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HIGHER ORDER POINCARÉ INEQUALITY AND CACCIOPPOLI INEQUALITY WITH ORLICZ NORMS FOR DIFFERENTIAL FORMS

JINLING NIU, GUANNAN SHI AND YUMING XING*

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Abstract. In this paper, we establish the local higher order Poincaré inequality and Caccioppoli inequality with Orlicz norms for solutions to the non-homogeneous A-harmonic equations on differential forms. Moreover, the global higher order Poincaré inequality and Caccioppoli inequality are derived. As applications, the higher order Caccioppoli-type inequality and a weak type inequality for homotopy operator are obtained.

1. Introduction

The higher integrability theory, which was introduced by N. Fusco and C. Sbordone [11] to study the regularity of minimizers of functionals, has emerged as a fascinating and interesting branch of mathematical and engineering sciences. The ideas and techniques are being applied in a variety of diverse areas of sciences, such as potential theory, quantum mechanics and partial differential equations, and proved to be productive and innovative, see [5, 12, 16, 18, 21] and the references therein. Especially, the higher integrability is a very important and core topic in the L^p theory of differential forms which can be used to give the upper bound estimates for the norms of various operators and investigate the qualitative and quantitative properties of the solutions to partial differential equations on differential forms. Recently, a number of significant studies have been undertaken in this regard, for example, the L^p higher integrability of singular integral, Green's operator, homotopy operator and iterated operators on differential forms, see [9, 19, 26]. These activities have motivated to study the higher order inequalities for differential forms. Inspired by the results about higher order Poincaré inequalities and higher integrability of operators on functions in [5, 11], we aim to study two types of higher order inequalities for solutions to the non-homogeneous A-harmonic equations on differential forms which are rather general equations including the usual p-harmonic equations and Laplace equations as special cases and play a key role in the fields of quasiconformal mappings and the theory of elasticity, see [2–4, 10, 14]. In this paper, we first establish the local higher order Poincaré

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* Corresponding author.

inequality and Caccioppoli inequality in terms of L^{φ} norms for solutions to the non-homogeneous A-harmonic equations with φ satisfying the non-standard growth conditions. Then, we extend the local higher order inequalities to the global cases using the well known Whitney covering lemma. When we choose $\varphi(t) = t^p$, the higher order Poincaré inequality and Caccioppoli inequality with L^{φ} norms will reduce to the corresponding higher order inequalities with L^p norms. These results derived in this paper generalize the classical Poincaré inequality and Caccioppoli inequality for differential forms in [1, 6, 17] and also can be viewed as important improvements of the previous results in [8, 17, 27]. Finally, we deduce the higher order Caccioppoli-type inequality and a weak type inequality for homotopy operator as the valuable applications of our main results. These results obtained in this paper will provide a further insight into the L^p theory and regularity theory of partial differential equations.

This work is organized as follows. In the next section, the preliminaries including some definitions and main lemmas are introduced. In Section 3, we first prove the Caccioppoli inequality with L^{φ} norms for solutions to the non-homogeneous A-harmonic equations in Theorem 3.1. Using the inequality, we establish the local higher order Poincaré inequality and Caccioppoli inequality in terms of L^{φ} norms for solutions to the non-homogeneous A-harmonic equations in Theorem 3.2 and Theorem 3.3, respectively. Based on the local results, the global higher order Poincaré inequality and Caccioppoli inequality are presented in Theorems 4.2 and Theorems 4.3 in Section 4. As applications of the main results, we give the higher order Caccioppoli-type inequality and a weak type inequality with L^{φ} norms for homotopy operator in Section 5.

2. Preliminaries

Throughout of this paper, let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $n \geqslant 2$, B and σB be the balls with the same center and $diam(\sigma B) = \sigma diam(B)$. We use $|E|$ to denote the Lebesgue measure of a set $E \subset \mathbb{R}^n$. Let $\Lambda^l(\mathbb{R}^n) = \Lambda^l$, $l = 1, 2, \cdots, n$, be the set of all l-forms $u(x) = \sum_I u_I(x) dx_I = \sum u_{i_1 \cdots i_l}(x) dx_{i_1} \wedge \cdots \wedge dx_{i_l}$ with summation over all ordered l-tuples $I = (i_1, i_2, \cdots, i_l)$, $1 \leqslant i_1 < \cdots < i_l \leqslant n$. $D'(\Omega, \Lambda^l)$ is the space of all differential l-forms on Ω, namely, the coefficient of the l-forms is differential on Ω. The operator $\star : \Lambda^l(\mathbb{R}^n) \to \Lambda^{n-l}(\mathbb{R}^n)$ is the Hodge-star operator as usual and the linear operator $d : D'(\Omega, \Lambda^l) \to D'(\Omega, \Lambda^{l+1})$, $0 \leqslant l \leqslant n-1$ is called the exterior differential operator. The Hodge codifferential operator $d^{\star} : D'(\Omega, \Lambda^{l+1}) \to D'(\Omega, \Lambda^l)$, the formal adjoint of d, is defined by $d^{\star} = (-1)^{nl+1} \star d \star$, see [20] for more introduction. We shall denote by $L^p(\Omega, \Lambda^l)$ the space of differential l-forms with coefficients in $L^p(\Omega, \mathbb{R}^n)$ and with norm $\|u\|_{p,\Omega} = \left(\int_{\Omega} \left(\sum_I |u_I(x)|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}$. The homotopy operator $T : C^{\infty}(\Omega, \Lambda^l) \to C^{\infty}(\Omega, \Lambda^{l-1})$ is a very important operator in differential form theory, given by

$$Tu = \int_{\Omega} \psi(y) K_y u dy,$$

where $\psi \in C_0^\infty(\Omega)$ is normalized by $\int_\Omega \psi(y)dy = 1$, and K_y is a liner operator defined by

$$(K_y u)(x;\xi_1,\cdots,\xi_{l-1}) = \int_0^1 t^{l-1} u(tx+y-ty; x-y; \xi_1,\cdots,\xi_{l-1})dt.$$

