Journal of Mathematical Inequalities
Volume 18, Number 4 (2024), 1247–1262
doi:10.7153/jmi-2024-18-71

COMPLETE CONVERGENCE FOR WEIGHTED SUMS OF m–WIDELY ACCEPTABLE RANDOM VARIABLES UNDER SUB–LINEAR EXPECTATIONS

SHUTING PAN, KE HU, XUEPING HU* AND KEMING YU

(*Communicated by X. Wang*)

Abstract. In this paper, under the assumption of the existence of Choquet integrals, the complete convergence properties for weighted sums of m-widely acceptable random variables in sub-linear expectation space are investigated. The results obtained in the paper generalize the corresponding ones for some dependent sequences.

1. Introduction

The limit theory has wide applications in the field of risk finance. However, classical limit theory has strict requirements and limitations. Peng [10] transformed the notions of probability and expectation in traditional space into capacity and sub-linear expectations, proposing the framework of sub-linear expectation space, which has attracted the attention of many statisticians. For example, Peng [8,9] obtained the central limit theorem in the framework of sub-linear expectation space. Zhang [17–19] extended moment inequalities and Rosenthal's inequality for negatively dependent (ND, for short) sequences from probability space to sub-linear expectation space. Wu and Jiang [13] studied independent sequences in sub-linear expectation space and proved strong laws of large numbers and a version of Chover type logarithmic law. Based on this work, Wu and Lu [14] derived a new form of the Chover type logarithmic law under sub-linear expectations.

Since Hsu and Robbins [2] introduced the concept of complete convergence, many scholars have discussed the complete convergence for sequences of random variables. Up to now, related research in classical probability space has become quite extensive, many fruitful and meaningful results have been obtained. See, for example, Yi et al. [16] proved the convergence rate for weighted sums of ψ-mixing random variables and gave its applications. Huang and Wu [5] studied the complete convergence and complete moment convergence for weighted sums of m-extented negatively dependent (m-END, for short) random variables and so on. Theoretically, it is feasible to generalize the study of classical probability space to sub-linear expectation space. However,

Mathematics subject classification (2020): 60F15.
Keywords and phrases: Sub-linear expectation, m-widely acceptable random variables, weighted sums, complete convergence.
* Corresponding author.

Paper JMI-18-71

due to the nonlinearity of sub-linear expectation space, this extension poses certain challenges. Nevertheless, researchers have conducted studies on independent random variable sequences [15, 20], END sequences [1, 11], and widely acceptable (WA, for short) sequences [3] in sub-linear expectation space.

Based on existing theoretical foundations, the complete convergence of weighted sums for m-WA sequences is investigated under sub-linear expectations, which extends the results of reference [3].

This paper is organized as follows: some preliminaries and lemmas are provided in Sect 2. The main results and their proofs are stated in Sect 3. Throughout this article, $\{X_n, n \geqslant 1\}$ is assumed to be a sequence of random variables in $\left(\Omega, \mathscr{H}, \hat{\mathbb{E}}\right)$, $S_n = \sum_{i=1}^{n} X_i$. Let c be a positive constant which may be different in various places. $a_x \ll b_x$ denotes the existence of a certain c such that for sufficiently large x, $a_x \leqslant c b_x$ holds true.

2. Preliminaries and Lemmas

We use the framework and notions of Peng [8–10]: Let $(\Omega, \mathscr{F})$ be a given measurable space, and $\mathscr{H}$ be a linear space of real functions defined on $(\Omega, \mathscr{F})$, such that if $X_1, X_2, \cdots, X_n \in \mathscr{H}$, then $\varphi(X_1, X_2, \cdots, X_n) \in \mathscr{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, where $C_{l,Lip}(\mathbb{R}^n)$ denotes the linear space of local Lipschitz functions φ satisfying

$$|\varphi(x) - \varphi(y)| \leqslant c\left(1 + |x|^m + |y|^m\right)|x - y|, \forall x, y \in \mathbb{R}^n,$$

for some $c > 0$, $m \in \mathbb{N}$ depending on φ. $\mathscr{H}$ is considered as a space of random variables.

DEFINITION 2.1. (see [17]) A sub-linear expectation $\hat{\mathbb{E}}$ on $\mathscr{H}$ is a function $\hat{\mathbb{E}} : \mathscr{H} \to \overline{\mathbb{R}}$ satisfying the following properties: for all $X, Y \in \mathscr{H}$, we have

(a) Monotonicity: if $X \geqslant Y$, then $\hat{\mathbb{E}}X \geqslant \hat{\mathbb{E}}Y$,

(b) Constant preserving: $\hat{\mathbb{E}}c = c$,

(c) Subadditivity: $\hat{\mathbb{E}}(X+Y) \leqslant \hat{\mathbb{E}}X + \hat{\mathbb{E}}Y$,

(d) Positive homogeneity: $\hat{\mathbb{E}}(\lambda X) = \lambda \hat{\mathbb{E}}(X)$, $\forall \lambda \geqslant 0$,

where $\overline{\mathbb{R}} = [-\infty, +\infty]$. The triple $\left(\Omega, \mathscr{H}, \hat{\mathbb{E}}\right)$ is called a sub-linear expectation space. The conjugate expectation $\hat{\varepsilon}$ of $\hat{\mathbb{E}}$ is defined by $\hat{\varepsilon}X := -\hat{\mathbb{E}}(-X), \forall X \in \mathscr{H}$. By Definition 2.1, for all $X, Y \in \mathscr{H}$, it follows that

$$\hat{\varepsilon}X \leqslant \hat{\mathbb{E}}X, \quad \hat{\mathbb{E}}(X+c) = \hat{\mathbb{E}}X + c,$$

$$\left|\hat{\mathbb{E}}(X-Y)\right| \leqslant \hat{\mathbb{E}}|X-Y|, \quad \hat{\mathbb{E}}(X-Y) \geqslant \hat{\mathbb{E}}X - \hat{\mathbb{E}}Y.$$

If $\hat{\mathbb{E}}Y = \hat{\varepsilon}Y$, for any $a \in \mathbb{R}$, we have $\hat{\mathbb{E}}(X + aY) = \hat{\mathbb{E}}X + a\hat{\mathbb{E}}Y$.

DEFINITION 2.2. (see [17]) Let $\mathscr{G} \subset \mathscr{F}$. A function $V : \mathscr{G} \to [0,1]$ is called a capacity if $V(\phi) = 0$, $V(\Omega) = 1$, $V(A) \leqslant V(B)$, for any $A \subset B$, $A, B \in \mathscr{G}$.

It is called subadditive, if $V(A \cup B) \leqslant V(A) + V(B)$ for all $A, B \in \mathscr{G}$.

$\mathbb{V}$ is defined as follows: $\mathbb{V}(A) := \inf\left\{\hat{\mathbb{E}}\xi; I(A) \leqslant \xi, \xi \in \mathscr{H}\right\}$, and we have

$$\mathscr{V}(A) := 1 - \mathbb{V}(A^c), \quad \mathscr{V}(A) \leqslant \mathbb{V}(A), \forall A \in \mathscr{F},$$

where A^c is the complement set of A. According to definition of $\mathbb{V}$ and $\mathscr{V}$, it is obvious that $\mathbb{V}$ is subadditive, and if $I(A) \in \mathscr{H}$, then $\mathbb{V}(A) = \hat{\mathbb{E}}(I(A))$, $\mathscr{V}(A) = \hat{\varepsilon}(I(A))$. If $f \leqslant I(A) \leqslant g$, and $f, g \in \mathscr{H}$, then $\hat{\mathbb{E}} f \leqslant \mathbb{V}(A) \leqslant \hat{\mathbb{E}} g$, $\hat{\varepsilon} f \leqslant \mathscr{V}(A) \leqslant \hat{\varepsilon} g$.

