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THE MINIMAL SYSTEM OF GENERATORS OF AN AFFINE, PLANE AND NORMAL SEMIGROUP

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Abstract. If X is a nonempty subset of $\mathbb{Q}^k$, the *cone* generated by X is $\mathrm{C}(X) = \{q_1x_1 + \cdots + q_nx_n \mid n \in \mathbb{N}\setminus\{0\}, \{q_1,\ldots,q_n\} \subseteq \mathbb{Q}_0^+$ and $\{x_1,\ldots,x_n\} \subseteq X\}$. In this work we present an algorithm which calculates from $\{(a_1,b_1),(a_2,b_2)\} \subseteq \mathbb{N}^2$, the minimal system of generators of the affine semigroup $\mathrm{C}(\{(a_1,b_1),(a_2,b_2)\}) \cap \mathbb{N}^2$. This algorithm is based on the study of proportionally modular Diophantine inequalities carried out in [1]. Also, we present an upper bound for the embedding dimension of this semigroup.

1. Introduction

Let $\mathbb{Z}$ be the set of integer numbers and $\mathbb{N} = \{x \in \mathbb{Z} \mid x \geqslant 0\}$. If $k \in \mathbb{N}\setminus\{0\}$ and A is a nonempty subset of $\mathbb{N}^k$, we denote by $\langle A\rangle$ the submonoid of $(\mathbb{N}^k,+)$ generated by A, that is, $\langle A\rangle = \{\lambda_1a_1 + \cdots + \lambda_na_n \mid n \in \mathbb{N}\setminus\{0\}, \{\lambda_1,\ldots,\lambda_n\} \subseteq \mathbb{N}$ and $\{a_1,\ldots,a_n\} \subseteq A\}$.

Let S be a submonoid of $(\mathbb{N}^k,+)$. If $S = \langle A\rangle$, we say that A is a *system of generators* of S. Moreover, if $S \neq \langle B\rangle$ for $B \subsetneq A$, then A is a *minimal* system of generators of S. It is well known, see for instance [5], that every submonoid of $(\mathbb{N}^k,+)$ admits a unique minimal system of generators. We will denote by $\mathrm{msg}(S)$ the minimal system of generators of S.

We say that a submonoid S of $(\mathbb{N}^k,+)$ is *finitely generated* if $\mathrm{msg}(S)$ is a finite set. An *affine semigroup* is a finitely generated submonoid of $(\mathbb{N}^k,+)$. If S is an affine semigroup, then the cardinality of $\mathrm{msg}(S)$ is called the *embedding dimension* of S and will be denoted by $\mathrm{e}(S)$.

Let $\mathbb{Q}$ be the set of rational numbers and $\mathbb{Q}_0^+ = \{x \in \mathbb{Q} \mid x \geqslant 0\}$. If X is a nonempty subset of $\mathbb{Q}^k$, the *cone* generated by X is $\mathrm{C}(X) = \{q_1x_1 + \cdots + q_nx_n \mid n \in \mathbb{N}\setminus\{0\}, \{q_1,\ldots,q_n\} \subseteq \mathbb{Q}_0^+$ and $\{x_1,\ldots,x_n\} \subseteq X\}$.

A submonoid S of $\mathbb{N}^k$ is *normal* if $S = \mathrm{C}(S) \cap \mathbb{N}^k$. This notation was introduced by Hochster in [3] where he proves that an affine semigroup S is normal if and only if its semigroup ring $\mathrm{K}[S]$ over a field K, is a normal ring.

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We say that an affine semigroup S is *plane* if $S \subseteq \mathbb{N}^2$ and the dimension of vectorial subspace of $\mathbb{Q}^2$ generated by S is two.

It is well known, and in the Section 2 we will prove it, that S is an affine, plane and normal semigroup if and only if

$$S = \mathrm{C}(\{(a_1,b_1),(a_2,b_2)\}) \cap \mathbb{N}^2 \text{ for some } \{(a_1,b_1),(a_2,b_2)\} \subseteq \mathbb{N}^2$$

where $\det\begin{pmatrix} a_2 & a_1 \\ b_2 & b_1 \end{pmatrix} \neq 0$ and $\gcd\{a_1,b_1\} = \gcd\{a_2,b_2\} = 1$.

Our aim in this paper is to give an alternative algorithm to the one presented by G. Lachaud in [4] to calculate from $\{(a_1,b_1),(a_2,b_2)\}$, the minimal system of generators of the affine semigroup $\mathrm{C}(\{(a_1,b_1),(a_2,b_2)\}) \cap \mathbb{N}^2$. Although the complexity of both algorithms are similar, there are some differences between them: The algorithm of Lachaud is based on the description of the covex hull of $(C \setminus \{0\}) \cap \mathbb{Z}^2$ (the Klein polygon of an angle C), using continued fractions. Our algorithm has a similar complexity to Euclid's algorithm to compute the greatest common divisor of two integers and it is based on the study of proportionally modular Diophantine inequalities carried out in [1] and the system of generators obtained is a minimal system of generators.

2. First results

The following result is easily deduced from the definition of cone generated by a set.

LEMMA 1. *If S is an affine semigroup generated by $\{\alpha_1,\alpha_2 \dots,\alpha_p\}$, then* $\mathrm{C}(S) = \mathrm{C}(\{\alpha_1,\alpha_2 \dots,\alpha_p\})$.

As a consequence of Cramer's Formula for the resolution of systems of linear equations, we have the following result.

LEMMA 2. *Let $\{(a_1,b_1),(a_2,b_2),(x,y)\} \subseteq \mathbb{N}^2$ such that* $\det\begin{pmatrix} a_2 & a_1 \\ b_2 & b_1 \end{pmatrix} > 0$. *Then* $(x,y) \in \mathrm{C}(\{(a_1,b_1),(a_2,b_2)\})$ *if and only if* $\det\begin{pmatrix} x & a_1 \\ y & b_1 \end{pmatrix} \geqslant 0$ *and* $\det\begin{pmatrix} a_2 & x \\ b_2 & y \end{pmatrix} \geqslant 0$.