See [13] for more of the function ψ and operator K_y. About the homotopy operator T, we have the following decomposition, which will be used repeatedly in this paper,

$$u = d(Tu) + T(du)$$

for any differential form $u \in L^p(\Omega,\Lambda^l)$, $1 \leqslant p < \infty$. A closed form u_Ω is defined by $u_\Omega = d(Tu)$, $l = 1,\cdots,n$, and when u is a differential 0-form, $u_\Omega = |\Omega|^{-1}\int_\Omega u(y)dy$.

The following nonlinear partial differential equation for differential forms

$$d^\star A(x,du) = B(x,du) \tag{2.1}$$

is called non-homogeneous A-harmonic equation, where $A : \Omega \times \Lambda^l(\mathbb{R}^n) \to \Lambda^l(\mathbb{R}^n)$ and $B : \Omega \times \Lambda^l(\mathbb{R}^n) \to \Lambda^{l-1}(\mathbb{R}^n)$ satisfy the conditions:

$$|A(x,\xi)| \leqslant a|\xi|^{p-1},\ A(x,\xi)\cdot\xi \geqslant |\xi|^p \ \text{ and } \ |B(x,\xi)| \leqslant b|\xi|^{p-1}$$

for $x \in \Omega$ a.e. and all $\xi \in \Lambda^l(\mathbb{R}^n)$. Here $p > 1$ is a constant related to the equation (2.1), and $a,b > 0$. See [6, 8, 15, 22–25] for recent results on the A-harmonic equations and related topics.

An Orlicz function is a continuously increasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$. The Orlicz space $L^\varphi(\Omega)$ consists of all measurable functions f on Ω such that $\int_\Omega \varphi\left(\frac{|f|}{\lambda}\right)dx < \infty$ for some $\lambda = \lambda(f) > 0$. $L^\varphi(\Omega)$ is equipped with the nonlinear Luxemburg functional

$$\|f\|_{L^\varphi(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \varphi\left(\frac{|f|}{\lambda}\right)dx \leqslant 1\right\}.$$

A convex Orlicz function φ is often called a Young function. If φ is a Young function, then $\|\cdot\|_{L^\varphi(\Omega)}$ defines a norm in $L^\varphi(\Omega)$, which is called the Luxemburg norm or Orlicz norm.

In order to prove our results, we recall the subclass of Young functions and two related lemmas given by N. Fusco and C. Sbordone in [11].

DEFINITION 2.1. A Young function $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is said to be in the class $NG(p,q)$ if φ satisfies the nonstandard growth condition

$$p\varphi(t) \leqslant t\varphi'(t) \leqslant q\varphi(t),\ \ 1 < p \leqslant q < \infty. \tag{2.2}$$

The first inequality in (2.2) is equivalent to that $\frac{\varphi(t)}{t^p}$ is increasing, and the second inequality in (2.2) is equivalent to $\triangle_2$-condition, i.e., for each $t > 0$, $\varphi(2t) \leqslant K\varphi(t)$, where $K > 1$, and $\frac{\varphi(t)}{t^q}$ is decreasing with t.

LEMMA 2.1. *Suppose φ is a continuous function in the class $NG(p,q)$, $1<p\leqslant q<\infty$. For any $t>0$, setting*

$$A(t)=\int_0^t\left(\frac{\varphi(s^{\frac{1}{q}})}{s}\right)^{\frac{n+q}{q}}ds,\quad K(t)=\frac{\left(\varphi(t^{\frac{1}{q}})\right)^{\frac{n+q}{q}}}{t^{\frac{n}{q}}}. \tag{2.3}$$

Then, $A(t)$ is a concave function, and there exists a constant C, such that

$$K(t)\leqslant A(t)\leqslant CK(t),\ \ \forall t>0. \tag{2.4}$$

LEMMA 2.2. *If ψ satisfies $\psi(2t)\leqslant k\psi(t)$ for all $t>0$, and there exists $p>1$ such that $\frac{\psi(t)}{t^p}$ is increasing and f is an $L^1_{loc}(\Omega)$ function, $f\geqslant 0$, such that, for any cube $Q\subset\Omega$ for which $2Q\subset\subset\Omega$,*

$$\int_Q\psi(f)dx\leqslant b_1\psi\left(\int_{2Q}f\right)+b_2, \tag{2.5}$$

then there exist $c_1,c_2>0$, $r>1$, depending only on b_1, b_2, n, k, p such that, for any $2Q\subset\subset\Omega$,

$$\int_Q\psi^r(f)dx\leqslant c_1\psi^r\left(\int_{2Q}f\right)+c_2. \tag{2.6}$$

The following Caccioppoli inequality for solutions to the non-homogeneous A-harmonic equations was given by S. Ding in [6].

LEMMA 2.3. *Let $u\in D'(\Omega,\Lambda^l)$, $l=0,1,\cdots,n-1$, be a solution to the non-homogeneous A-harmonic equation (2.1) in Ω. Then, there exists a constant C, independent of u, such that*

$$\|du\|_{p,B}\leqslant C|B|^{-\frac{1}{n}}\|u-c\|_{p,\sigma B} \tag{2.7}$$

for all balls B with $\sigma B\subset\Omega$ for some $\sigma>1$ and all closed forms c. Here $1<p<\infty$.