Noting that $I(|X| \geqslant x) \leqslant \frac{|X|^p}{x^p} \in \mathscr{H}$, which implies Markov inequality [4]: $\forall X \in \mathscr{H}$,

$$\mathbb{V}(|X| \geqslant x) \leqslant \frac{\hat{\mathbb{E}}|X|^p}{x^p}, \forall x > 0, \ p > 0.$$

DEFINITION 2.3. (see [17]) The Choquet integral is defined as follows:

$$C_{\mathbb{V}}(X) = \int_0^\infty \mathbb{V}(X \geqslant t) dt + \int_{-\infty}^0 [\mathbb{V}(X \geqslant t) - 1] dt,$$

where is a similar definition for $\mathscr{V}$.

DEFINITION 2.4. (see [17]) Let X_1, X_2 be two n-dimensional random vectors defined, respectively, in sub-linear expectation spaces $\left(\Omega_1, \mathscr{H}_1, \hat{\mathbb{E}}_1\right)$ and $\left(\Omega_2, \mathscr{H}_2, \hat{\mathbb{E}}_2\right)$, which are called identical distribution if

$$\hat{\mathbb{E}}_1(\varphi(X_1)) = \hat{\mathbb{E}}_2(\varphi(X_2)), \forall \varphi \in C_{l,Lip}(\mathbb{R}^n),$$

whenever the sub-linear expectation is finite. A sequence $\{X_n; n \geqslant 1\}$ of random variables is said to be identically distributed if for each $i \geqslant 1, X_i$ and X_1 are identical distribution.

DEFINITION 2.5. (see [12]) A sequence $\{X_n; n \geqslant 1\}$ of random variables in a sub-linear expectation space $\left(\Omega, \mathscr{H}, \hat{\mathbb{E}}\right)$ is called to be upper (resp. lower) WA if there exists a positive sequence $\{g(n), n \geqslant 1\}$ of dominating coefficients such that for each $n \geqslant 1$,

$$\hat{\mathbb{E}} \exp\left(\sum_{i=1}^n a_{ni} f_i(X_i)\right) \leqslant g(n) \prod_{i=1}^n \hat{\mathbb{E}} \exp(a_{ni} f_i(X_i)), \tag{2.1}$$

where $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ is an array of nonnegative constants and $f_i(\cdot) \in C_{b,Lip}(\mathbb{R})$, $i = 1, 2, \cdots, n$, are all non-decreasing (resp. all non-increasing) real valued truncation functions. We say that $\{X_n, n \geqslant 1\}$ is WA if it is both upper WA and lower WA.

Especially, it follows that if for $\forall t \geqslant 0$, $n \in \mathbb{N}$,

$$\hat{\mathbb{E}} \exp\left(\sum_{i=1}^n t X_i\right) \leqslant g(n) \prod_{i=1}^n \hat{\mathbb{E}} \exp(t X_i), \tag{2.2}$$

a sequence $\{X_n, n \geqslant 1\}$ of random variables is also WA.

DEFINITION 2.6. (see [12]) A sequence $\{X_n, n \geqslant 1\}$ is called to be m-WA, if for some integer $m \geqslant 1$, for any $n \geqslant 2$ and $i_1, \cdots, i_n$ satisfying $\left|i_k - i_j\right| \geqslant m$ $(1 \leqslant k \neq j \leqslant n)$, $\{X_{i_1}, \cdots, X_{i_n}\}$ is a sequence of WA random variables.

According to Definition 2.6, it can be found that the sequence of m-WA is more general than the sequence of WA, and it serves as an extension of the sequence of WA. The sequence of WA random variables is a special case of the m-WA when m=1. In fact, if $\{X_n, n \geqslant 1\}$ is a sequence of m-WA, then it is also the sequence of m'-WA for any $m' > m$.

LEMMA 2.1. (see [6]) *Let $\{X_n, n \geqslant 1\}$ be a sequence of WA random variables in $\left(\Omega, \mathscr{H}, \hat{\mathbb{E}}\right)$ with $\hat{\mathbb{E}}(X_i) \leqslant 0$, $1 \leqslant i \leqslant n$, for all $x > 0$, $d > 0$, we have*

$$\mathbb{V}(S_n \geqslant x) \leqslant \mathbb{V}\left(\max_{1 \leqslant i \leqslant n} X_i > d\right) + g(n) \exp\left(\frac{x}{d} - \frac{x}{d} \ln\left(1 + \frac{xd}{\sum_{i=1}^{n} \hat{\mathbb{E}}|X_i|^2}\right)\right).$$

LEMMA 2.2. *Let $\{X_n, n \geqslant 1\}$ be a sequence of m-WA random variables in $\left(\Omega, \mathscr{H}, \hat{\mathbb{E}}\right)$, with $\hat{\mathbb{E}}(X_i) \leqslant 0$, $1 \leqslant i \leqslant n$, then for all $x > 0$, $d > 0$, it follows that*

$$\mathbb{V}(S_n \geqslant x) \leqslant m\mathbb{V}\left(\max_{1 \leqslant i \leqslant n} X_i > d\right) + mg(n) \exp\left(\frac{x}{md} - \frac{x}{md} \ln\left(1 + \frac{xd}{m\sum_{i=1}^{n} \hat{\mathbb{E}}|X_i|^2}\right)\right).$$

Proof. For any $1 \leqslant k \leqslant n$, denote $\tau = \left[\frac{n}{m}\right]$. Let

$$Y_j = \begin{cases} X_j, & 1 \leqslant j \leqslant n, \\ 0, & j > n. \end{cases}$$

$$T_{nj} = \sum_{i=0}^{\tau} Y_{mi+j}, \quad 1 \leqslant j \leqslant m.$$

Obviously, $\left\{Y_{mi+j}, i = 0, 1, \cdots, \tau\right\}$ is a sequence of WA for $1 \leqslant j \leqslant m$, $m \leqslant n$. Thus,

$$\{S_n \geqslant x\} \subset \left(\left\{T_{n1} \geqslant \frac{x}{m}\right\} \cup \left\{T_{n2} \geqslant \frac{x}{m}\right\} \cup \cdots \cup \left\{T_{nj} \geqslant \frac{x}{m}\right\}\right).$$

By Lemma 2.1, it is easily checked that

$$\begin{aligned}
\mathbb{V}(S_n \geqslant x) \leqslant & \mathbb{V}\left(\bigcup_{j=1}^{m}\left(T_{nj} \geqslant \frac{x}{m}\right)\right) \leqslant \sum_{j=1}^{m} \mathbb{V}\left(T_{nj} \geqslant \frac{x}{m}\right) \\
& \leqslant \sum_{j=1}^{m} \mathbb{V}\left(\max_{0 \leqslant i \leqslant \tau} Y_{mi+j} > d\right) \\
& \quad + g(n) \sum_{j=1}^{m} \exp\left(\frac{x}{md} - \frac{x}{md} \ln\left(1 + \frac{xd}{m\sum_{i=0}^{\tau} \hat{\mathbb{E}}\left|Y_{mi+j}\right|^2}\right)\right)
\end{aligned}$$

$$\leqslant m\mathbb{V}\left(\max_{1\leqslant i\leqslant n} X_i > d\right) + mg(n)\exp\left(\frac{x}{md} - \frac{x}{md}\ln\left(1 + \frac{xd}{m\sum_{i=1}^{n}\hat{\mathbb{E}}|X_i|^2}\right)\right). \quad \square$$

LEMMA 2.3. *Suppose $X \in \mathscr{H}$, $p > 0$, for any constant $c > 0$, then*
(i) (*see* [11]) *if* $\alpha > 0$,

$$C_{\mathbb{V}}\left(|X|^p\right) < \infty \Leftrightarrow \sum_{n=1}^{\infty} n^{\alpha p-1}\mathbb{V}\left(|X| > cn^{\alpha}\right) < \infty.$$

(ii) (*see* [7]) *if* $C_{\mathbb{V}}\left(|X|^p\right) < \infty$, *for any* $a > 1$,

$$\sum_{k=1}^{\infty} a^k\mathbb{V}\left(|X| > ca^{\frac{k}{p}}\right) < \infty.$$

3. The main results and their proofs

Before formulating the main results, we first give some notations and assumptions. For $1 \leqslant i \leqslant n$, $n \geqslant 1$, let

$$Y_{ni} = -n^{\alpha}I\left(X_i < -n^{\alpha}\right) + X_iI\left(|X_i| \leqslant n^{\alpha}\right) + n^{\alpha}I\left(X_i > n^{\alpha}\right), \quad (\alpha > 0). \tag{3.1}$$

Suppose that $\{X_n, n \geqslant 1\}$ is a sequence of m-WA random variables under sub-linear expectations.

