a^b F(x)w(x)d\lambda(x)}{\widehat{c_3}}, \tag{20}$$

where the first inequality follows from the convex part of Theorem 3, and the second inequality holds since $\widehat{c_3} \geqslant \int_a^b w(x)d\lambda(x)$ (see (6) for $f(x) = 1$). Again we have the same conclusions: (6) holds with the same inequality and that the improvement of the form (13) does not hold. This particularly shows that for strongly concave functions improvements of the form (13) do not hold (compare to (19) and Lemma 7).

There is an another interesting feature of the right-hand side of the Clausing inequality. Suppose that G is a concave function on $[a,b]$ such that $\int_a^b G(x)d\lambda(x) > 0$, which implies, using (3), $\int_a^b G(x)w(x)d\lambda(x) > 0$. This gives that (6) can be written equivalently as:

$$\frac{\int_a^b G(x)w(x)d\lambda(x)}{\int_a^b G(x)d\lambda(x)} \leqslant \frac{\widehat{c_3}}{2\lambda(b)} = \frac{\int_a^b \min\{x-a, b-x\}w(x)d\lambda(x)}{\int_a^b \min\{x-a, b-x\}d\lambda(x)}, \tag{21}$$

where we again emphasized the form of the maximum value of the left-hand side of (21) (see [11]). The claim is that if F is a convex function with $\int_a^b F(x)d\lambda(x) > 0$, then

$$\frac{\int_a^b F(x)w(x)d\lambda(x)}{\int_a^b F(x)d\lambda(x)} \leqslant \frac{\int_a^b G(x)w(x)d\lambda(x)}{\int_a^b G(x)d\lambda(x)}, \tag{22}$$

which, taking into account (21), gives that the generalization of the right-hand side of the Clausing inequality (6) (as it is) also holds for convex F with $\int_a^b F(x)d\lambda(x) > 0$.

We give a short proof of (22) independent of Theorem 3. It is obviously enough to prove $\int_a^b H(t)w(t)d\lambda(t) \geqslant 0$ for concave H with $\int_a^b H(t)d\lambda(t) = 0$. Using integration by parts we get:

$$\begin{aligned}\int_a^b H(t)w(td\lambda(t) &= -\int_a^b w'(x)\int_a^x H(t)d\lambda(t)dx \\ &= -\int_a^{(a+b)/2} w'(x)\int_a^x H(t)d\lambda(t)dx \\ &\quad -\int_a^{(a+b)/2} w'(a+b-x)\int_a^{a+b-x} H(t)d\lambda(t)dx \\ &= \int_a^{(a+b)/2} w'(x)\int_x^{a+b-x} H(t)d\lambda(t).\end{aligned}$$

Notice that if w is not in $C^1([a,b])$ some uniform approximation argument (the Bernstein polynomials) is assumed.

It remains to prove

$$\psi(x) = \int_x^{a+b-x} H(t)d\lambda(t) \geqslant 0,\ \ x \in [a,(a+b)/2].$$

Obviously, $\psi(a) = \psi((a+b)/2) = 0$. Since

$$\psi(x) = \frac{1}{2}\int_x^{a+b-x}(H(t)+H(a+b-t))d\lambda(t) = \int_x^{(a+b)/2} g(t)d\lambda(t),$$

with g concave and increasing, such that $\int_a^{(a+b)/2} g(t)d\lambda(t) = 0$. It follows that there is $x_0 \in (a,(a+b)/2)$ such $g(x) \leqslant 0$ on (a,x_0) and $g(x) \geqslant 0$ on $(x_0,(a+b)/2)$. Trivially $\psi(x) \geqslant 0$ on $(x_0,(a+b)/2)$. Suppose that $x \in ((a+b)/2,x_0)$. Set:

$$\tilde{g}(t) = \begin{cases} g(x) & a \leqslant t \leqslant x \\ g(t) & x \leqslant t \leqslant (a+b)/2 \end{cases}.$$

We have:

$$0 = \int_a^{(a+b)/2} g(t)d\lambda(t) \leqslant \int_a^{(a+b)/2} \tilde{g}(t)d\lambda(t) = g(x)(\lambda(x)-\lambda(a)) + \psi(x),$$

which, by $g(x) \leqslant 0$, certainly gives $\psi(x) \geqslant 0$.

It remains to discuss the convex case of Theorem 4 if $F(a)+F(b) > 0$ and $\int_a^b F(x)w(x)d\lambda(x) < 0$ (again we slightly changed the notation because of the strongly $\mathscr{F}$-concave context). It follows from the following example that generally there is no decisive answer on validity of (6) in this case.

EXAMPLE 8. Let $w(x) = x(1-x)$, $d\lambda(x) = dx$, $F_a(x) = (x-a)(x-1+a)$, $x \in [0,1]$ and $0 < a < 1$. Notice $F(0)+F(1) = 2a(1-a) > 0$. Straightforwardly:

$$\widehat{c_3} = \int_0^1 \widehat{w}(x)dx = \frac{5}{24}, \quad \int_0^1 F_a(x)dx = -\frac{1}{6}+a-a^2,$$

$$\int_0^1 F_a(x)w(x)dx = \frac{1}{6}\left(-\frac{1}{5}+a-a^2\right),$$

$$\int_0^1 F_a(x)w(x)dx - \widehat{c_3}\int_0^1 F_a(x)dx = \frac{1}{720}\left(1-30a+30a^2\right).$$

We get (among other cases):

$$\int_0^1 F_a(x)w(x)dx \geqslant \widehat{c_3}\int_0^1 F_a(x)dx, \text{ if } 0<a<\frac{15-\sqrt{195}}{30},$$

$$\int_0^1 F_a(x)w(x)dx \leqslant \widehat{c_3}\int_0^1 F_a(x)dx, \text{ if } \frac{15-\sqrt{195}}{30}<a<\frac{3-\sqrt{3}}{6}.$$

In both cases holds $\int_0^1 F_a(x)w(x)dx<0$, $\int_0^1 F_a(x)dx<0$.

In the first case the reverse inequality in (6) holds (or the same as in the convex case of Theorem 4) and an improvement is obtained in Theorem 6 (if control function is F_a). In the second case the opposite conclusions hold.

Note that $\int_0^1 F_a(x)dx>0$ for $a\in\left(\left(3-\sqrt{3}\right)/6,\left(3+\sqrt{3}\right)/6\right)\approx(0.2113,0.7887)$, and $\int_0^1 F_a(x)w(x)dx>0$ for $a\in\left(\left(5-\sqrt{5}\right)/10,\left(5+\sqrt{5}\right)/10\right)\approx(0.2764,0.7236)$, which illustrates the fact that (22) is proved (and henceforth (6)) under weaker assumption than (20).

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