The following result appears in [8].

LEMMA 2.4. *Let u be a solution to the non-homogeneous A-harmonic equation (2.1) in Ω and $0<s,t<\infty$. Then, there exists a constant C, independent of u, such that*

$$\|du\|_{s,B}\leqslant C|B|^{\frac{t-s}{st}}\|du\|_{t,\sigma B} \tag{2.8}$$

for all balls B with $\sigma B\subset\Omega$ for some $\sigma>1$.

In [17], the authors extended the Poincaré inequality for differential norms with L^p-norms to the following version with L^φ-norms.

LEMMA 2.5. *Suppose $u \in D'(\Omega,\Lambda^l)$ and $du \in L^p(\Omega,\Lambda^{l+1})$, φ is a Young function in the class $NG(p,q)$ with $q(n-p) < np$, $1 < p \leqslant q < \infty$. Then, there exists a constant C, independent of u, such that*

$$\int_B \varphi\big(|u-u_B|\big)dx \leqslant C\int_B \varphi\big(|du|\big)dx, \tag{2.9}$$

where B is a ball in Ω.

For the upcoming main results, we also need the following lemmas, given by T. Iwaniec and A. Lutoborski in [13].

LEMMA 2.6. *Let $u \in D'(\Omega,\Lambda^l)$ and $du \in L^p(\Omega,\Lambda^{l+1})$. Then $u-u_Q$ is in $L^{\frac{np}{n-p}}(\Omega,\Lambda^l)$ and there exists a constant C, independent of u, such that*

$$\left(\int_Q |u-u_Q|^{\frac{np}{n-p}}dx\right)^{\frac{n-p}{np}} \leqslant C\left(\int_Q |du|^p dx\right)^{1/p} \tag{2.10}$$

for Q a cube or ball in $\mathbb{R}^n$, $l = 0,1,\cdots,n-1$, and $1 < p < n$.

LEMMA 2.7. *Let $u \in L^p_{loc}(\Omega,\Lambda^l)$, $l = 1,2,\cdots,n$, $1 < p < \infty$, be a differential form and $T : L^p(\Omega,\Lambda^l) \to W^{1,p}(\Omega,\Lambda^{l-1})$ be the homotopy operator. Then, we have*

$$\|Tu\|_{p,\Omega} \leqslant C|\Omega|diam(\Omega)\|u\|_{p,\Omega} \tag{2.11}$$

holds for any bounded and convex domain Ω, where C is a constant, independent of u.

3. Local higher order inequalities

In this section, we will mainly establish the local higher order Poincaré inequality and the local higher order Caccioppoli inequality with L^φ norms for solutions to the non-homogeneous A-harmonic equations. Before that, we first prove the Caccioppoli inequality with L^φ norms for solutions to the non-homogeneous A-harmonic equations which will be used to establish the local higher order Poincaré inequality.

THEOREM 3.1. *Let φ be a Young function in the class $NG(p,q)$ with $q(n-p) < np$, $1 < p \leqslant q < \infty$, $u \in L^p(\Omega,\Lambda^l)$ be a solution to the non-homogeneous A-harmonic equation (2.1). If $\varphi(|u|) \in L^1_{loc}(\Omega)$, then, there exists a constant C, independent of u, such that*

$$\|du\|_{L^\varphi(B)} \leqslant C\|u-c\|_{L^\varphi(\sigma B)} \tag{3.1}$$

for all balls B with $\sigma B \subset \Omega$, where $\sigma > 1$ is a constant, c is any closed form.

Proof. Using the Hölder's inequality with $1 = \frac{q}{n+q} + \frac{n}{n+q}$, we obtain

$$\begin{aligned}
&\int_B \varphi\big(|du|\big)dx \\
&= \int_B \frac{\varphi\big(|du|\big)}{|du|^{\frac{nq}{n+q}}} |du|^{\frac{nq}{n+q}} dx \\
&\leqslant \Big(\int_B \frac{\varphi\big(|du|\big)^{\frac{n+q}{q}}}{|du|^n} dx\Big)^{\frac{q}{n+q}} \Big(\int_B |du|^q dx\Big)^{\frac{n}{n+q}}.
\end{aligned} \tag{3.2}$$

Using Lemma 2.1 and noticing $A(t)$ is a concave function, it follows that

$$\begin{aligned}
&\int_B \varphi\big(|du|\big)dx \\
&\leqslant \left(\int_B K\big(|du|^q\big)dx\right)^{\frac{q}{n+q}} \left(\int_B |du|^q dx\right)^{\frac{n}{n+q}} \\
&\leqslant \left(\int_B A\big(|du|^q\big)dx\right)^{\frac{q}{n+q}} \left(\int_B |du|^q dx\right)^{\frac{n}{n+q}} \\
&\leqslant A^{\frac{q}{n+q}} \left(\int_B |du|^q dx\right) \left(\int_B |du|^q dx\right)^{\frac{n}{n+q}} \\
&\leqslant C_1(n,q) K^{\frac{q}{n+q}} \left(\int_B |du|^q dx\right) \left(\int_B |du|^q dx\right)^{\frac{n}{n+q}} \\
&= C_1(n,q) \frac{\varphi\left((\int_B |du|^q dx)^{\frac{1}{q}} \right)}{(\int_B |du|^q dx)^{\frac{n}{n+q}}} \left(\int_B |du|^q dx\right)^{\frac{n}{n+q}} \\
&= C_1(n,q)\varphi\left(\left(\int_B |du|^q dx\right)^{\frac{1}{q}} \right).
\end{aligned} \tag{3.3}$$