(a) Suppose that $g(x)$ is a nondecreasing positive function on $[0,\infty)$, and $g(x) = g(n)$ when $x = n$, $\frac{g(x)}{x^{\tau}} \downarrow$ for some $0 < \tau < 1$.

(b) There exists a nondecreasing positive function $h(x)$ on $[0,\infty)$, such that $\frac{h(x)}{x} \downarrow$ and $\sum_{n=1}^{\infty} \frac{g(n)}{n^{2-\alpha}h^{\gamma}(\mu n^{\alpha})} < \infty$ for some $\gamma > 0$, $0 < \mu < 1$.

THEOREM 3.1. *Let $\{X, X_n, n \geqslant 1\}$ be a sequence of m-WA and identically distributed random variables under sub-linear expectations, $\hat{\mathbb{E}}$ is countably sub-additive. Assume that*

$$C_{\mathbb{V}}\left(|X|^p\right) < \infty. \tag{3.2}$$

For $0 < \frac{1}{p} < \alpha < 1$, $g(x)$ satisfies (a) or (b), and $\{Y_{ni}, 1 \leqslant i \leqslant n, n > 1\}$ is m-WA.

Let $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of positive constant satisfying

$$\sum_{i=1}^{n} a_{ni}^2 = O\left(n^{-\alpha}\right), \tag{3.3}$$

and

$$\max_{1\leqslant i\leqslant n} a_{ni} = O\left(n^{-\alpha}\right). \tag{3.4}$$

Then, for all $\varepsilon > 0$, *we have*

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left(\sum_{i=1}^{n} a_{ni}\left(X_i - \hat{\mathbb{E}}X_i\right) > \varepsilon\right) < \infty, \tag{3.5}$$

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left(\sum_{i=1}^{n} a_{ni}\left(X_i - \hat{\varepsilon}X_i\right) < -\varepsilon\right) < \infty. \tag{3.6}$$

In particular, if $\hat{\mathbb{E}}X_i = \hat{\varepsilon}X_i$, *then*

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left(\left|\sum_{i=1}^{n} a_{ni}\left(X_i - \hat{\mathbb{E}}X_i\right)\right| > \varepsilon\right) < \infty. \tag{3.7}$$

THEOREM 3.2. *Let* $\{X, X_n, n \geqslant 1\}$ *be a sequence of* m*-WA and identically distributed random variables under sub-linear expectations,* $\hat{\mathbb{E}}$ *is countably sub-additive. For* $p = \alpha = 1$, $g(x)$ *satisfies (a) or (b), and* $\{Y_{ni}, 1 \leqslant i \leqslant n, n > 1\}$ *is* m*-WA.*

If (a) holds, and for some $0 < \delta < 1$,

$$C_{\mathbb{V}}\left(|X|^{1+\delta}\right) < \infty. \tag{3.8}$$

If (b) holds, and statisfying

$$\hat{\mathbb{E}}\left(|X|\,h\left(|X|\right)\right) \leqslant C_{\mathbb{V}}\left(|X|\,h\left(|X|\right)\right) < \infty. \tag{3.9}$$

Suppose that $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ *is an array of positive constant satisfying* (3.3) *and* (3.4), *then* (3.5)–(3.7) *are also hold.*

REMARK 3.1. The assumptions in (3.3) and (3.4) are commonly used conditions, which are also similar to the assumptions of Huang and Wu [16] and Yi et al. [5]. The results in this paper can be compared with Hu and Wu [16], which improve and generalize the conclusions of this literature.

Proof of Theorem 3.1. It can be checked that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left(\sum_{i=1}^{n} a_{ni}\left(X_i - \hat{\mathbb{E}}X_i\right) > \varepsilon\right)\\
&\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2}\sum_{i=1}^{n}\mathbb{V}(|X_i| > n^{\alpha}) + \sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left(\sum_{i=1}^{n} a_{ni}\left(Y_{ni} - \hat{\mathbb{E}}X_i\right) > \varepsilon\right)\\
&\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2}\sum_{i=1}^{n}\mathbb{V}(|X_i| > n^{\alpha}) + \sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left(\sum_{i=1}^{n} a_{ni}\left(Y_{ni} - \hat{\mathbb{E}}Y_{ni}\right) > \frac{\varepsilon}{2}\right)\\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{V}\left(\sum_{i=1}^{n} a_{ni}\left(\hat{\mathbb{E}}Y_{ni} - \hat{\mathbb{E}}X_i\right) > \frac{\varepsilon}{2}\right)\\
&=: H_1 + H_2 + H_3,
\end{aligned}$$

to prove (3.5), it suffices to show $H_i < \infty,\ i = 1,2,3$.

First, we prove $H_1 < \infty$. For $0 < \mu < 1$, let $z(x) \in C_{l,Lip}(\mathbb{R}),\ 0 \leqslant z(x) \leqslant 1$, for all x, $z(x) = 1$ if $|x| \leqslant \mu,\ z(x) = 0$ if $|x| > 1$, and $z(x)$ is nonincreasing for any $x > 0$. Then

$$I(|x| \leqslant \mu) \leqslant z(|x|) \leqslant I(|x| \leqslant 1),\ I(|x| > 1) \leqslant 1 - z(|x|) \leqslant I(|x| > \mu). \tag{3.10}$$

Applying Lemma 2.3 and together with (3.10), we can get

$$\begin{aligned} H_1 &\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{i=1}^{n} \hat{\mathbb{E}}\left(1 - z\left(\frac{|X_i|}{n^{\alpha}}\right)\right) \\ &= \sum_{n=1}^{\infty} n^{\alpha p-1} \hat{\mathbb{E}}\left(1 - z\left(\frac{|X|}{n^{\alpha}}\right)\right) \\ &\leqslant \sum_{n=1}^{\infty} n^{\alpha p-1} \mathbb{V}(|X| > \mu n^{\alpha}) \\ &< \infty. \end{aligned}$$

Next, in order to prove $H_2 < \infty$, let $\{T_i, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be a set of random variables, it follows by Markov inequality that

$$\mathbb{V}\left(\max_{1\leqslant i\leqslant n} T_i > d\right) \leqslant \sum_{i=1}^{n} \mathbb{V}(|T_i| > d) \leqslant \sum_{i=1}^{n} \frac{\hat{\mathbb{E}}|T_i|^q}{d^q},\quad q > 0. \tag{3.11}$$