If x is a positive integer, then we admit the fraction $\frac{x}{0} = +\infty$, and we assume that it is greater than every integer number. With this agreement, we can rewrite the previous lemma in the following form. Note that it is also a reformulation of Lemma 4 of [7].

LEMMA 3. *Let $\{(a_1,b_1),(a_2,b_2),(x,y)\} \subseteq \mathbb{N}^2 \setminus \{(0,0)\}$ such that $\frac{a_1}{b_1} < \frac{a_2}{b_2}$. Then* $(x,y) \in \mathrm{C}(\{(a_1,b_1),(a_2,b_2)\})$ *if and only if* $\frac{a_1}{b_1} \leqslant \frac{x}{y} \leqslant \frac{a_2}{b_2}$.

As an immediate consequence of previous lemma, we have the following result.

LEMMA 4. *If* $\{(a_1,b_1),(a_2,b_2),\dots,(a_p,b_p)\}\subseteq\mathbb{N}^2\setminus\{(0,0)\}$ *and* $\frac{a_1}{b_1}\leqslant\frac{a_2}{b_2}\leqslant\dots\leqslant\frac{a_p}{b_p}$, *then* $\mathrm{C}(\{(a_1,b_1),(a_2,b_2),\dots,(a_p,b_p)\})=\mathrm{C}(\{(a_1,b_1),(a_p,b_p)\})$.

The following result is deduced from [6].

LEMMA 5. *If* $\{(a_1,b_1),(a_2,b_2)\}\subseteq\mathbb{N}^2$, *then* $\mathrm{C}(\{(a_1,b_1),(a_2,b_2)\})\cap\mathbb{N}^2$ *is an affine semigroup.*

The following result has an immediate proof.

LEMMA 6. *If* $\{(a_1,b_1),(a_2,b_2)\}\subseteq\mathbb{N}^2\setminus\{(0,0)\}$, $d_1=\gcd\{a_1,b_1\}$ *and* $d_2=\gcd\{a_2,b_2\}$, *then* $\mathrm{C}(\{(a_1,b_1),(a_2,b_2)\})=\mathrm{C}\left(\left\{\left(\frac{a_1}{d_1},\frac{b_1}{d_1}\right),\left(\frac{a_2}{d_2},\frac{b_2}{d_2}\right)\right\}\right)$.

As an immediate consequence of previous results, we have the following proposition.

PROPOSITION 7. *The following conditions are equivalent:*

1) S *is an affine, plane and normal semigroup.*

2) $S=\mathrm{C}(\{(a_1,b_1),(a_2,b_2)\})\cap\mathbb{N}^2$ *for some* $\{(a_1,b_1),(a_2,b_2)\}\subseteq\mathbb{N}^2$ *such that* $\det\begin{pmatrix}a_2 & a_1\\ b_2 & b_1\end{pmatrix}\neq 0$ *and* $\gcd\{a_1,b_1\}=\gcd\{a_2,b_2\}=1$.

3. Triangulations

In this section, and unless we say otherwise, we suppose that $\{(a_1,b_1),(a_2,b_2)\}\subseteq\mathbb{N}^2$, $\det\begin{pmatrix}a_2 & a_1\\ b_2 & b_1\end{pmatrix}>0$ and $\gcd\{a_1,b_1\}=\gcd\{a_2,b_2\}=1$.

Observe that $\frac{a_1}{b_1}<\frac{a_2}{b_2}$. Besides, if $a_i=0$, then $i=1$ and $b_1=1$. Thus, $(a_1,b_1)=(0,1)$. Analogously, if $b_i=0$ then $i=2$ and $a_2=1$. Therefore, $(a_2,b_2)=(1,0)$.

If $q\in\mathbb{Q}$, then $\lfloor q\rfloor=\max\{x\in\mathbb{Z}\mid x\leqslant q\}$ and $\lceil q\rceil=\min\{x\in\mathbb{Z}\mid q\leqslant x\}$. If $\{a,b\}\subseteq\mathbb{Z}$ and $b\neq 0$, we denote by $a \mod b$ the remainder of the division of a by b. Note that $a=\left\lfloor\frac{a}{b}\right\rfloor b+(a \mod b)$. Also, let us look at $\left\lfloor\frac{a}{b}\right\rfloor=\left\lceil\frac{a}{b}\right\rceil$ if and only if $a \mod b=0$, otherwise $\left\lceil\frac{a}{b}\right\rceil=\left\lfloor\frac{a}{b}\right\rfloor+1$.

The following result is the key for the development of this section.

LEMMA 8. *Let* $S=\mathrm{C}(\{(a_1,b_1),(a_2,b_2)\})\cap\mathbb{N}^2$. *Then* $S=\langle\{(a_1,b_1),(a_2,b_2)\}\rangle$ *if and only if* $\det\begin{pmatrix}a_2 & a_1\\ b_2 & b_1\end{pmatrix}=1$.

Proof. Sufficiency. If $(x,y) \in S$, then $(x,y) \in \mathrm{C}(\{(a_1,b_1),(a_2,b_2)\})$ and thus, there is $\{\lambda,\mu\} \subseteq \mathbb{Q}_0^+$ such that $(x,y) = \lambda(a_1,b_1) + \mu(a_2,b_2)$. Therefore, $x = a_1\lambda + a_2\mu$, and $y = b_1\lambda + b_2\mu$. As $\det\begin{pmatrix} a_2 & a_1 \\ b_2 & b_1 \end{pmatrix} = 1$, if we apply now the Cramers's Formula, we deduce that $\{\lambda,\mu\} \subseteq \mathbb{Z}$. Hence, $\{\lambda,\mu\} \subseteq \mathbb{Q}_0^+ \cap \mathbb{Z} = \mathbb{N}$ and consequently $(x,y) \in \langle\{(a_1,b_1),(a_2,b_2)\}\rangle$.