Combining Lemma 2.4 and Lemma 2.3 gives

$$\begin{aligned}
\left(\int_B |du|^q dx\right)^{1/q} &\leqslant C_2 |B|^{\frac{p-q}{pq}} \left(\int_{\sigma_1 B} |du|^p dx\right)^{1/p} \\
&\leqslant C_3 |B|^{\frac{p-q}{pq}} |B|^{-\frac{1}{n}} \left(\int_{\sigma_2 B} |u-c|^p dx\right)^{1/p},
\end{aligned} \tag{3.4}$$

where $\sigma_2 > \sigma_1 > 1$ is a constant. Combining (3.3), (3.4) and and noticing φ is increasing and satisfies Δ_2-condition, we have

$$\int_B \varphi\big(|du|\big)dx \leqslant C_4 \varphi\left(\left(\int_{\sigma_2 B} |u-c|^p dx\right)^{1/p} \right). \tag{3.5}$$

Taking $h(t) = \int_0^t \frac{\varphi(s)}{s} ds$ and using the fact that $\frac{\varphi(t)}{t^q}$ is decreasing with t, we obtain

$$h(t) = \int_0^t \frac{\varphi(s)}{s} ds = \int_0^t \frac{\varphi(s)}{s^q} s^{q-1} ds \geqslant \frac{\varphi(t)}{t^q} \frac{1}{q} s^q |_0^t = \frac{1}{q}\varphi(t).$$

Similarly, we have $h(t) \leqslant \frac{1}{p}\varphi(t)$ since $\frac{\varphi(t)}{t^p}$ is increasing with t. Hence,

$$\frac{1}{q}\varphi(t) \leqslant h(t) \leqslant \frac{1}{p}\varphi(t). \tag{3.6}$$

Let $g(t) = h(t^{\frac{1}{p}})$, then $\left(h(t^{\frac{1}{p}})\right)' = \frac{1}{p}\frac{\varphi(t^{\frac{1}{p}})}{t}$ is increasing. Thus, g is a convex function. According to definitions of g and h and using Jensen's inequality to g, we have

$$h\left(\left(\int_B |u|^p dx\right)^{\frac{1}{p}}\right) = g\left(\int_B |u|^p dx\right) \leqslant \int_B g(|u|^p)dx = \int_B h(|u|)dx. \tag{3.7}$$

Combining (3.5), (3.6) and (3.7), we have

$$\begin{aligned}
\int_B \varphi\big(|du|\big)dx &\leqslant C_4\varphi\left(\left(\int_{\sigma_2 B}|u-c|^p dx\right)^{\frac{1}{p}}\right)\\
&\leqslant C_5 h\left(\left(\int_{\sigma_2 B}|u-c|^p dx\right)^{\frac{1}{p}}\right)\\
&\leqslant C_5\int_{\sigma_2 B} h\big(|u-c|\big)dx\\
&\leqslant C_6\int_{\sigma_2 B}\varphi\big(|u-c|\big)dx,
\end{aligned} \tag{3.8}$$

which implies (3.1) holds. This completes the proof of Theorem 3.1. □

Now, we are ready to prove the local higher order Poincaré inequality with L^{φ} norms for solutions to the non-homogeneous A-harmonic equation (2.1).

THEOREM 3.2. *Let φ be a Young function in the class $NG(p,q)$ with $q(n-p) < np$, $1 < p \leqslant q < \infty$, $u \in L^p(\Omega,\Lambda^l)$ be a solution to the non-homogeneous A-harmonic equation (2.1). If $\varphi(|du|) \in L^1_{loc}(\Omega)$, then, there exist constants $r > 1$, $C > 0$, independent of u, such that*

$$\left(\int_B \varphi^r\big(|u-u_B|\big)dx\right)^{\frac{1}{r}} \leqslant C\int_{\sigma B}\varphi\big(|du|\big)dx \tag{3.9}$$

for all balls B with $\sigma B \subset \Omega$, where $\sigma > 2$ is a constant.

Proof. From Theorem 3.1, we have

$$\int_B \varphi\big(|du|\big)dx \leqslant C_1\int_{\sigma_1 B}\varphi\big(|u-c|\big)dx, \tag{3.10}$$

where $\sigma_1 > 1$ is a constant, c is any closed form. Here, taking $c = u_{\sigma_1 B}$ yields that

$$\int_B \varphi\big(|du|\big)dx \leqslant C_1\int_{\sigma_1 B}\varphi\big(|u-u_{\sigma_1 B}|\big)dx. \tag{3.11}$$

Using the Lemma 2.5, (3.11) and the Hölder inequality with $1 = \frac{q}{n+q} + \frac{n}{n+q}$, we obtain

$$\begin{aligned}
\int_B \varphi\left(|u-u_B|\right)dx &\leqslant C_2 \int_B \varphi\left(|du|\right)dx \\
&\leqslant C_3 \int_{\sigma_1 B} \varphi\left(|u-u_{\sigma_1 B}|\right)dx \\
&= C_3 \int_{\sigma_1 B} \frac{\varphi\left(|u-u_{\sigma_1 B}|\right)}{|u-u_{\sigma_1 B}|^{\frac{nq}{n+q}}} |u-u_{\sigma_1 B}|^{\frac{nq}{n+q}} dx \\
&\leqslant C_3 \left(\int_{\sigma_1 B} \frac{\varphi\left(|u-u_{\sigma_1 B}|\right)^{\frac{n+q}{q}}}{|u-u_{\sigma_1 B}|^{n}} dx\right)^{\frac{q}{n+q}} \left(\int_{\sigma_1 B} |u-u_{\sigma_1 B}|^{q} dx\right)^{\frac{n}{n+q}}.
\end{aligned} \tag{3.12}$$