Since $\{Y_{ni}, 1 \leqslant i \leqslant n, n > 1\}$ is an array of m-WA random variables, it follows that for any $n \geqslant 2$ and $ni_{k_1}, \cdots, ni_{k_n}$ satisfying $\left|ni_{k_p} - ni_{k_q}\right| \geqslant m \ (1 \leqslant p \neq q \leqslant n)$, $\left\{Y_{ni_{k_l}}, 1 \leqslant l \leqslant n\right\}$ is a sequence of WA. To prove $\{Y_{ni} - \hat{\mathbb{E}}Y_{ni}\}$ is m-WA, we only need to show that for all $ni_{k_1}, \cdots, ni_{k_n}$ satisfying $\left|ni_{k_p} - ni_{k_q}\right| \geqslant m \ (1 \leqslant p \neq q \leqslant n)$, $\left\{Y_{ni_{k_l}} - \hat{\mathbb{E}}Y_{ni_{k_l}}, 1 \leqslant l \leqslant n\right\}$ is WA. It is easily checked by (2.2) that for $\lambda > 0$

$$\begin{aligned} \hat{\mathbb{E}}\exp\left(\sum_{l=1}^{n} \lambda\left(Y_{ni_{k_l}} - \hat{\mathbb{E}}Y_{ni_{k_l}}\right)\right) &= \hat{\mathbb{E}}\exp\left(\sum_{l=1}^{n} \lambda Y_{ni_{k_l}} - \sum_{l=1}^{n} \lambda \hat{\mathbb{E}}Y_{ni_{k_l}}\right) \\ &= \hat{\mathbb{E}}\left(\exp\left(\sum_{l=1}^{n} \lambda Y_{ni_{k_l}}\right) \exp\left(-\sum_{l=1}^{n} \lambda \hat{\mathbb{E}}Y_{ni_{k_l}}\right)\right) \\ &= \exp\left(-\sum_{l=1}^{n} \lambda \hat{\mathbb{E}}Y_{ni_{k_l}}\right) \hat{\mathbb{E}}\left(\exp\left(\sum_{l=1}^{n} \lambda Y_{ni_{k_l}}\right)\right) \\ &\leqslant \exp\left(-\sum_{l=1}^{n} \lambda \hat{\mathbb{E}}Y_{ni_{k_l}}\right) g(n) \prod_{l=1}^{n} \hat{\mathbb{E}}\exp\left(\lambda Y_{ni_{k_l}}\right) \\ &= g(n) \prod_{l=1}^{n} \exp\left(-\lambda \hat{\mathbb{E}}Y_{ni_{k_l}}\right) \hat{\mathbb{E}}\exp\left(\lambda Y_{ni_{k_l}}\right) \end{aligned}$$

$$
\begin{aligned}
&= g(n) \prod_{l=1}^{n} \hat{\mathbb{E}}\left(\exp\left(-\lambda \hat{\mathbb{E}} Y_{ni_{k_l}}\right) \exp\left(\lambda Y_{ni_{k_l}}\right)\right) \\
&= g(n) \prod_{l=1}^{n} \hat{\mathbb{E}}\left(\exp\left(\lambda\left(Y_{ni_{k_l}} - \hat{\mathbb{E}} Y_{ni_{k_l}}\right)\right)\right).
\end{aligned}
$$

We can see from the above equation that $\left\{Y_{ni_{k_l}} - \hat{\mathbb{E}} Y_{ni_{k_l}}, 1 \leqslant l \leqslant n\right\}$ is WA. Thus, $\left\{Y_{ni} - \hat{\mathbb{E}} Y_{ni}, 1 \leqslant i \leqslant n, n > 1\right\}$ is a sequence of m-WA. According to Lemma 2.2, let $x = \frac{\varepsilon}{2}$, $d > 0$, taking $q > p$, we have

$$
\begin{aligned}
H_2 \leqslant& \sum_{n=1}^{\infty} n^{\alpha p-2} \left(m \mathbb{V}\left(\max_{1 \leqslant i \leqslant n} \left(a_{ni}\left(Y_{ni} - \hat{\mathbb{E}} Y_{ni}\right)\right) > d\right)\right) \\
&+ \sum_{n=1}^{\infty} n^{\alpha p-2} \left(m g(n) \exp\left(\frac{\varepsilon}{2md} - \frac{\varepsilon}{2md} \ln\left(1 + \frac{\varepsilon d}{2m \sum_{i=1}^{n} \hat{\mathbb{E}}\left|a_{ni}\left(Y_{ni} - \hat{\mathbb{E}} Y_{ni}\right)\right|^2}\right)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant m \sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{i=1}^{n} \frac{\hat{\mathbb{E}}\left|a_{ni}\left(Y_{ni} - \hat{\mathbb{E}} Y_{ni}\right)\right|^q}{d^q} \\
&\quad + m \sum_{n=1}^{\infty} n^{\alpha p-2} g(n) \exp\left(\frac{\varepsilon}{2md} - \frac{\varepsilon}{2md} \ln\left(1 + \frac{\varepsilon d}{2m \sum_{i=1}^{n} \hat{\mathbb{E}}\left|a_{ni}\left(Y_{ni} - \hat{\mathbb{E}} Y_{ni}\right)\right|^2}\right)\right) \\
&=: H_{21} + H_{22}.
\end{aligned}
$$

In the following, we prove $H_{21} < \infty$. For any $r > 0$, which together with (3.10) yields that

$$
\begin{aligned}
|Y_{ni}|^r =& |X_i|^r I(|X_i| \leqslant n^\alpha) + n^{\alpha r} I(|X_i| > n^\alpha) \\
\leqslant& |X_i|^r z\left(\frac{\mu |X_i|}{n^\alpha}\right) + n^{\alpha r}\left(1 - z\left(\frac{|X_i|}{n^\alpha}\right)\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\hat{\mathbb{E}}|Y_{ni}|^r \leqslant& \hat{\mathbb{E}}\left(|X_i|^r z\left(\frac{\mu |X_i|}{n^\alpha}\right)\right) + n^{\alpha r} \hat{\mathbb{E}}\left(1 - z\left(\frac{|X_i|}{n^\alpha}\right)\right) \\
=& \hat{\mathbb{E}}\left(|X|^r z\left(\frac{\mu |X|}{n^\alpha}\right)\right) + n^{\alpha r} \hat{\mathbb{E}}\left(1 - z\left(\frac{|X|}{n^\alpha}\right)\right) \\
\leqslant& \hat{\mathbb{E}}\left(|X|^r z\left(\frac{\mu |X|}{n^\alpha}\right)\right) + n^{\alpha r} \mathbb{V}(|X| > \mu n^\alpha). \qquad (3.12)
\end{aligned}
$$

By (3.4), (3.12) and C_r inequality, it follows that

$$\begin{aligned}
H_{21} &\leqslant cm\sum_{n=1}^{\infty} n^{\alpha p-2}\sum_{i=1}^{n} a_{ni}^{q}\hat{\mathbb{E}}|Y_{ni}|^{q}\\
&\leqslant cm\sum_{n=1}^{\infty} n^{\alpha p-2}\sum_{i=1}^{n} a_{ni}^{q}\left(\hat{\mathbb{E}}\left(|X|^{q}z\left(\frac{\mu|X|}{n^{\alpha}}\right)\right)+n^{\alpha q}\mathbb{V}\left(|X|>\mu n^{\alpha}\right)\right)\\
&\leqslant cm\sum_{n=1}^{\infty} n^{\alpha p-2}n\left(\max_{1\leqslant i\leqslant n} a_{ni}\right)^{q}\left(\hat{\mathbb{E}}\left(|X|^{q}z\left(\frac{\mu|X|}{n^{\alpha}}\right)\right)+n^{\alpha q}\mathbb{V}\left(|X|>\mu n^{\alpha}\right)\right)\\
&\leqslant cm\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha q}\left(\hat{\mathbb{E}}\left(|X|^{q}z\left(\frac{\mu|X|}{n^{\alpha}}\right)\right)+n^{\alpha q}\mathbb{V}\left(|X|>\mu n^{\alpha}\right)\right)\\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha q}\hat{\mathbb{E}}\left(|X|^{q}z\left(\frac{\mu|X|}{n^{\alpha}}\right)\right)+\sum_{n=1}^{\infty} n^{\alpha p-1}\mathbb{V}\left(|X|>\mu n^{\alpha}\right)\\
&=: H_{211}+H_{212}.
\end{aligned}$$

Applying Lemma 2.3 we can get $H_{212}<\infty$. To prove $H_{21}<\infty$, it suffices to show $H_{211}<\infty$.