Necessity. If $\det\begin{pmatrix} a_2 & a_1 \\ b_2 & b_1 \end{pmatrix} \neq 1$, as $\det\begin{pmatrix} a_2 & a_1 \\ b_2 & b_1 \end{pmatrix} > 0$, then $\det\begin{pmatrix} a_2 & a_1 \\ b_2 & b_1 \end{pmatrix} \notin \{1,-1\}$ and so $\{(a_1,b_1),(a_2,b_2)\}$ is not a basis of $\mathbb{Z}^2$ as free $\mathbb{Z}$-module. Thus, $\{e_1 = (1,0), e_2 = (0,1)\} \not\subseteq G = \{z_1(a_1,b_1) + z_2(a_2,b_2) \mid \{z_1,z_2\} \subseteq \mathbb{Z}\}$. Let $i \in \{1,2\}$ such that $e_i \notin G$. As $\det\begin{pmatrix} a_2 & a_1 \\ b_2 & b_1 \end{pmatrix} \neq 0$, then $\{(a_1,b_1),(a_2,b_2)\}$ is a basis of $\mathbb{Q}^2$ as $\mathbb{Q}$-vectorial space. Therefore, there is $\{\lambda,\mu\} \subseteq \mathbb{Q}$ such that $e_i = \lambda(a_1,b_1) + \mu(a_2,b_2)$. Consequently, $e_i - \lfloor\lambda\rfloor(a_1,b_1) - \lfloor\mu\rfloor(a_2,b_2) = (\lambda - \lfloor\lambda\rfloor)(a_1,b_1) + (\mu - \lfloor\mu\rfloor)(a_2,b_2) \in S$ because it belongs to $\mathrm{C}(\{(a_1,b_1),(a_2,b_2)\})$ and it also belongs to $\mathbb{Z}^2$. But, $e_i - \lfloor\lambda\rfloor(a_1,b_1) - \lfloor\mu\rfloor(a_2,b_2) \notin \langle\{(a_1,b_1),(a_2,b_2)\}\rangle$ because, otherwise we would deduce that $e_i \in G$, which is absurd. □

A *triangulation* is a sequence $(x_1,y_1),(x_2,y_2),\ldots,(x_p,y_p)$ of elements from $\mathbb{N}^2$ such that $\det\begin{pmatrix} x_{i+1} & x_i \\ y_{i+1} & y_i \end{pmatrix} = 1$ for all $i \in \{1,\ldots,p-1\}$. In this case, we will say that the triangulation has *lenght* p and the elements (x_1,y_1) and (x_p,y_p) will be called the *ends* of the triangulation. We will say that the triangulation is *proper* if $\det\begin{pmatrix} a_{i+h} & a_i \\ b_{i+h} & b_i \end{pmatrix} \neq 1$ for all $h \in \mathbb{N} \setminus \{0,1\}$ such that $i + h \leqslant p$. It is clear that every triangulation can be refined to a proper triangulation with the same ends.

PROPOSITION 9. *If* $(x_1,y_1),(x_2,y_2),\ldots,(x_p,y_p)$ *is a triangulation, then*

$$S = \mathrm{C}(\{(x_1,y_1),(x_p,y_p)\}) \cap \mathbb{N}^2 =$$

$$\langle\{(x_1,y_1),(x_2,y_2)\}\rangle \cup \langle\{(x_2,y_2),(x_3,y_3)\}\rangle \cup \ldots \cup \langle\{(x_{p-1},y_{p-1}),(x_p,y_p)\}\rangle.$$

Proof. As $\det\begin{pmatrix} x_{i+1} & x_i \\ y_{i+1} & y_i \end{pmatrix} = 1$, then $\frac{x_i}{y_i} < \frac{x_{i+1}}{y_{i+1}}$ and so $\frac{x_1}{y_1} < \frac{x_2}{y_2} < \ldots < \frac{x_p}{y_p}$. By applying Lemma 4, $\{(x_1,y_1),(x_2,y_2),\ldots,(x_p,y_p)\} \subseteq S$.

If $(x,y) \in S \setminus \{(0,0)\}$, then by Lemma 3, we know that $\frac{x_1}{y_1} \leqslant \frac{x}{y} \leqslant \frac{x_p}{y_p}$. Thus, there exits $i \in \{1,\ldots,p-1\}$ such that $\frac{x_i}{y_i} \leqslant \frac{x}{y} \leqslant \frac{x_{i+1}}{y_{i+1}}$ and by Lemma 3 again, we have $(x,y) \in \mathrm{C}(\{(x_i,y_i),(x_{i+1},y_{i+1})\})$. Finally, Lemma 8 asserts that

$$(x,y) \in \langle\{(x_i,y_i),(x_{i+1},y_{i+1})\}\rangle. \quad \square$$

As an immediate consequence from previous proposition, we have the following result.

COROLLARY 10. *If* $(x_1,y_1),(x_2,y_2),\dots,(x_p,y_p)$ *is a triangulation, then the set* $\{(x_1,y_1),(x_2,y_2),\dots,(x_p,y_p)\}$ *is a system of generators of the semigroup* $\mathrm{C}(\{(x_1,y_1),(x_p,y_p)\})\cap\mathbb{N}^2$.

The Stern-Brocot tree (see [2]) allows us an ingenious method to build all the fractions $\frac{x}{y}$, where $\{x,y\}\subseteq\mathbb{N}$ and $\gcd\{x,y\}=1$. The idea is to begin with the fractions $\frac{0}{1}<\frac{1}{0}$ and then we insert $\frac{x+x'}{y+y'}$ between the two consecutive fractions $\frac{x}{y}<\frac{x'}{y'}$. So the first steps are:

Step 1*:* $\frac{0}{1}<\frac{1}{0}$.

Step 2*:* $\frac{0}{1}<\frac{1}{1}<\frac{1}{0}$.

Step 3*:* $\frac{0}{1}<\frac{1}{2}<\frac{1}{1}<\frac{2}{1}<\frac{1}{0}$.

Step 4*:* $\frac{0}{1}<\frac{1}{3}<\frac{1}{2}<\frac{2}{3}<\frac{1}{1}<\frac{3}{2}<\frac{2}{1}<\frac{3}{1}<\frac{1}{0}$.