Using lemma 2.6 with $p = q_* = \frac{nq}{n+q}$ and $Q = \sigma_1 B$, we have

$$\left(\int_{\sigma_1 B} |u-u_{\sigma_1 B}|^{q} dx\right)^{\frac{n}{n+q}} \leqslant C_4 \int_{\sigma_1 B} |du|^{q_*}. \tag{3.13}$$

Substituting (3.13) into (3.12), then using Lemma 2.1 and noticing $A(t)$ is a concave and increasing function, it follows that

$$\begin{aligned}
\int_B \varphi\left(|u-u_B|\right)dx &\leqslant C_5 \left(\int_{\sigma_1 B} \frac{\varphi\left(|u-u_{\sigma_1 B}|\right)^{\frac{n+q}{q}}}{|u-u_{\sigma_1 B}|^{n}} dx\right)^{\frac{q}{n+q}} \int_{\sigma_1 B} |du|^{q_*} \\
&\leqslant C_6 \left(\int_{\sigma_1 B} K(|u-u_{\sigma_1 B}|^{q}) dx\right)^{\frac{q}{n+q}} \int_{\sigma_1 B} |du|^{q_*} dx \\
&\leqslant C_6 \left(\int_{\sigma_1 B} A(|u-u_{\sigma_1 B}|^{q}) dx\right)^{\frac{q}{n+q}} \int_{\sigma_1 B} |du|^{q_*} dx \\
&\leqslant C_6 A^{\frac{q}{n+q}} \left(\int_{\sigma_1 B} |u-u_{\sigma_1 B}|^{q} dx\right) \int_{\sigma_1 B} |du|^{q_*} dx \\
&\leqslant C_7 A^{\frac{q}{n+q}} \left(\left(\int_{\sigma_1 B} |du|^{q_*} dx\right)^{\frac{q}{q_*}}\right) \int_{\sigma_1 B} |du|^{q_*} dx \\
&\leqslant C_8 K^{\frac{q}{n+q}} \left(\left(\int_{\sigma_1 B} |du|^{q_*} dx\right)^{\frac{q}{q_*}}\right) \int_{\sigma_1 B} |du|^{q_*} dx \\
&= C_8 \frac{\varphi\left(\left(\int_{\sigma_1 B} |du|^{q_*} dx\right)^{\frac{1}{q_*}}\right)}{\int_{\sigma_1 B} |du|^{q_*} dx} \int_{\sigma_1 B} |du|^{q_*} dx \\
&= C_8 \varphi\left(\left(\int_{\sigma_1 B} |du|^{q_*} dx\right)^{\frac{1}{q_*}}\right).
\end{aligned} \tag{3.14}$$

From Lemma 2.3, we have

$$\|du\|_{q_*,\sigma_1 B} \leqslant C_9 |\sigma_1 B|^{\frac{-1}{n}} \|u-c\|_{q_*,\sigma_2 B}, \tag{3.15}$$

where $\sigma_2 > \sigma_1 > 1$. Selecting $c = u_B$ in (3.15), and noticing that φ is a increasing function, we obtain

$$\varphi\left(\left(\int_{\sigma_1 B} |du|^{q_*} dx\right)^{\frac{1}{q_*}}\right) \leqslant C_{10}(n,q)\varphi\left(\left(\int_{\sigma_2 B} |u-u_B|^{q_*} dx\right)^{\frac{1}{q_*}}\right). \tag{3.16}$$

Combining (3.14) and (3.16), we have

$$\int_B \varphi\big(|u-u_B|\big)dx \leqslant C_{11}\varphi\left(\left(\int_{\sigma_2 B} |u-u_B|^{q_*} dx\right)^{\frac{1}{q_*}}\right). \tag{3.17}$$

Let $\psi(t) = \varphi\left(t^{\frac{1}{q_*}}\right)$, we have

$$\psi(2t) \leqslant K\psi(t), \tag{3.18}$$

$$\psi'(t) \geqslant \frac{p}{q_*}\frac{\psi(t)}{t}, \quad p/q_* > 1, \tag{3.19}$$

and so with $f(x) = |u-u_B|^{q_*}$, we deduce by (3.17) that

$$\int_B \psi(f)dx \leqslant C_{11}\psi\left(\int_{\sigma_2 B} f dx\right), \tag{3.20}$$

which shows that ψ and f satisfy the assumptions of Lemma 2.2 when $\sigma_2 = 2$. Thus there exists $r > 1$ such that

$$\int_B \psi^r(f)dx \leqslant C_{12}\psi^r\left(\int_{2B} f dx\right), \tag{3.21}$$

that is,

$$\int_B \varphi^r\big(|u-u_B|\big)dx \leqslant C_{12}\varphi^r\left(\left(\int_{2B} |u-u_B|^{q_*} dx\right)^{\frac{1}{q_*}}\right). \tag{3.22}$$

Taking $h(t) = \int_0^t \frac{\varphi(s)}{s} ds$ and using the fact that $\frac{\varphi(t)}{t^q}$ is decreasing with t, we obtain

$$h(t) = \int_0^t \frac{\varphi(s)}{s} ds = \int_0^t \frac{\varphi(s)}{s^q} s^{q-1} ds \geqslant \varphi(t)/t^q \frac{1}{q} s^q\Big|_0^t = \frac{1}{q}\varphi(t).$$

Similarly, we have $h(t) \leqslant \frac{1}{p}\varphi(t)$ since $\frac{\varphi(t)}{t^p}$ is increasing with t. Hence,

$$\frac{1}{q}\varphi(t) \leqslant h(t) \leqslant \frac{1}{p}\varphi(t). \tag{3.23}$$