For $j\geqslant 1$, let $z_j(x)\in C_{l,Lip}(\mathbb{R})$, $0\leqslant z_j(x)\leqslant 1$, for all x, and $z_j\left(\frac{x}{2^{j\alpha}}\right)=1$ if $2^{(j-1)\alpha}<|x|\leqslant 2^{j\alpha}$, $z_j\left(\frac{x}{2^{j\alpha}}\right)=0$ if $|x|\leqslant \mu 2^{(j-1)\alpha}$ or $|x|>(1+\mu)2^{j\alpha}$, and z_j is an even function. Then

$$z_j\left(\frac{X}{2^{j\alpha}}\right)\leqslant I\left(\mu 2^{(j-1)\alpha}<|X|\leqslant(1+\mu)2^{j\alpha}\right), \tag{3.13}$$

$$|X|^{r}z_j\left(\frac{X}{2^{k\alpha}}\right)\leqslant 1+\sum_{j=1}^{k}|X|^{r}z_j\left(\frac{X}{2^{j\alpha}}\right),\ \forall r>0. \tag{3.14}$$

Together with (3.13), (3.14) and $q>p$, $z(x)$ is nonincreasing for any $x>0$, such that

$$\begin{aligned}
H_{211} &\leqslant \sum_{k=1}^{\infty}\sum_{2^{k-1}\leqslant n<2^{k}} 2^{k(\alpha p-1-\alpha q)}\hat{\mathbb{E}}\left(|X|^{q}z\left(\frac{\mu|X|}{n^{\alpha}}\right)\right)\\
&\ll \sum_{k=1}^{\infty}2^{\alpha k(p-q)}\hat{\mathbb{E}}\left(|X|^{q}z\left(\frac{\mu|X|}{n^{\alpha}}\right)\right)\\
&\leqslant \sum_{k=1}^{\infty}2^{\alpha k(p-q)}\sum_{j=1}^{k}\hat{\mathbb{E}}\left(|X|^{q}z_j\left(\frac{\mu|X|}{2^{\alpha j}}\right)\right)\\
&\leqslant \sum_{k=1}^{\infty}2^{\alpha k(p-q)}\sum_{j=1}^{k}\hat{\mathbb{E}}|X|^{q}I\left(2^{(j-1)\alpha}<|X|\leqslant 2^{j\alpha}\right)\\
&\leqslant \sum_{k=1}^{\infty}2^{\alpha k(p-q)}\sum_{j=1}^{k}2^{j\alpha q}\hat{\mathbb{E}}I\left(|X|>2^{(j-1)\alpha}\right)\\
&= \sum_{j=1}^{\infty}2^{j\alpha q}\sum_{k=j}^{\infty}2^{\alpha k(p-q)}\mathbb{V}\left(|X|>2^{(j-1)\alpha}\right)
\end{aligned}$$

$$
\begin{aligned}
&\ll \sum_{j=1}^{\infty} 2^{j\alpha q}\mathbb{V}\left(|X| > 2^{(j-1)\alpha}\right)\\
&= \sum_{j=1}^{\infty} (2^{\alpha p})^j \mathbb{V}\left(|X| > \frac{1}{2^{\alpha}}(2^{\alpha p})^{\frac{j}{p}}\right).
\end{aligned}
$$

It can be inferred $H_{211} < \infty$ by Lemma 2.3.

Next, we will show $H_{22} < \infty$. Consider the following two cases:

(i) $p \geqslant 2$.

From equation (3.11) in Reference [6], we have $\hat{\mathbb{E}}|Y_{ni}| \leqslant C_{\mathbb{V}}(|Y_{ni}|)$, combine with Definition 2.3, it follows that

$$
C_{\mathbb{V}}\left(|Y_{ni}|^2\right) = \int_0^{\infty} \mathbb{V}\left(|Y_{ni}|^2 \geqslant t\right)\mathrm{d}t + \int_{-\infty}^{0}\left[\mathbb{V}\left(|Y_{ni}|^2 \geqslant t\right) - 1\right]\mathrm{d}t \geqslant \hat{\mathbb{E}}|Y_{ni}|^2,
$$

which implies $\hat{\mathbb{E}}|Y_{ni}|^2 \leqslant C_{\mathbb{V}}\left(|Y_{ni}|^2\right)$. We can also get $C_{\mathbb{V}}(|Y_{ni}|) \leqslant C_{\mathbb{V}}(|X|)$ from Definition 2.3. By (3.1) we have $|Y_{ni}| \leqslant |X|$, thus $C_{\mathbb{V}}\left(|Y_{ni}|^2\right) \leqslant C_{\mathbb{V}}\left(|X|^2\right)$. Therefore, we can immediately get $\hat{\mathbb{E}}|Y_{ni}|^2 \leqslant C_{\mathbb{V}}\left(|Y_{ni}|^2\right) \leqslant C_{\mathbb{V}}\left(|X|^2\right) < \infty$. Combining (3.1) and (3.3), taking $d = \frac{\varepsilon\alpha}{4m(1+\alpha p-\alpha+\alpha\gamma)}$ and noting that $\tau < 1$, if the conditions (a) or (b) hold, we have

$$
\begin{aligned}
H_{22} =& m\sum_{n=1}^{\infty} n^{\alpha p-2} g(n) \exp\left(\frac{\varepsilon}{2md} - \frac{\varepsilon}{2md}\ln\left(1 + \frac{\varepsilon d}{2m\sum_{i=1}^{n}\hat{\mathbb{E}}\left|a_{ni}\left(Y_{ni} - \hat{\mathbb{E}}Y_{ni}\right)\right|^2}\right)\right)\\
\leqslant& cm\sum_{n=1}^{\infty} n^{\alpha p-2} g(n) \exp\left(-\frac{\varepsilon}{2md}\ln\left(1 + \frac{\varepsilon d}{c\sum_{i=1}^{n} a_{ni}^2}\right)\right)\\
\leqslant& cm\sum_{n=1}^{\infty} n^{\alpha p-2} g(n) \exp\left(-\frac{\varepsilon}{2md}\ln\left(\frac{\varepsilon d}{cn^{-\alpha}}\right)\right)\\
=& cm\sum_{n=1}^{\infty} n^{\alpha p-2} g(n) \left(\frac{\varepsilon d}{cn^{-\alpha}}\right)^{-\frac{\varepsilon}{2md}}\\
\ll& \sum_{n=1}^{\infty} n^{\alpha p-2} g(n) n^{-\frac{\varepsilon\alpha}{2md}}\\
\leqslant& \begin{cases} \sum_{n=1}^{\infty} n^{\alpha p-2+\tau-\frac{\varepsilon\alpha}{2md}}, & \text{if (a) holds,}\\ \sum_{n=1}^{\infty} n^{\alpha p-2-\frac{\varepsilon\alpha}{2md}} \frac{g(n)h^{\gamma}(n^{\alpha})n^{2-\alpha+\alpha\gamma}}{n^{2-\alpha}h^{\gamma}(n^{\alpha})n^{\alpha\gamma}}, & \text{if (b) holds,}\end{cases}\\
\leqslant& \begin{cases} \sum_{n=1}^{\infty} n^{\alpha p-1-\frac{\varepsilon\alpha}{2md}}, & \text{if (a) holds,}\\ \sum_{n=1}^{\infty} n^{\alpha p-\frac{\varepsilon\alpha}{2md}-\alpha+\alpha\gamma}, & \text{if (b) holds,}\end{cases}
\end{aligned}
$$

$$= \begin{cases} \sum_{n=1}^{\infty} n^{\alpha p-1-\frac{\varepsilon\alpha}{2m\frac{\varepsilon\alpha}{4m(1+\alpha p-\alpha+\alpha\gamma)}}}, & \text{if (a) holds,} \\ \sum_{n=1}^{\infty} n^{\alpha p-\frac{\varepsilon\alpha}{2m\frac{\varepsilon\alpha}{4m(1+\alpha p-\alpha+\alpha\gamma)}}-\alpha+\alpha\gamma}, & \text{if (b) holds,} \end{cases}$$

$$\leqslant \sum_{n=1}^{\infty} n^{-1-\alpha p-\alpha\gamma} < \infty.$$

(ii) $1 < p < 2$.

