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OPERATOR QUADRATIC MEAN AND POSITIVE LINEAR MAPS

HAMID REZA MORADI, SHIGERU FURUICHI AND MOHAMMAD SABABHEH

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Abstract. This paper presents several bounds for unital positive linear mappings and the so-called quadratic mean. Some of these results can be viewed as sub-multiplicative inequalities, while others are Cauchy-Schwarz-type inequalities. Related results that treat the tensor products will be presented, too. Dragomir's result is improved by using the inequalities by Fujii–Nakamoto. As applications, we present some numerical radius inequalities by the obtained results. Finally, we study Ando-type inequalities for the quadratic mean.

1. Introduction and preliminiaries

Let $\mathscr{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a Hilbert space $\mathscr{H}$, with zero element $O_{\mathscr{H}}$, and identity $I_{\mathscr{H}}$. If no confusion arises, we will write O and I to denote the zero element and the identity operator, respectively. A linear mapping $\Phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{K})$ is said to be positive if $\Phi(A) \geqslant O_{\mathscr{K}}$ whenever $A \geqslant O_{\mathscr{H}}$. In this context, if $T \in \mathscr{B}(\mathscr{H})$, we write $T \geqslant O$ when $\langle Tx, x\rangle \geqslant 0$, for all $x \in \mathscr{H}$. Such an operator is called a positive operator. If $T \geqslant O$ is invertible, it is said to be strictly positive and denoted as $T > O$. If $\Phi(I_{\mathscr{H}}) = I_{\mathscr{K}}$, it is said to be unital.

Inequalities governing unital positive linear mappings have been in the core interest of numerous researchers in the literature, as one can see in [1, 5, 6, 7, 16, 18, 19, 21, 22, 26].

It is well known that unital positive linear mappings are not sub-multiplicative nor super-multiplicative. That is, if $A, B \in \mathscr{B}(\mathscr{H})$, then neither $\Phi(AB) \leqslant \Phi(A)\Phi(B)$ nor $\Phi(AB) \geqslant \Phi(A)\Phi(B)$, in general. In this context, we say that $X \leqslant Y$, for self-adjoint $X, Y \in \mathscr{B}(\mathscr{H})$, if $Y - X \geqslant O$.

However, in [14] it is shown that if $T \in \mathscr{B}(\mathscr{H})$ is self-adjoint, then

$$\Phi(T)^2 \leqslant \Phi(T^2). \tag{1.1}$$

Extending (1.1), Davis showed that if $f : J \to \mathbb{R}$ is operator convex, and if $T \in \mathscr{B}(\mathscr{H})$ is self-adjoint with spectrum in J, then [7]

$$f(\Phi(T)) \leqslant \Phi(f(T)), \tag{1.2}$$

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provided that Φ is completely positive. Latter, Choi in [6] proved (1.2) for any unital positive linear mapping Φ.

Further, Choi proved in [5, 6] that if $T \in \mathscr{B}(\mathscr{H})$, then

$$\Phi(T)^*\Phi(T) \leqslant \Phi(T^*T), \tag{1.3}$$

provided that Φ is 2-positive unital. Here, Φ is called 2-positive if $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \geqslant O_{\mathscr{H}\oplus\mathscr{H}}$ implies $\begin{bmatrix} \Phi(A) & \Phi(B) \\ \Phi(C) & \Phi(D) \end{bmatrix} \geqslant O_{\mathscr{K}\oplus\mathscr{K}}$.

Reversing (1.3), it is shown in [2, Theorem 3.1] that

$$\Phi(T^*T) - \Phi(T)^*\Phi(T) \leqslant \inf_{\lambda \in \mathbb{C}} \|T - \lambda I\|^2 I, \tag{1.4}$$

for all $T \in \mathscr{B}(\mathscr{H})$ and a unital positive linear mapping Φ, where $\|\cdot\|$ denotes the usual operator norm. This paper will show a sharper bound than (1.4). This will be done in Theorem 2.1 below.

When treating $\mathscr{B}(\mathscr{H})$, the operator means acquire considerable attention. The geometric mean is among the most important operator means, defined for $A, B > O$ by

$$A\sharp_v B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^v A^{\frac{1}{2}}, \ \ 0 \leqslant v \leqslant 1.$$

In [11], many mixed Schwarz inequalities via the geometric mean were shown.