Let $g(t)=h(t^{\frac{1}{p}})$, then $\left(h(t^{\frac{1}{p}})\right)'=\frac{1}{p}\frac{\varphi(t^{\frac{1}{p}})}{t}$ is increasing. Thus, g is a convex function. From a result of [9], page 169, it follows that

$$\left(\int_{2B}|u-u_B|^p dx\right)^{\frac{1}{p}}\leqslant h^{-1}\left(\int_{2B}h(|u-u_B|)dx\right). \tag{3.24}$$

Taking into account that $p>q_*$ and using (3.23), (3.24), we have

$$\begin{aligned}\frac{1}{q}\varphi\left(\left(\int_{2B}|u-u_B|^{q^*}dx\right)^{\frac{1}{q_*}}\right)&\leqslant\frac{1}{q}\varphi\left(\left(\int_{2B}|u-u_B|^p dx\right)^{\frac{1}{p}}\right)\\&\leqslant h\left(\left(\int_{2B}|u-u_B|^p dx\right)^{\frac{1}{p}}\right)\\&\leqslant\int_{2B}h(|u-u_B|)dx\\&\leqslant C_{13}\int_{2B}\varphi(|u-u_B|)dx.\end{aligned}\tag{3.25}$$

Thus, we have

$$\varphi^r\left(\left[\int_{2B}|u-u_B|^{q_*}dx\right]^{\frac{1}{q_*}}\right)\leqslant C_{14}\left(\int_{2B}\varphi(|u-u_B|)dx\right)^r. \tag{3.26}$$

Substituting (3.26) into (3.22) yields that

$$\left(\int_B\varphi^r\left(|u-u_B|\right)dx\right)^{\frac{1}{r}}\leqslant C_{15}\int_{2B}\varphi(|u-u_B|)dx. \tag{3.27}$$

Since $u=Tdu+dTu$, and notice that $u_B=dTu$, (3.27) follows

$$\left(\int_B\varphi^r\left(|u-u_B|\right)dx\right)^{\frac{1}{r}}\leqslant C_{15}\int_{2B}\varphi(|Tdu|)dx. \tag{3.28}$$

Using the similar process developed in inequality (3.3), we obtain

$$\int_{2B}\varphi(|Tdu|)dx\leqslant C_{16}\varphi\left(\left(\int_{2B}|Tdu|^q dx\right)^{\frac{1}{q}}\right). \tag{3.29}$$

By Lemma 2.4 and Lemma 2.7, we get

$$\begin{aligned}\left(\int_{2B}|Tdu|^q dx\right)^{\frac{1}{q}}&=\|Tdu\|_{q,2B}\\&\leqslant C_{17}|2B|diam(2B)\|du\|_{q,2B}\\&\leqslant C_{18}|2B|^{1+\frac{1}{n}+\frac{p-q}{pq}}\left(\int_{\sigma B}|du|^p dx\right)^{\frac{1}{p}},\end{aligned}\tag{3.30}$$

where $\sigma > 2$ is a constant. Substituting (3.30) into (3.29) and repeating the process from (3.5) to (3.8), we have

$$\int_{2B} \varphi(|Tdu|)dx \leqslant C_{19} \int_{\sigma B} \varphi\big(|du|\big)dx. \tag{3.31}$$

Combining (3.28) and (3.31) yields

$$\left(\int_B \varphi^r\big(|u-u_B|\big)dx\right)^{\frac{1}{r}} \leqslant C_{20} \int_{\sigma B} \varphi(|du|)dx. \tag{3.32}$$

This completes the proof of Theorem 3.2. $\square$

Note that the above inequality (3.9) can be written as the following version

$$\|\varphi(|u-u_B|)\|_{r,B} \leqslant C\|\varphi(|du|)\|_{1,\sigma B}, \tag{3.33}$$

which indicates that $\varphi(|u-u_B|) \in L^r_{loc}(\Omega)$ if $\varphi(|u|) \in L^1_{loc}(\Omega)$ for $r > 1$. This gives the higher order estimate of Poincaré-type inequality with L^φ norm for the solution to the non-homogeneous A-harmonic equation.

By the proof of Theorem 3.2, we can easily establish the local higher order Caccioppoli inequality with L^φ norms for solutions to the non-homogeneous A-harmonic equation as follows.

THEOREM 3.3. *Let φ be a Young function in the class $NG(p,q)$ with $q(n-p) < np$, $1 < p \leqslant q < \infty$, $u \in L^p(\Omega,\Lambda^l)$ be a solution to the non-homogeneous A-harmonic equation (2.1). If $\varphi(|u|) \in L^1_{loc}(\Omega)$, then, there exist constants $r > 1$, $C > 0$, independent of u, such that*

$$\left(\int_B \varphi^r\big(|du|\big)dx\right)^{\frac{1}{r}} \leqslant C \int_{\sigma B} \varphi\big(|u-c|\big)dx\big) \tag{3.34}$$

for all balls B with $\sigma B \subset \Omega$, where $\sigma > 2$ is a constant, c is any closed form.