It is easily checked that $|Y_{ni}| \leqslant n^{\alpha}$ by (3.1). Taking $d = \frac{\varepsilon\alpha(p-1)}{4m(1+\alpha p-\alpha+\alpha\gamma)}$, if (a) or (b) hold, we can get

$$\begin{aligned} H_{22} \leqslant & cm \sum_{n=1}^{\infty} n^{\alpha p-2} g(n) \exp\left(-\frac{\varepsilon}{2md} \ln\left(1+\frac{\varepsilon d}{2m\sum_{i=1}^{n} a_{ni}^{2}\hat{\mathbb{E}}|Y_{ni}|^{p}|Y_{ni}|^{2-p}}\right)\right) \\ \leqslant & cm \sum_{n=1}^{\infty} n^{\alpha p-2} g(n) \exp\left(-\frac{\varepsilon}{2md} \ln\left(\frac{\varepsilon d}{cn^{\alpha(2-p)}\sum_{i=1}^{n} a_{ni}^{2}}\right)\right) \\ \leqslant & cm \sum_{n=1}^{\infty} n^{\alpha p-2} g(n) \exp\left(-\frac{\varepsilon}{2md} \ln\left(\frac{\varepsilon d}{cn^{\alpha(1-p)}}\right)\right) \\ = & cm \sum_{n=1}^{\infty} n^{\alpha p-2} g(n) \left(\frac{\varepsilon d}{cn^{\alpha(1-p)}}\right)^{-\frac{\varepsilon}{2md}} \\ \ll & \sum_{n=1}^{\infty} n^{\alpha p-2} g(n) n^{-\frac{\varepsilon}{2md}\alpha(p-1)} \\ \leqslant & \begin{cases} \sum_{n=1}^{\infty} n^{\alpha p-2+\tau-\frac{\varepsilon\alpha}{2md}(p-1)}, & \text{if (a) holds,} \\ \sum_{n=1}^{\infty} n^{\alpha p-2-\frac{\varepsilon\alpha}{2md}(p-1)} \frac{g(n)h^{\gamma}(n^{\alpha})n^{2-\alpha+\alpha\gamma}}{n^{2-\alpha}h^{\gamma}(n^{\alpha})n^{\alpha\gamma}}, & \text{if (b) holds,} \end{cases} \\ \leqslant & \begin{cases} \sum_{n=1}^{\infty} n^{\alpha p-1-\frac{\varepsilon\alpha}{2md}(p-1)}, & \text{if (a) holds,} \\ \sum_{n=1}^{\infty} n^{\alpha p-\frac{\varepsilon\alpha}{2md}(p-1)-\alpha+\alpha\gamma}, & \text{if (b) holds,} \end{cases} \\ = & \begin{cases} \sum_{n=1}^{\infty} n^{\alpha p-1-\frac{\varepsilon\alpha}{2m\frac{\varepsilon\alpha(p-1)}{4m(1+\alpha p-\alpha+\alpha\gamma)}}(p-1)}, & \text{if (a) holds,} \\ \sum_{n=1}^{\infty} n^{\alpha p-\frac{\varepsilon\alpha}{2m\frac{\varepsilon\alpha(p-1)}{4m(1+\alpha p-\alpha+\alpha\gamma)}}(p-1)-\alpha+\alpha\gamma}, & \text{if (b) holds,} \end{cases} \\ \leqslant & \sum_{n=1}^{\infty} n^{-1-\alpha p-\alpha\gamma} < \infty. \end{aligned}$$

Thus, it follows that $H_2 < \infty$.

Finally, we prove $H_3<\infty$. Combine with (3.2) and (3.10), obviously,

$$\begin{aligned}\left|\hat{\mathbb{E}}Y_{ni}-\hat{\mathbb{E}}X_i\right| \leqslant & \hat{\mathbb{E}}\left|Y_{ni}-X_i\right| \\ \leqslant & \hat{\mathbb{E}}\left|(X_i+n^\alpha)I(X_i<-n^\alpha)+(X_i-n^\alpha)I(X_i>n^\alpha)\right| \\ \ll & \hat{\mathbb{E}}\left(|X_i|\left(1-z\left(\frac{|X_i|}{n^\alpha}\right)\right)\right) \\ \leqslant & \hat{\mathbb{E}}|X|\left(\frac{|X|}{\mu n^\alpha}\right)^{p-1} \\ = & cn^{-\alpha(p-1)}\hat{\mathbb{E}}|X|^p. \end{aligned} \tag{3.15}$$

It implies by (3.4) and (3.15) that,

$$\begin{aligned}\left|\sum_{i=1}^n a_{ni}\left(\hat{\mathbb{E}}Y_{ni}-\hat{\mathbb{E}}X_i\right)\right| \leqslant & \max_{1\leqslant i\leqslant n} a_{ni}\sum_{i=1}^n \hat{\mathbb{E}}\left|Y_{ni}-X_i\right| \\ \ll & n\max_{1\leqslant i\leqslant n} a_{ni}n^{-\alpha(p-1)}\hat{\mathbb{E}}|X|^p \\ = & n^{-\alpha+1}n^{-\alpha(p-1)}\hat{\mathbb{E}}|X|^p \\ = & n^{1-\alpha p}\hat{\mathbb{E}}|X|^p\to 0,\ \ n\to\infty. \end{aligned} \tag{3.16}$$

By (3.16), it can be seen $\mathbb{V}\left(\sum\limits_{i=1}^n a_{ni}\left(\hat{\mathbb{E}}Y_{ni}-\hat{\mathbb{E}}X_i\right)>\frac{\varepsilon}{2}\right)=0$ for sufficiently large n, thus $H_3<\infty$. Hence, (3.5) holds. Obviously, $\{-X_n, n\geqslant 1\}$ also satisfies the conditions of Theorem 3.1. Considering $\{-X_n, n\geqslant 1\}$ instead of $\{X_n, n\geqslant 1\}$ in (3.5), we can obtain (3.6).

In particular, if $\hat{\mathbb{E}}X_i=\hat{\varepsilon}X_i$, then

$$\begin{aligned}&\mathbb{V}\left(\left|\sum_{i=1}^n a_{ni}\left(X_i-\hat{\varepsilon}X_i\right)\right|>\varepsilon\right) \\ \leqslant & \mathbb{V}\left(\sum_{i=1}^n a_{ni}\left(X_i-\hat{\varepsilon}X_i\right)>\varepsilon\right)+\mathbb{V}\left(\sum_{i=1}^n a_{ni}\left(X_i-\hat{\varepsilon}X_i\right)<-\varepsilon\right).\end{aligned}$$

Thus (3.7) holds. The proof of Theorem 3.1 is completed. □

REMARK 3.2. In the proof of H_{22}, condition (a) does not involve γ, but taking $d=\frac{\varepsilon\alpha}{4m(1+\alpha p-\alpha+\alpha\gamma)}$ does not affect the proof of the result.