$\vdots$

The following result is deduced from [2].

LEMMA 11. *It is verified that*

1) *All the fractions which appear in the Stern-Brocot tree are irreducibles.*

2) *Every rational nonnegative number appears exactly once in the Stern-Brocot tree.*

3) *If* $\frac{x_1}{y_1}<\frac{x_2}{y_2}<\dots<\frac{x_p}{y_p}$ *is the Step* k *of the construction of the Stern-Brocot tree, then* $x_{i+1}y_i-x_iy_{i+1}=1$ *for all* $i\in\{1,\dots,p-1\}$.

As an immediate consequence from previous lemma, we have the following result.

PROPOSITION 12. *If* $\{(a_1,b_1),(a_2,b_2)\}\subseteq\mathbb{N}^2$, $\gcd\{a_1,b_1\}=\gcd\{a_2,b_2\}=1$ *and* $\det\begin{pmatrix}a_2 & a_1\\ b_2 & b_1\end{pmatrix}>0$, *then there is a triangulation with ends* (a_1,b_1) *and* (a_2,b_2).

The construction of the Stern-Brocot tree provides us a first algorithm to compute a system of generators of an affine, plane and normal semigroup. Indeed, if we want to calculate a system of generators of the semigroup $S=\mathrm{C}(\{(a_1,b_1),(a_2,b_2)\})\cap\mathbb{N}^2$, we compute step by step the Stern-Brocot tree until, at a step given, the fractions $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ appear. Let us suppose that Step k is $\frac{x_1}{y_1}<\frac{x_2}{y_2}<\dots<\frac{x_p}{y_p}$ with $\frac{x_i}{y_i}=\frac{a_1}{b_1}$ and

$\frac{x_j}{y_j} = \frac{a_2}{b_2}$, then $(x_i,y_i),(x_{i+1},y_{i+1}),\ldots,(x_j,y_j)$ is a triangulation with ends (a_1,b_1) and (a_2,b_2). Therefore, applying Corollary 10, we have that

$$\{(x_i,y_i),(x_{i+1},y_{i+1}),\ldots,(x_j,y_j)\}$$

is a system of generators of S.

EXAMPLE 13. We are going to calculate a system of generators of the semigroup $S = \mathrm{C}(\{(1,2),(3,2)\}) \cap \mathbb{N}^2$. In order that fractions $\frac{1}{2}$ appear and $\frac{3}{2}$, it is necessary to build up to Step 4 in the Stern-Brocot tree. That is,

Step 4*:* $\ldots < \frac{1}{2} < \frac{2}{3} < \frac{1}{1} < \frac{3}{2} < \ldots$

Then $\{(1,2),(2,3),(1,1),(3,2)\}$ is a system of generators of the semigroup S.

4. The minimal system of generators

If we analyze Example 13, we observe that $\{(1,2),(2,3),(1,1),(3,2)\}$ is not a minimal system of generators of S because $(2,3) = (1,2)+(1,1)$. Note also that $(1,2),(2,3),(1,1),(3,2)$ is not a proper triangulation because $\det\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = 1$. If we refine the triangulation, then we obtain the proper triangulation $(1,2),(1,1),(3,2)$. Moreover, $\{(1,2),(1,1),(3,2)\}$ is the minimal system of generators of S. Our main aim in this section will be to show that what happens in this example is true in general. In fact, we prove that if $(x_1,y_1),(x_2,y_2),\ldots,(x_p,y_p)$ is a proper triangulation, then $\{(x_1,y_1),(x_2,y_2),\ldots,(x_p,y_p)\}$ is the minimal system of generators of the semigroup $\mathrm{C}(\{(x_1,y_1),(x_p,y_p)\}) \cap \mathbb{N}^2$.

The results of this section are inspired by and are closely parallel to some of the results of [7]. In fact, at first, we planned to carry out some of the proofs of the results that appear below, based on the results of [7]. But in this attempt, we saw that clarity was compromised and the scope of the work was excessively extended. This is why we have made this section a self-contained section.

LEMMA 14. *If* $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ *is a triangulation and* $\det\begin{pmatrix} x_3 & x_1 \\ y_3 & y_1 \end{pmatrix} = k$, *then* $(x_2,y_2) = \left(\frac{x_1+x_3}{k}, \frac{y_1+y_3}{k}\right)$.

Proof. By Corollary 10, we know that there is $\{\lambda,\mu\} \subseteq \mathbb{Q}_0^+$ such that $(x_2,y_2) = \lambda(x_1,y_1)+\mu(x_3,y_3)$. That is, $x_2 = \lambda x_1 + \mu x_3$ and $y_2 = \lambda y_1 + \mu y_3$.

As $\det\begin{pmatrix} x_3 & x_2 \\ y_3 & y_2 \end{pmatrix} = 1$, using Cramer's Formula, we have that $\lambda = \dfrac{\det\begin{pmatrix} x_2 & x_3 \\ y_2 & y_3 \end{pmatrix}}{\det\begin{pmatrix} x_1 & x_3 \\ y_1 & y_3 \end{pmatrix}} = \dfrac{1}{k}$. In a similar way, as $\det\begin{pmatrix} x_2 & x_1 \\ y_2 & y_1 \end{pmatrix} = 1$, we have that $\mu = \dfrac{\det\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}}{\det\begin{pmatrix} x_1 & x_3 \\ y_1 & y_3 \end{pmatrix}} = \dfrac{1}{k}$. □

The following result is key for the development of this section.

LEMMA 15. *If* $(x_1,y_1),(x_2,y_2),\ldots,(x_p,y_p)$ *is a proper triangulation, then* $\max\{x_1,x_2,\ldots,x_p\} \in \{x_1,x_p\}$.