Proof. From the inequalities (3.12) to (3.14) in Theorem 3.2, we also have

$$\int_B \varphi\big(|du|\big)dx \leqslant C_1 \varphi\left(\left(\int_{\sigma_1 B} |du|^{q_*} dx\right)^{\frac{1}{q_*}}\right), \tag{3.35}$$

where $\sigma_1 > 1$. Choosing $\psi(t) = \varphi\big(t^{1/q_*}\big)$, $f(x) = |du|^{q_*}$ and using the similar proof of Theorem 3.2, we get

$$\left(\int_B \varphi^r\big(|du|\big)dx\right)^{\frac{1}{r}} \leqslant C \int_{2B} \varphi(|du|)dx. \tag{3.36}$$

Combining (3.36) and Theorem 3.1, we obtain

$$\left(\int_B \varphi^r\big(|du|\big)dx\right)^{\frac{1}{r}} \leqslant C \int_{\sigma B} \varphi\big(|u-c|\big)dx\big) \tag{3.37}$$

for all balls B with $\sigma B \subset \Omega$, where $\sigma > 2$ is a constant, c is any closed form. □

Note that the above inequality (3.34) can be written as the following version

$$\|\varphi(|du|)\|_{r,B} \leqslant C\|\varphi(|u-c|)\|_{1,\sigma B}, \tag{3.38}$$

which indicates that $\varphi(|du|) \in L^r_{loc}(\Omega)$ if $\varphi(|u-c|) \in L^1_{loc}(\Omega)$ for $r > 1$. This result shows the higher order estimate than the result in Theorem 3.1.

4. Global higher order inequalities

In this section, we are going to prove the global higher order Poincaré inequality and Caccioppoli inequality for solutions to the non-homogeneous A-harmonic equation. We need the following well known Whitney covering lemma from [20].

LEMMA 4.1. *Each domain M has a modified Whitney cover of cubes $\mathscr{V} = \{Q_i\}$ such that*

$$\cup_i Q_i = M, \sum_{Q_i \in \mathscr{V}} \chi_{\sqrt{\frac{5}{4}}Q_i} \leqslant N\chi_\Omega$$

and some $N > 1$, and if $Q_i \cap Q_j \neq \emptyset$, then there exists a cube R (this cube need not be a member of $\mathscr{V}$) in $Q_i \cap Q_j$ such that $Q_i \cup Q_j \subset NR$. Moreover, if M is δ-John, then there is a distinguished cube $Q_0 \in \mathscr{V}$ which can be connected with every cube $Q \in \mathscr{V}$ by a chain of cubes $Q_0, Q_1, \cdots, Q_k = Q$ from $\mathscr{V}$ and such that $Q \subset \rho Q_i$, $i = 0,1,2,\cdots,k$, for some $\rho = \rho(n,\delta)$.

We now extend the Theorem 3.2 to the global higher order Poincaré inequality with L^φ norms for solutions to the non-homogeneous A-harmonic equations.

THEOREM 4.2. *Let φ be a Young function in the class $NG(p,q)$ with $q(n-p) < np$, $1 < p \leqslant q < \infty$, $u \in L^p(\Omega,\Lambda^l)$ be a solution to the non-homogeneous A-harmonic equation (2.1). If $\varphi(|du|) \in L^1(\Omega)$, then, there exist constants $r > 1$, $C > 0$, independent of u, such that*

$$\left(\int_\Omega \varphi^r\big(|u-u_\Omega|\big)dx\right)^{\frac{1}{r}} \leqslant C\int_\Omega \varphi(|du|)dx \tag{4.1}$$

for any bounded domain $\Omega \subset \mathbb{R}^n$.

Proof. Notice that $u - u_\Omega = T(du)$. By Lemma 4.1 and Theorem 3.2, we have

$$\begin{aligned}
\left(\int_\Omega \varphi^r\big(|u-u_\Omega|\big)dx\right)^{\frac{1}{r}} &= \|\varphi(|u-u_\Omega|)\|_{r,\Omega} \\
&= \|\varphi(|T(du)|)\|_{r,\Omega} \\
&\leqslant \sum_{B\in v} \|\varphi(|T(du)|)\|_{r,B}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \sum_{B\in \mathcal{V}} (C_1\|\varphi(|du|)\|_{1,\sigma B}) \\
&\leqslant C_2 N\|\varphi(|du|)\|_{1,\Omega} \\
&\leqslant C_3\|\varphi(|du|)\|_{1,\Omega} \\
&= C_3\int_\Omega \varphi(|du|)dx,
\end{aligned} \tag{4.2}
$$

which finishes the proof of Theorem 4.2. □

Similarly, Theorem 3.3 can be extended into the following global higher order Caccioppoli inequality with L^φ norms for solutions to the non-homogeneous A-harmonic equations.

THEOREM 4.3. *Let φ be a Young function in the class $NG(p,q)$ with $q(n-p) < np$, $1 < p \leqslant q < \infty$, $u \in L^p(\Omega,\Lambda^l)$ be a solution to the non-homogeneous A-harmonic equation* (2.1)*. If $\varphi(|u|) \in L^1(\Omega)$, then, there exist constants $r > 1$, $C > 0$, independent of u, such that*

$$
\left(\int_\Omega \varphi^r(|du|)dx\right)^{\frac{1}{r}} \leqslant C\int_\Omega \varphi(|u-c|)dx) \tag{4.3}
$$

for any bounded domain $\Omega \subset \mathbb{R}^n$, where c is any closed form.

By choosing $\varphi(t) = t^p$ in Theorem 4.2 and simple deduction, we can obtain the following version of the higher order Poincaré inequality inequality with L^p-norms which is the special case of Theorem 4.2.

THEOREM 4.4. *Let $\varphi(t) = t^p$, $p \geqslant 1$, $u \in L^p(\Omega,\Lambda^l)$ be a solution to the non-homogeneous A-harmonic equation* (2.1) *in Ω. If $\varphi(|du|) \in L^1(\Omega)$, then, there exist constants $r > 1$, $C > 0$, independent of u, such that*

$$
\|u-u_\Omega\|_{rp,\Omega} \leqslant C\|du\|_{p,\Omega} \tag{4.4}
$$

for any bounded domain $\Omega \subset \mathbb{R}^n$.