Proof of Theorem 3.2. Similar to the proof of Theorem 3.1, it can be inferred $H_1<\infty$ when $p=1$. Take $x=\frac{\varepsilon}{2}, d=\frac{\varepsilon\delta}{4m}, q>p$ in Lemma 2.2, we have $H_{21}<\infty$. Then we show $H_{22}<\infty$. Similar to the proof of $\hat{\mathbb{E}}|Y_{ni}|^2\leqslant C_{\mathbb{V}}\left(|Y_{ni}|^2\right)\leqslant C_{\mathbb{V}}\left(|X|^2\right)<\infty$, we can get $\hat{\mathbb{E}}|Y_{ni}|^{1+\delta}\leqslant C_{\mathbb{V}}\left(|X|^{1+\delta}\right)<\infty$. If (a) holds, $|Y_{ni}|\leqslant n^\alpha$, combine with (3.8), thus

$$
\begin{aligned}
H_{22} &\ll \sum_{n=1}^{\infty} n^{\alpha-2} g(n) \exp\left(\frac{\varepsilon}{2md} - \frac{\varepsilon}{2md}\ln\left(1+\frac{\varepsilon d}{2m\sum_{i=1}^{n}\hat{\mathbb{E}}\left|a_{ni}\left(Y_{ni}-\hat{\mathbb{E}}Y_{ni}\right)\right|^2}\right)\right)\\
&\leqslant c\sum_{n=1}^{\infty} n^{\alpha-2} g(n) \exp\left(-\frac{\varepsilon}{2md}\ln\left(1+\frac{\varepsilon d}{c\sum_{i=1}^{n} a_{ni}^2\hat{\mathbb{E}}|Y_{ni}|^2}\right)\right)\\
&\leqslant c\sum_{n=1}^{\infty} n^{\alpha-2} g(n) \exp\left(-\frac{\varepsilon}{2md}\ln\left(1+\frac{\varepsilon d}{c\sum_{i=1}^{n} a_{ni}^2\hat{\mathbb{E}}|Y_{ni}|^{1+\delta}|Y_{ni}|^{1-\delta}}\right)\right)\\
&\leqslant c\sum_{n=1}^{\infty} n^{\alpha-2} g(n) \exp\left(-\frac{\varepsilon}{2md}\ln\left(\frac{\varepsilon d}{cn^{\alpha(1-\delta)}\sum_{i=1}^{n} a_{ni}^2}\right)\right)\\
&\leqslant c\sum_{n=1}^{\infty} n^{\alpha-2} g(n) \exp\left(-\frac{\varepsilon}{2md}\ln\left(\frac{\varepsilon d}{cn^{-\delta\alpha}}\right)\right)\\
&\leqslant c\sum_{n=1}^{\infty} n^{\alpha-2} g(n)\, n^{-\frac{\varepsilon}{2md}\delta\alpha} \leqslant c\sum_{n=1}^{\infty} n^{\alpha-1-\frac{\varepsilon}{2m\frac{\varepsilon\delta}{4m}}\delta\alpha}\\
&\leqslant c\sum_{n=1}^{\infty} n^{-1-\alpha} < \infty.
\end{aligned}
$$

If (b) holds, by (3.13) and (3.14), it follows that

$$
\begin{aligned}
H_{22} &\leqslant cm\sum_{n=1}^{\infty} n^{\alpha-2} g(n) \exp\left(-\frac{\varepsilon}{2md}\ln\left(1+\frac{\varepsilon d}{c\sum_{i=1}^{n} a_{ni}^2\hat{\mathbb{E}}|Y_{ni}|^2}\right)\right)\\
&\ll \sum_{n=1}^{\infty} n^{\alpha-2} g(n)\left(\sum_{i=1}^{n} a_{ni}^2\hat{\mathbb{E}}|Y_{ni}|^2\right)^{\frac{\varepsilon}{2md}}\\
&\leqslant \sum_{n=1}^{\infty} n^{\alpha-2-\frac{\varepsilon\alpha}{2md}} g(n)\left(\hat{\mathbb{E}}|X|^2 z\left(\frac{\mu|X|}{n^{\alpha}}\right) + n^{2\alpha}\hat{\mathbb{E}}\left(1-z\left(\frac{|X|}{n^{\alpha}}\right)\right)\right)^{\frac{\varepsilon}{2md}}\\
&\leqslant \sum_{n=1}^{\infty} n^{\alpha-2-\frac{\varepsilon\alpha}{2md}} g(n)\left(\hat{\mathbb{E}}|X|^2 z\left(\frac{\mu|X|}{n^{\alpha}}\right)\right)^{\frac{\varepsilon}{2md}}\\
&\quad+\sum_{n=1}^{\infty} n^{\alpha-2-\frac{\varepsilon\alpha}{2md}} g(n)\left(n^{2\alpha}\hat{\mathbb{E}}\left(1-z\left(\frac{|X|}{n^{\alpha}}\right)\right)\right)^{\frac{\varepsilon}{2md}}\\
&=: H_{221}+H_{222}.
\end{aligned}
$$

Next we will show $H_{221} < \infty$ and $H_{222} < \infty$. By (3.10) we have $z\left(\frac{\mu|X|}{n^\alpha}\right) \leqslant I\left(|X| \leqslant \frac{n^\alpha}{\mu}\right)$. Since $h(x) \uparrow$ and $\frac{h(x)}{x} \downarrow$ in condition (b), it is apparent that $xh(x) \uparrow$ and $\frac{x}{h(x)} \uparrow$, combine with $\sum_{n=1}^{\infty} \frac{g(n)}{n^{2-\alpha} h^\gamma(\mu n^\alpha)} < \infty$ in condition (b) and (3.9), taking $d = \frac{\varepsilon}{2m\gamma}$, we can obtain

$$
\begin{aligned}
H_{221} &= \sum_{n=1}^{\infty} n^{\alpha-2-\frac{\varepsilon\alpha}{2md}} g(n) \left(\hat{\mathbb{E}} \frac{|X| h(X) |X|}{h(X)} z\left(\frac{\mu |X|}{n^\alpha}\right)\right)^{\frac{\varepsilon}{2md}} \\
&\leqslant \sum_{n=1}^{\infty} n^{\alpha-2-\frac{\varepsilon\alpha}{2md}} g(n) \left(\frac{n^\alpha/\mu}{h(n^\alpha/\mu)} \hat{\mathbb{E}} |X| h(X)\right)^{\frac{\varepsilon}{2md}} \\
&\ll \sum_{n=1}^{\infty} \frac{g(n)}{n^{2-\alpha} h^{\frac{\varepsilon}{2md}}(\mu n^\alpha)} \left(\hat{\mathbb{E}} |X| h(X)\right)^{\frac{\varepsilon}{2md}} \\
&< \infty.
\end{aligned}
$$

We can also conclude $1 - z\left(\frac{|X|}{n^\alpha}\right) \leqslant I(|X| \geqslant \mu n^\alpha)$ by (3.10). Combine with $\frac{x}{h(x)} \uparrow$ and (3.9), such that

$$
\begin{aligned}
H_{222} &= \sum_{n=1}^{\infty} n^{\alpha-2-\frac{\varepsilon\alpha}{2md}} g(n) \left(n^{2\alpha} \hat{\mathbb{E}} \frac{|X| h(X)}{|X| h(X)} \left(1 - z\left(\frac{|X|}{n^\alpha}\right)\right)\right)^{\frac{\varepsilon}{2md}} \\
&\leqslant \sum_{n=1}^{\infty} n^{\alpha-2-\frac{\varepsilon\alpha}{2md}} g(n) \left(n^{2\alpha} \hat{\mathbb{E}} \frac{|X| h(X)}{\mu n^\alpha h(\mu n^\alpha)}\right)^{\frac{\varepsilon}{2md}} \\
&\ll \sum_{n=1}^{\infty} n^{\alpha-2-\frac{\varepsilon\alpha}{2md}} g(n) \left(n^{\alpha} \hat{\mathbb{E}} \frac{|X| h(X)}{h(\mu n^\alpha)}\right)^{\frac{\varepsilon}{2md}} \\
&= \sum_{n=1}^{\infty} \frac{g(n)}{n^{2-\alpha} h^{\frac{\varepsilon}{2md}}(\mu n^\alpha)} \left(\hat{\mathbb{E}} |X| h(X)\right)^{\frac{\varepsilon}{2md}} \\
&< \infty.
\end{aligned}
$$