Proof. We proceed by induction on p. For $p=2$, the statement is trivially true. We assume as induction hypothesis that $\max\{x_2,\ldots,x_p\} \in \{x_2,x_p\}$. We next show that $\max\{x_1,x_2,\ldots,x_p\} \in \{x_1,x_p\}$. If $\max\{x_2,\ldots,x_p\} = x_p$, then $\max\{x_1,x_2,\ldots,x_p\} \in \{x_1,x_p\}$. Let us assume then $\max\{x_2,\ldots,x_p\} = x_2$. As $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ is a proper triangulation, then $\det\begin{pmatrix} x_3 & x_1 \\ y_3 & y_1 \end{pmatrix} = k \geqslant 2$ and by applying Lemma 14, $(x_2,y_2) = \left(\dfrac{x_1+x_3}{k}, \dfrac{y_1+y_3}{k}\right)$. As $k \geqslant 2$, then $x_2 \leqslant \dfrac{x_1+x_3}{2} \leqslant \dfrac{2\max\{x_1,x_3\}}{2} = \max\{x_1,x_3\}$. We distinguish two cases depending on the value of $\max\{x_1,x_3\}$.

- If $\max\{x_1,x_3\} = x_1$, then $x_2 \leqslant x_1$ and so $\max\{x_1,x_2,\ldots,x_p\} = x_1 \in \{x_1,x_p\}$.
- If $\max\{x_1,x_3\} = x_3$, then $x_2 \leqslant x_3$. By using that $\max\{x_2,\ldots,x_p\} = x_2$, we obtain $x_2 = x_3$. As $\det\begin{pmatrix} x_3 & x_2 \\ y_3 & y_2 \end{pmatrix} = 1$, then $x_2 = x_3 = 1$. If $x_1 \geqslant 1$, then $\max\{x_1,x_2,\ldots,x_p\} \in \{x_1,x_p\}$. If $x_1 = 0$, then we have that $1 = \det\begin{pmatrix} x_2 & x_1 \\ y_2 & y_1 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 \\ y_2 & y_1 \end{pmatrix} = y_1$. Thus, $(x_1,y_1) = (0,1)$. Since $x_3 = 1$, then $(x_3,y_3) = (1,y_3)$. Hence, $\det\begin{pmatrix} x_3 & x_1 \\ y_3 & y_1 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 \\ y_3 & 1 \end{pmatrix} = 1$, which is in contradiction with the fact that $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ is a proper triangulation. □

As an immediate consequence of previous lemma, we obtain the following result.

PROPOSITION 16. *If* $(x_1,y_1),(x_2,y_2),\ldots,(x_p,y_p)$ *is a proper triangulation, then* $x_1,x_2,\ldots,x_p$ *is a convex sequence, that is, one the following assertions is verified:*

1) $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_p$,

2) $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_p$, *3) There exists* $h \in \{2,\dots,p-1\}$ *such that* $x_1 \geqslant x_2 \geqslant \dots \geqslant x_h \leqslant x_{h+1} \leqslant \dots \leqslant x_p$.

LEMMA 17. *If* $(x_1,y_1),(x_2,y_2),\dots,(x_p,y_p)$ *is a triangulation, then* $(x_1,y_1) \notin \langle\{(x_2,y_2),\dots,(x_p,y_p)\}\rangle$ *and* $(x_p,y_p) \notin \langle\{(x_1,y_1),\dots,(x_{p-1},y_{p-1})\}\rangle$.

Proof. It is clear that $\frac{x_1}{y_1} < \frac{x_2}{y_2} < \dots < \frac{x_p}{y_p}$. To conclude the proof it is enough to apply Lemma 3 and Corollary 10. □

The following result has an immediate proof.

LEMMA 18. *Let* A *be a subset nonempty of* $\mathbb{N}^k$. *Then* A *is the minimal system of generators of* $\langle A\rangle$ *if and only if* $a \notin \langle A \setminus \{a\}\rangle$ *for every* $a \in A$.

At this point, after these results, it is possible to validate the result announced at the beginning of this section.

THEOREM 19. *If* $(x_1,y_1),(x_2,y_2),\dots,(x_p,y_p)$ *is a proper triangulation, then* $\{(x_1,y_1),(x_2,y_2),\dots,(x_p,y_p)\}$ *is the minimal system of generators of the semigroup* $S = \mathrm{C}(\{(a_1,b_1),(a_p,b_p)\}) \cap \mathbb{N}^2$.

Proof. Let $A = \{(x_1,y_1),(x_2,y_2)\dots,(x_p,y_p)\}$. By Corollary10 and Lemma18, to prove the theorem it is enough to show that if $a \in A$, then $a \notin \langle A \setminus \{a\}\rangle$. We will prove this fact by induction on p. For $p = 2$, the result clearly true. By Lemma 17, we know that $(x_1,y_1) \notin \langle\{(x_2,y_2),\dots,(x_p,y_p)\}\rangle$ and $(x_p,y_p) \notin \langle\{(x_1,y_1),\dots,(x_{p-1},y_{p-1})\}\rangle$. Thus, to conclude the proof, it suffices to show that if $i \in \{2,\dots,p-1\}$ then $(x_i,y_i) \notin \langle A \setminus \{(x_i,y_i)\}\rangle$. From Lemma 15, we know that $\max\{x_1,x_2,\dots,x_p\} \in \{x_1,x_p\}$. We distinguish two cases.