REMARK 4.1. When we choose $\varphi(t) = t^p$ in Theorem 3.2, 3.3, 4.2, 4.3, the higher order Poincaré inequalities and Caccioppoli inequalities with L^φ norms will reduce to the corresponding higher order Poincaré inequalities and Caccioppoli inequalities with L^p norms. Here, we just take Theorem 4.2 as an example.

5. Applications

Using Theorem 3.2, 3.1 and the fact that $u - u_B = T(du)$, we can easily show the following local higher order Caccioppoli-type inequality for homotopy operator acting on solutions to the non-homogeneous A-harmonic equations.

THEOREM 5.1. *Let φ be a Young function in the class $NG(p,q)$ with $q(n-p) < np$, $1 < p \leqslant q < \infty$, $u \in L^p(\Omega, \Lambda^l)$ be a solution to the non-homogeneous A-harmonic equation (2.1), $T : L^s(\Omega, \Lambda^l) \to W^{1,s}(\Omega, \Lambda^{l-1})$ be the homotopy operator. If $\varphi(|u|) \in L^1_{loc}(\Omega)$, then, there exist constants $r > 1$, $C > 0$, independent of u, such that*

$$\left(\int_B \varphi^r\big(|T(du)|\big)dx\right)^{\frac{1}{r}} \leqslant C\int_{\sigma B} \varphi(|u-c|)dx \tag{5.1}$$

for all balls B with $\sigma B \subset \Omega$, where $\sigma > 2$ is a constant, c is any closed form.

Applying the analogous method developed in Theorem 4.2, we obtain the global higher order Caccioppoli-type inequality for homotopy operator as follows.

THEOREM 5.2. *Let φ be a Young function in the class $NG(p,q)$ with $q(n-p) < np$, $1 < p \leqslant q < \infty$, $u \in L^p(\Omega, \Lambda^l)$ be a solution to the non-homogeneous A-harmonic equation (2.1), $T : L^s(\Omega, \Lambda^l) \to W^{1,s}(\Omega, \Lambda^{l-1})$ be the homotopy operator. If $\varphi(|u|) \in L^1(\Omega)$, then, there exist constants $r > 1$, $C > 0$, independent of u, such that*

$$\left(\int_\Omega \varphi^r\big(|T(du)|\big)dx\right)^{\frac{1}{r}} \leqslant C\int_{\Omega} \varphi(|u-c|)dx \tag{5.2}$$

for any bounded domain $\Omega \subset \mathbb{R}^n$, where c is any closed form.

Next, we will derive a weak type inequality for homotopy operator with the help of Theorem 5.1 and the famous Chebyshev's inequality.

THEOREM 5.3. *Let φ be a Young function in the class $NG(p,q)$ with $q(n-p) < np$, $1 < p \leqslant q < \infty$, $u \in L^p(\Omega, \Lambda^l)$ be a solution to the non-homogeneous A-harmonic equation (2.1), $T : L^s(\Omega, \Lambda^l) \to W^{1,s}(\Omega, \Lambda^{l-1})$ be the homotopy operator. If $\varphi(|u|) \in L^1_{loc}(\Omega)$, then, there exist constants $r > 1$, $C > 0$, independent of u, such that*

$$\Big|\{x \in B : |T(du) \geqslant t|\}\Big|^{1/r} \leqslant \frac{C}{\varphi(t)}\int_{\sigma B} \varphi(|u-c|)dx \tag{5.3}$$

for all balls B with $\sigma B \subset \Omega$, where $\sigma > 2$ is a constant, c is any closed form.

Proof. From Chebyshev's inequality, we have

$$\mu\Big(\{x \in X : |f(x) \geqslant t|\}\Big) \leqslant \frac{1}{g(t)}\int_X g\circ f d\mu. \tag{5.4}$$

Let X be the ball B with $\sigma B \subset \Omega$ and choose $g(x) = \varphi^r(x)$, $f = T(du)$ in inequality (5.4), we obtain

$$\Big|\{x \in B : |T(du) \geqslant t|\}\Big| \leqslant \frac{1}{\varphi^r(t)}\int_B \varphi^r(|T(du)|)dx. \tag{5.5}$$

By Theorem 5.1, we have

$$\int_B \varphi^r\big(|T(du)|\big)dx \leqslant \left(\int_{\sigma B} \varphi(|u-c|)dx\right)^r \tag{5.6}$$

for all balls B with $\sigma B \subset \Omega$, where $\sigma > 2$ is a constant, c is any closed form. Substituting (5.6) into (5.5) yields that

$$\Big|\{x \in B : |T(du) \geqslant t|\}\Big| \leqslant \frac{C}{\varphi^r(t)}\left(\int_{\sigma B} \varphi(|u-c|)dx\right)^r, \tag{5.7}$$

which indicates that the inequality (5.3) holds. □

REMARK 5.1. It is worth pointing out that the higher order inequalities with L^φ norms obtained in this paper can be used to study the regularity properties of the solutions to non-homogeneous A-harmonic equations. Additionally, the techniques developed in this paper also provide an effective mean to study the higher order estimates with L^φ norms for solutions to the Dirac-harmonic equations in [7] which are more general and complicated.

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Jinling Niu
Department of Mathematics and Statistics
Changshu Institute of Technology
Changshu, 215500, P. R. China
e-mail: njlhit@163.com

Guannan Shi
Department of Mathematics
Northeast Petroleum University
Daqin, 163318, P. R. China
e-mail: shig@seattleu.edu

Yuming Xing
Department of Mathematics
Harbin Institute of Technology
Harbin, 150001, P. R. China
e-mail: xyuming@hit.edu.cn