Dragomir [8] introduced the quadratic weighted mean in the following form:

$$\begin{aligned} A \circledS_v B &= A^*\left((A^*)^{-1}B^*BA^{-1}\right)^v A \\ &= A^*\left((A^*)^{-1}B^*\left((A^*)^{-1}B^*\right)^*\right)^v A \\ &= A^*\left|BA^{-1}\right|^{2v} A \\ &= A^*\left|BA^{-1}\right|^v \left|BA^{-1}\right|^v A \\ &= A^*\left|BA^{-1}\right|^v \left(A^*\left|BA^{-1}\right|^v\right)^* \\ &= \left|\left|BA^{-1}\right|^v A\right|^2 \end{aligned}$$

for any $A, B \in \mathscr{B}(\mathscr{H})$ with invertible A and $0 \leqslant v \leqslant 1$. One can see that

$$A \circledS_v B = |A|^2 \sharp_v |B|^2.$$

It may be remarkable that we have the following inequalities from [13, Eq. (1.4)] (see [24] originally).

$$T \circledS_v V \leqslant L_v^{\circledS}(T,V) \leqslant \frac{1}{2}\left(T \circledS_v V + |T|^2 \nabla_v |V|^2\right) \leqslant |T|^2 \nabla_v |V|^2,$$

where

$$L_\nu^{\circledS}(T,V) := \frac{1-\nu}{\nu}\int_0^\nu T \circledS_x V\, dx + \frac{\nu}{1-\nu}\int_\nu^1 T \circledS_x V\, dx, \quad (0<\nu<1).$$

This latter quantity $L_\nu^{\circledS}(T,V)$ may be called a quadratic weighted operator logarithmic mean. The above inequalities complement the first inequality in [8, Eq. (1.7)].

We will present many Schwarz-type inequalities via the quadratic mean, extending some of the results in [11] to this mean.

Then, Schwarz-type inequalities involving the tensor product will be shown. Finally, some numerical radius inequalities with the Schwarz-type will be shown.

We recall that the numerical radius $\omega(T)$ of $T \in \mathscr{B}(\mathscr{H})$ is defined as $\omega(T) = \sup\limits_{\substack{x\in\mathscr{H} \\ \|x\|=1}} |\langle Tx,x\rangle|$, and the Crawford number of T is defined as $m(T) = \inf\limits_{\substack{x\in\mathscr{H} \\ \|x\|=1}} |\langle Tx,x\rangle|$.

2. Main results

In this section, we present our main results. For organizational purposes, we present these results in four subsections. We present results with the sub or super multiplicativity behavior in the first subsection. More precisely, we study possible relations between $\Phi(B^*A)$ and $\Phi(B)^*\Phi(A)$, for general $A,B \in \mathscr{B}(\mathscr{H})$. A particular case when $B = A$ will be more attractive in the way it refines (1.4). Then, Schwarz-type inequalities involving the quadratic mean and tensor product are discussed.

2.1. On $\Phi(B^*A)$ and $\Phi(B)^*\Phi(A)$

THEOREM 2.1. *Let $T \in \mathscr{B}(\mathscr{H})$ and let Φ be a unital positive linear map. If $\lambda \in \mathbb{C}$, then*

$$\Phi(T^*T) - \Phi(T)^*\Phi(T) \leqslant \Phi\left(|T-\lambda I|^2\right) - m^2\left(\Phi(T-\lambda I)\right) I.$$

Proof. One can see that for any $\lambda \in \mathbb{C}$,

$$\Phi(T^*T) - \Phi(T)^*\Phi(T) = \Phi\left(|T-\lambda I|^2\right) - |\Phi(T)-\lambda I|^2.$$

Thus, for any unit vector $x \in \mathscr{H}$,

$$\begin{aligned}
&\left\langle \left(\Phi(T^*T) - \Phi(T)^*\Phi(T)\right)x,x\right\rangle + |\langle(\Phi(T)-\lambda I)x,x\rangle|^2 \\
&\leqslant \left\langle \left(\Phi(T^*T) - \Phi(T)^*\Phi(T)\right)x,x\right\rangle + \|(\Phi(T)-\lambda I)x\|^2 \\
&= \left\langle \left(\Phi(T^*T) - \Phi(T)^*\Phi(T)\right)x,x\right\rangle + \left\langle |\Phi(T)-\lambda I|^2 x,x\right\rangle \\
&= \left\langle \Phi\left(|T-\lambda I|^2\right)x,x\right\rangle.
\end{aligned}$$

That is,

$$\left\langle \left(\Phi(T^*T) - \Phi(T)^*\Phi(T)\right)x,x\right\rangle + |\langle(\Phi(T)-\lambda I)x,x\rangle|^2 \leqslant \left\langle \Phi\left(|T-\lambda I|^2\right)x,x\right\rangle.$$