1) If $\max\{x_1,x_2,\dots,x_p\} = x_1$, then $x_1 \neq 0$ because otherwise, $x_1 = x_2 = 0$ and $\det\begin{pmatrix} x_2 & x_1 \\ y_2 & y_1 \end{pmatrix} = \det\begin{pmatrix} 0 & 0 \\ y_2 & y_1 \end{pmatrix} = 0$, which contradicts the fact that $(x_1,y_1),(x_2,y_2)$ is a triangulation. As $x_1 \neq 0$ and $\frac{x_1}{y_1} < \frac{x_2}{y_2} < \dots < \frac{x_p}{y_p}$, then $x_j \neq 0$ for all $j \in \{1,\dots,p\}$. If $(x_i,y_i) \in \langle A \setminus \{(x_i,y_i)\}\rangle$, then
$$(x_i,y_i) = \lambda_1(x_1,y_1) + \dots + \lambda_{i-1}(x_{i-1},y_{i-1}) + \lambda_{i+1}(x_{i+1},y_{i+1}) + \dots + \lambda_p(x_p,y_p)$$
for some $\{\lambda_1,\dots,\lambda_{i-1},\lambda_{i+1},\dots,\lambda_p\} \subseteq \mathbb{N}$. By induction hypothesis, $\{(x_2,y_2),\dots,(x_p,y_p)\}$ is the minimal system of generators of the semigroup $\langle\{(x_2,y_2),\dots,(x_p,y_p)\}\rangle$ and so $\lambda_1 \neq 0$. As $x_1 = \max\{x_1,x_2,\dots,x_p\}$ and $x_j \neq 0$ for all $j \in \{1,\dots,p\}$, we deduce that $\lambda_1 = 1$ and $\lambda_j = 0$ for all $j \in \{2,\dots,i-1,i+1,\dots,p\}$. Hence, $(x_i,y_i) = (x_1,y_1)$, which is absurd because $\frac{x_1}{y_1} < \frac{x_i}{y_i}$.

2) Suppose now that $\max\{x_1,x_2,\dots,x_p\} = x_p$. If $(x_i,y_i) \in \langle A \setminus \{(x_i,y_i)\}\rangle$, then reasoning in a similar way to that yielded in case 1), we assert that $(x_1,y_1) =$

$(0,1)$ and $(x_i,y_i) = \lambda(0,1)+(x_p,y_p)$ for some $\lambda \in \mathbb{N}\setminus\{0\}$. Therefore, $x_i = x_p = \max\{x_1,x_2,\dots,x_p\}$. By applying Proposition 16, we obtain $x_{p-1} = x_p$. As $\det\begin{pmatrix} x_p & x_{p-1} \\ y_p & y_{p-1}\end{pmatrix} = 1$, then $x_p = x_{p-1} = 1$. Thus, $\det\begin{pmatrix} x_p & x_1 \\ y_p & y_1\end{pmatrix} = \det\begin{pmatrix} 1 & 0 \\ y_p & 1\end{pmatrix} = 1$, which contradicts the fact that $(x_1,y_1),\dots,(x_p,y_p)$ is a proper triangulation. □

5. The algorithm

At the end of Section 3, we have presented an algorithmic method, based in the construction of the Stern-Brocot tree, to compute a triangulation conecting two elements of $\mathbb{N}^2$. The aim of this section will be to show an alternative algorithm to solve this problem. The algorithm which we present in this section is based on the results of [1] and it is more efficient than the one mentioned above. In fact, it has a similar complexity to the Euclidean Algorithm to compute the great common divisor of two integer numbers.

According to the terminology introduced in [7], a Bézout sequence is a sequence of rational numbers $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \dots < \frac{a_p}{b_p}$ such that $\{a_1,b_1,\dots,a_p,b_p\} \subseteq \mathbb{N}\setminus\{0\}$ and $a_{i+1}b_i - a_ib_{i+1} = 1$ for all $i \in \{1,\dots,p-1\}$. The number p is called the *length* of the sequence and $\frac{a_1}{b_1}$ and $\frac{a_p}{b_p}$ are the *ends* of the sequence.

A Bézout sequence $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \dots < \frac{a_p}{b_p}$ is *proper* if $a_{i+h}b_i - a_ib_{i+h} \neq 1$ for all $i \in \{1,\dots,p-2\}$ and for all $h \in \mathbb{N}\setminus\{0,1\}$ such that $i+h \in \{1,\dots,p\}$.

The following result has an immediate proof.

PROPOSITION 20. *Let* $\{a_1,b_1,\dots,a_p,b_p\} \subseteq \mathbb{N}\setminus\{0\}$. *Then*

1) $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \dots < \frac{a_p}{b_p}$ *is a Bézout sequence if and only if* $(a_1,b_1),(a_2,b_2),\dots,(a_p,b_p)$ *is a triangulation.*

2) $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \dots < \frac{a_p}{b_p}$ *is a proper Bézout sequence if and only if* $(a_1,b_1),(a_2,b_2),\dots,(a_p,b_p)$ *is a proper triangulation.*

The following result is deduced from [1, Theorem 2.7].

PROPOSITION 21. *If* $\{a,b,c,d\} \subseteq \mathbb{N}\setminus\{0\}$, $\frac{a}{b} < \frac{c}{d}$ *and* $\gcd\{a,b\} = \gcd\{c,d\} = 1$, *then there exits a unique proper Bézout sequence with ends* $\frac{a}{b}$ *and* $\frac{c}{d}$.

As an immediate consequence from Propositions 20 and 21, we have the following result.

COROLLARY 22. *If* $\{a,b,c,d\} \subseteq \mathbb{N}\setminus\{0\}$, $\frac{a}{b} < \frac{c}{d}$ *and* $\gcd\{a,b\} = \gcd\{c,d\} = 1$, *then there is a unique proper triangulation with ends* (a,b) *and* (c,d). *Moreover, if*

$\frac{a_1}{b_1} < \frac{a_2}{b_2} < \ldots < \frac{a_p}{b_p}$ *is the unique proper Bézout sequence with ends* $\frac{a}{b}$ *and* $\frac{c}{d}$, *then* $(a_1,b_1),(a_2,b_2),\ldots,(a_p,b_p)$ *is the unique proper triangulation with ends* (a,b) *and* (c,d).

Algorithm 3.5 from [1] allows us to calculate, with a similar complexity to the Euclidean Algorithm, the unique proper Bézout sequence with ends $\frac{a}{b}$ and $\frac{c}{d}$, where $\{a,b,c,d\} \subseteq \mathbb{N}\setminus\{0\}$, $\frac{a}{b} < \frac{c}{d}$ and $\gcd\{a,b\} = \gcd\{c,d\} = 1$. So, by applying Corollary 22, we have an algorithm to calculate a proper triangulation with ends (a,b) and (c,d).

Now, we will focus on studying the cases where $(a,b) = (0,1)$ or $(c,d) = (1,0)$.