Thus we can infer $H_{22} < \infty$. Next we provide the proof of $H_3 < \infty$. We have $|X| \geqslant \mu n^\alpha$ from $0 \leqslant 1 - z\left(\frac{|X|}{n^\alpha}\right) \leqslant I\left(\frac{|X|}{n^\alpha} > \mu\right)$. It follows from $1 - z\left(\frac{|X|}{n^\alpha}\right) \leqslant 1$ and (3.15) that

$$
\begin{aligned}
\left|\hat{\mathbb{E}} Y_{ni} - \hat{\mathbb{E}} X_i\right| &\leqslant \hat{\mathbb{E}}\left(|X| \left(1 - z\left(\frac{|X|}{n^\alpha}\right)\right)\right) \\
&\leqslant C_{\mathbb{V}}\left(|X| \left(1 - z\left(\frac{|X|}{n^\alpha}\right)\right)\right) \\
&= \int_0^\infty \mathbb{V}\left(|X| \left(1 - z\left(\frac{|X|}{n^\alpha}\right)\right) > x\right) \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{\mu n^\alpha} \mathbb{V}\left(|X|\left(1 - z\left(\frac{|X|}{n^\alpha}\right)\right) > x\right) dx \\
&\quad + \int_{\mu n^\alpha}^{\infty} \mathbb{V}\left(|X|\left(1 - z\left(\frac{|X|}{n^\alpha}\right)\right) > x\right) \mathrm{d}x \\
&\leqslant \int_0^{\mu n^\alpha} \mathbb{V}(|X| > \mu n^\alpha) dx + \int_{\mu n^\alpha}^{\infty} \mathbb{V}(|X| > x) \mathrm{d}x \\
&= \mu n^\alpha \mathbb{V}(|X| > \mu n^\alpha) + \int_{\mu n^\alpha}^{\infty} \mathbb{V}(|X| > x) \mathrm{d}x \\
&=: H_{31} + H_{32}.
\end{aligned}
$$

By $\sum_{n=1}^{\infty} \mathbb{V}(|X| > \mu n^\alpha) < \infty$ in Lemma 2.3 and $\mathbb{V}(|X| > \mu n^\alpha) \downarrow$, we can obtain $H_{31} \to 0$. According to $C_{\mathbb{V}}\left(|X|^{1+\delta}\right) < \infty$ we have $C_{\mathbb{V}}(|X|) < \infty$, hence $H_{32} \to 0$. It is easily checked that $\left|\sum_{i=1}^{n} a_{ni}\left(\hat{\mathbb{E}}Y_{ni} - \hat{\mathbb{E}}X_i\right)\right| \to 0$ if $n \to \infty$, we can get $H_3 < \infty$. All above, (3.5) is established. Similar to the proof of (3.6) and (3.7) in Theorem 3.1, we can also prove that (3.6) and (3.7) are hold, thus the proof of Theorem 3.2 is finished. □

Funding. The work is supported by National Social Science Fundation (Grant No. 21BTJ040), Project of Outstanding Young People in University of Anhui Province (No. 2023AH020037) and International Joint Research Center of Simulation and Control for Population Ecology of Yangtze River in Anhui (No. SLXY2024A001).

REFERENCES

[1] X. W. FENG, *Law of the logarithm for weighted sums of negatively dependent random variables under sub-linear expectation*, Statistics & Probability Letters, 2019, **149**: 132–141.

[2] P. L. HSU, H. ROBBINS, *Complete convergence and the law of large numbers*, Proceedings of the National Academy of Sciences of the United States of America, USA, 1947, **33** (2): 25–31.

[3] R. HU, Q. Y. WU, *Complete convergence for weighted sums of widely acceptable random variables under sub-linear expectations*, Progress in Mathematics, 2022, **51** (06): 1132–1144.

[4] W. H. HUANG, P. Y. WU, *Strong laws of large numbers for general random variables in sublinear expectation spaces*, Journal of Inequalities and Applications, 2019 (1): 1–18.

[5] X. HUANG, Y. F. WU, *Complete convergence and complete moment convergence for weighted sums of m-extended negatively dependent random variables*, Journal of Mathematical Inequalities, 2023, **17** (2): 707–719.

[6] A. KUCZMASZEWSKA, *Complete convergence for widely acceptable random variables under the sublinear expectations*, Journal of Mathematical Analysis and Applications, 2020, **484** (1): 123662, 15 pp.

[7] X. C. MA, Q. Y. WU, *On some conditions for strong law of large numbers for weighted sums of END random variables under sublinear expectations*, Discrete Dynamics in Nature and Society, 2019 (8): 1–8.

[8] S. G. PENG, *A new central limit theorem under sub-linear expectations*, 2008, arXiv:0803.2656.

[9] S. G. PENG, *G-expection, G-Brownian motion and related stochastic calculus of type*, Stochastic Analysis and Applications, 2007, **2**: 541–567.

[10] S. G. PENG, *Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation*, Stochastic Processes and Their Applications, 2008, **118** (12): 2223–2253.

[11] R. X. WANG, Q. Y. WU, *Some types of convergence for negatively dependent random variables under sub-linear expectations*, Discrete Dynamics in Nature and Society, 2019, 1–7.

[12] Y. WU, X. DENG, X. J. WANG, *Capacity inequalities and strong laws for m-widely acceptable random variables under sub-linear expectations*, Journal of Mathematical Analysis and Applications, 2023, **525** (2): 127282.

[13] Q. Y. WU, Y. Y. JIANG, *Strong law of large numbers and Chover's law of the iterated logarithm under sub-linear expectations*, Journal of Mathematical Analysis Applications, 2018, **460** (1): 252–270.

[14] Q. Y. WU, J. F. LU, *Another form of Chover's law of the iterated logarithm under sub-linear expectations*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales, Serie A, Matemáticas, 2020, **114** (1): 1–22.

[15] J. P. XU, L. X. ZHANG, *Three series theorem for independent random variables under sub-linear expectations with applications*, Acta Mathematica Sinica, 2019, **35** (02): 172–184.

[16] Y. C. YI, P. Y. CHEN, S. H. SUNG, *Convergence rate for weighted sums of ψ-mixing random variables and applications*, Journal of Mathematical Inequalities, 2022, **16** (3): 851–865.

[17] L. X. ZHANG, *Exponential inequalities under the sub-linear expectations with applications to laws of the iterated logarithm*, Science China Mathematics, 2016, **59** (12): 2503–2526.

[18] L. X. ZHANG, *Rosenthal's inequalities for independent and negatively dependent random variables under sub-linear expectations with applications*, Science China, 2016, **59** (4): 751–768.

[19] L. X. ZHANG, *Strong limit theorems for extended independent and extended negatively dependent random variables under non-linear expectations*, Acta Mathematica Scientia, 2022, **42** (2): 467–490.

[20] L. X. ZHANG, *The convergence of the sums of independent random variables under the sub-linear expectations*, Acta Mathematica Sinica, 2020, **36** (03): 224–244.

(Received September 14, 2023)

Shuting Pan
School of Mathematics and Physics
Anqing Normal University
Anqing Anhui, China
and
School of Mathematics and Statistics
Huangshan University
Huangshan Anhui, China

Ke Hu
Capital Normal University
Beijing, 100037, China

Xueping Hu
School of Mathematics and Physics
Anqing Normal University
Anqing Anhui, China
e-mail: hxprob@163.com

Keming Yu
Department of Mathematics
Brunel University
London UB8 3PH, UK