PROPOSITION 23. *The following hold:*

1) $(0,1),(1,0)$ *is a proper triangulation with ends* $(0,1)$ *and* $(1,0)$.

2) If $\{c,d\} \subseteq \mathbb{N}\setminus\{0\}$, $\gcd\{c,d\} = 1$ *and* $\left\lceil \frac{d}{c} \right\rceil = \frac{d}{c}$, *then* $(0,1),(1,d)$ *is a proper triangulation with ends* $(0,1)$ *and* (c,d).

3) If $\{c,d\} \subseteq \mathbb{N}\setminus\{0\}$, $\gcd\{c,d\} = 1$, $\left\lceil \frac{d}{c} \right\rceil \neq \frac{d}{c}$ *and* $(x_1,y_1),\ldots,(x_p,y_p)$ *is the proper triangulation with ends* $\left(1, \left\lceil \frac{d}{c} \right\rceil\right)$ *and* (c,d), *then* $(0,1),(x_1,y_1),\ldots,(x_p,y_p)$ *is the proper triangulation with ends* $(0,1)$ *and* (c,d).

4) If $\{a,b\} \subseteq \mathbb{N}\setminus\{0\}$, $\gcd\{a,b\} = 1$, $\left\lceil \frac{a}{b} \right\rceil = \frac{a}{b}$, *then* $(a,1),(1,0)$ *is a proper triangulation with ends* (a,b) *and* $(1,0)$.

5) If $\{a,b\} \subseteq \mathbb{N}\setminus\{0\}$, $\gcd\{a,b\} = 1$, $\left\lceil \frac{a}{b} \right\rceil \neq \frac{a}{b}$ *and* $(x_1,y_1),(x_2,y_2),\ldots,(x_p,y_p)$ *is the proper triangulation with ends* (a,b) *and* $\left(\left\lceil \frac{a}{b} \right\rceil, 1\right)$, *then* $(x_1,y_1),\ldots,(x_p,y_p)$, $(1,0)$ *is the proper triangulation with ends* (a,b) *and* $(1,0)$.

Proof.

1) Trivial.

2) If $\left\lceil \frac{d}{c} \right\rceil = \frac{d}{c}$, then $\frac{d}{c} \in \mathbb{Z}$ and by applying that $\gcd\{c,d\} = 1$, we deduce that $c = 1$. Thus, $(0,1),(1,d)$ is a proper triangulation with ends $(0,1)$ and (c,d).

3) If $\left\lceil \frac{d}{c} \right\rceil \neq \frac{d}{c}$, then $\frac{d}{c} < \left\lceil \frac{d}{c} \right\rceil$ and so $\frac{1}{\lceil \frac{d}{c} \rceil} < \frac{c}{d}$. If $(x_1,y_1),(x_2,y_2),\ldots,(x_p,y_p)$ is the proper triangulation with ends $\left(1, \left\lceil \frac{d}{c} \right\rceil\right)$ and (c,d), then it is clear that $(0,1),(x_1,y_1),\ldots,(x_p,y_p)$ is a triangulation with ends $(0,1)$ and (c,d). To prove that this triangulation is a proper triangulation, we will see that if $i \in \{2,\dots,p\}$ then $(0,1),(x_i,y_i)$ is not a triangulation. Otherwise, $x_i = 1$ and $\frac{0}{1} < \frac{1}{y_i} \leqslant \frac{c}{d}$. Therefore, $y_i \geqslant \frac{d}{c}$ and thus $y_i \geqslant \left\lceil \frac{d}{c} \right\rceil$. Consequently, $\frac{x_i}{y_i} = \frac{1}{y_i} \leqslant \frac{1}{\lceil \frac{d}{c} \rceil} = \frac{x_1}{y_1}$, that does not make sense because $\frac{x_1}{y_1} < \frac{x_2}{y_2} < \dots < \frac{x_p}{y_p}$.

4) If $\left\lceil \frac{a}{b} \right\rceil = \frac{a}{b}$, then $\frac{a}{b} \in \mathbb{Z}$ and by applying that $\gcd\{a,b\} = 1$, we deduce that $b = 1$. Thus, $(a,1),(1,0)$ is a proper triangulation with ends (a,b) and $(1,0)$.

5) If $\left\lceil \frac{a}{b} \right\rceil \neq \frac{a}{b}$, then $\frac{a}{b} < \frac{\lceil \frac{a}{b} \rceil}{1}$. If $(x_1,y_1),\dots,(x_p,y_p)$ is the proper triangulation with ends (a,b) and $(\lceil \frac{a}{b} \rceil, 1)$, then $(x_1,y_1),\dots,(x_p,y_p),(1,0)$ is a triangulation with ends (a,b) and $(1,0)$. To prove that this triangulation is a proper triangulation, we should see that if $i \in \{1,\dots,p-1\}$ then $(x_i,y_i),(1,0)$ is not a triangulation. Otherwise, $y_i = 1$. Then $\frac{a}{b} \leqslant \frac{x_i}{y_i} = \frac{x_i}{1} < \frac{\lceil \frac{a}{b} \rceil}{1} = \frac{x_p}{y_p}$. Therefore, $x_i \in \mathbb{Z}$ and $\frac{a}{b} \leqslant x_i < \left\lceil \frac{a}{b} \right\rceil$, that does not make sense. $\square$

If $\{a,b,c,d\} \subseteq \mathbb{N} \setminus \{0\}$, $\frac{a}{b} < \frac{c}{d}$ and $\gcd\{a,b\} = \gcd\{c,d\} = 1$, then we denote by $\mathrm{PBS}\left(\frac{a}{b}, \frac{c}{d}\right)$ the output of Algorithm 3.5 from [1] with input $\frac{a}{b}$ and $\frac{c}{d}$. Therefore, $\mathrm{PBS}\left(\frac{a}{b}, \frac{c}{d}\right)$ is the unique proper Bézout sequence with ends $\frac{a}{b}$ and $\frac{c}{d}$.

ALGORITHM 24.

INPUT: $\{(a,b),(c,d)\} \subseteq \mathbb{N}^2 \setminus \{(0,0)\}$ such that $\frac{a}{b} < \frac{c}{d}$ and $\gcd\{a,b\} = \gcd\{c,d\} = 1$.
OUTPUT: A proper triangulation with ends (a,b) and (c,d).

1. If $(a,b) = (0,1)$ and $c = 1$, then return $(0,1),(1,d)$.
2. If $(c,d) = (1,0)$ and $b = 1$, then return $(a,1),(1,0)$.
3. If $\{a,b,c,d\} \subseteq \mathbb{N} \setminus \{0\}$ and $\mathrm{PBS}\left(\frac{a}{b}, \frac{c}{d}\right)$ is $\frac{x_1}{y_1} < \frac{x_2}{y_2} < \dots < \frac{x_p}{y_p}$, then return $(x_1,y_1),(x_2,y_2),\dots,(x_p,y_p)$.
4. If $(a,b) = (0,1)$, $c \neq 1$ and $\mathrm{PBS}\left(\frac{1}{\lceil \frac{d}{c} \rceil}, \frac{c}{d}\right)$ is $\frac{x_1}{y_1} < \frac{x_2}{y_2} < \dots < \frac{x_p}{y_p}$, then return $(0,1),(x_1,y_1),(x_2,y_2),\dots,(x_p,y_p)$.
5. If $(c,d) = (1,0)$, $b \neq 1$ and $\mathrm{PBS}\left(\frac{a}{b}, \frac{\lceil \frac{a}{b} \rceil}{1}\right)$ is $\frac{x_1}{y_1} < \frac{x_2}{y_2} < \dots < \frac{x_p}{y_p}$, then return $(x_1,y_1),(x_2,y_2),\dots,(x_p,y_p),(1,0)$.

Now we are going to illustrate how the previous algorithm works with an example.

EXAMPLE 25. A proper triangulation with left and right ends (a,b) and (c,d) respectively, are:

1. If $(a,b)=(0,1)$ and $(c,d)=(1,5)$, then the proper triangulation is $(0,1),(1,5)$.
2. If $(a,b)=(7,1)$ and $(c,d)=(1,0)$, then the proper triangulation is $(7,1),(1,0)$.
3. If $(a,b)=(4,11)$ and $(c,d)=(12,5)$, as

$$\mathrm{PBS}\left(\frac{4}{11},\frac{12}{5}\right)=\frac{4}{11}<\frac{3}{8}<\frac{2}{5}<\frac{1}{2}<\frac{1}{1}<\frac{2}{1}<\frac{7}{3}<\frac{12}{5},$$

then the proper triangulation is

$$(4,11),(3,8),(2,5),(1,2),(1,1),(2,1),(7,3),(12,5).$$

4. If $(a,b)=(0,1)$ and $(c,d)=(15,8)$, then

$$\mathrm{PBS}\left(\frac{1}{\lceil\frac{8}{15}\rceil},\frac{15}{8}\right)=\frac{1}{1}<\frac{3}{2}<\frac{5}{3}<\frac{7}{4}<\frac{9}{5}<\frac{11}{6}<\frac{13}{7}<\frac{15}{8}.$$

Thus the proper triangulation is

$$(0,1),(1,1),(3,2),(5,3),(7,4),(9,5),(11,6),(13,7),(15,8).$$

5. If $(a,b)=(127,46)$ and $(c,d)=(1,0)$, then

$$\mathrm{PBS}\left(\frac{127}{46},\frac{\lceil\frac{127}{46}\rceil}{1}\right)=\frac{127}{46}<\frac{58}{21}<\frac{47}{17}<\frac{36}{13}<\frac{25}{9}<\frac{14}{5}<\frac{3}{1}.$$

Thus the proper triangulation is

$$(127,46),(58,21),(47,17),(36,13),(25,9),(14,5),(3,1),(1,0).$$

We end this work giving an upper bound for the embedding dimension of the semigroup $\mathrm{C}(\{(a,b),(c,d)\})\cap\mathbb{N}^2$.

The following result is Theorem 7 from [7].

PROPOSITION 26. *If* $\{a,b,c,d\}\subseteq\mathbb{N}\setminus\{0\}$, $\frac{a}{b}<\frac{c}{d}$ *and* $\gcd\{a,b\}=\gcd\{c,d\}=1$, *then there exists a Bézout sequence with length less than or equal to* $cd-ad+1$ *and ends* $\frac{a}{b}$ *and* $\frac{c}{d}$.

COROLLARY 27. *If* $\{(a,b),(c,d)\}\subseteq\mathbb{N}^2\setminus\{(0,0)\}$, $\frac{a}{b}<\frac{c}{d}$, $\gcd\{a,b\}=\gcd\{c,d\}=1$ *and* $S=\mathrm{C}(\{(a,b),(c,d)\})\cap\mathbb{N}^2$, *then* $\mathrm{e}(S)\leqslant bc-ad+1$.

Proof. We consider the following cases:

- If $(a,b)=(0,1)$ and $c=1$, then the result is trivially true.
- If $(c,d)=(1,0)$ and $b=1$, then the result is trivially true.
- If $\{a,b,c,d\}\subseteq \mathbb{N}\setminus\{0\}$, then the result is a consequence of Proposition 26.
- If $(a,b)=(0,1)$ and $c\neq 1$, then $\mathrm{e}(S)$ is less or equal than $c\left\lceil\frac{d}{c}\right\rceil-d+1+1=c\left\lfloor\frac{d}{c}\right\rfloor+c-d+2=d-(d \mod c)+c-d+2=c-(d \mod c)+2\leqslant c+1.$
- If $(c,d)=(1,0)$ and $b\neq 1$, then $\mathrm{e}(S)$ is less or equal than $b\left\lceil\frac{a}{b}\right\rceil-a+1+1=b\left\lfloor\frac{a}{b}\right\rfloor+b-a+2=a-(a \mod b)+b-a+2\leqslant b+1.$ □

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