Therefore,

$$\left\langle \left(\Phi\left(T^*T\right) - \Phi\left(T\right)^*\Phi\left(T\right)\right)x,x\right\rangle + m^2\left(\Phi\left(T\right) - \lambda I\right) \leqslant \left\langle \Phi\left(\left|T-\lambda I\right|^2\right)x,x\right\rangle$$

for any unit vector $x \in \mathscr{H}$. This implies the desired result and completes the proof. □

REMARK 2.1. Bhatia and Sharma [2, Theorem 3.1] proved that

$$\Phi\left(T^*T\right) - \Phi\left(T\right)^*\Phi\left(T\right) \leqslant \inf_{\lambda\in\mathbb{C}} \left\|T-\lambda I\right\|^2 I.$$

Notice that

$$\begin{aligned}\Phi\left(\left|T-\lambda I\right|^2\right) - m^2\left(\Phi\left(T-\lambda I\right)\right)I &\leqslant \Phi\left(\left|T-\lambda I\right|^2\right)\\ &\leqslant \Phi\left(\left\|T-\lambda I\right\|^2 I\right)\\ &= \left\|T-\lambda I\right\|^2 I.\end{aligned}$$

So,

$$\inf_{\lambda\in\mathbb{C}} \left\|\Phi\left(\left|T-\lambda I\right|^2\right) - m^2\left(\Phi\left(T-\lambda I\right)\right)I\right\| \leqslant \inf_{\lambda\in\mathbb{C}} \left\|T-\lambda I\right\|^2.$$

Thus, Theorem 2.1 provides a refinement of (1.4).

Next, we present an upper bound for the difference $\left|\Phi\left(B^*A\right) - \Phi\left(B\right)^*\Phi\left(A\right)\right|$.

COROLLARY 2.1. Let $A, B \in \mathscr{B}(\mathscr{H})$, and let Φ be a unital 3–positive linear map. Then

$$\begin{aligned}&\left|\Phi\left(B^*A\right) - \Phi\left(B\right)^*\Phi\left(A\right)\right|\\ &\leqslant \sqrt{\inf_{\mu\in\mathbb{C}} \left\|\Phi\left(\left|B-\mu I\right|^2\right) - m^2\left(\Phi\left(B-\mu I\right)\right)I\right\| \cdot \inf_{\lambda\in\mathbb{C}} \left\|\Phi\left(\left|A-\lambda I\right|^2\right) - m^2\left(\Phi\left(A-\lambda I\right)\right)I\right\|}.\end{aligned}$$

Proof. The following matrix is positive [17, Theorem 1]:

$$\begin{bmatrix} \Phi\left(A^*A\right) - \Phi\left(A\right)^*\Phi\left(A\right) & \Phi\left(A^*B\right) - \Phi\left(A\right)^*\Phi\left(B\right) \\ \left(\Phi\left(A^*B\right) - \Phi\left(A\right)^*\Phi\left(B\right)\right)^* & \Phi\left(B^*B\right) - \Phi\left(B\right)^*\Phi\left(B\right) \end{bmatrix} \geqslant O.$$

Positivity of this operator matrix immediately implies $\Phi\left(B^*B\right) - \Phi\left(B\right)^*\Phi\left(B\right) \geqslant O$. On the other hand, we know that [5, Lemma 2.1]

$$\begin{bmatrix} S & X \\ X^* & T \end{bmatrix} \geqslant O \Leftrightarrow S \geqslant XT^{-1}X^*.$$

Therefore, by Theorem 2.1, we have

$$\begin{aligned}&\Phi\left(A^*A\right) - \Phi^*\left(A\right)\Phi\left(A\right)\\ &\geqslant \left(\Phi\left(A^*B\right) - \Phi^*\left(A\right)\Phi\left(B\right)\right)\left(\Phi\left(B^*B\right) - \Phi^*\left(B\right)\Phi\left(B\right)\right)^{-1}\left(\Phi\left(A^*B\right) - \Phi^*\left(A\right)\Phi\left(B\right)\right)^*\\ &\geqslant \left\|\Phi\left(\left|B-\mu I\right|^2\right) - m^2\left(\Phi\left(B-\mu I\right)\right)I\right\|^{-1}\left|\Phi\left(B^*A\right) - \Phi^*\left(B\right)\Phi\left(A\right)\right|^2.\end{aligned}$$

So, by applying again Theorem 2.1, we obtain

$$\begin{aligned}&|\Phi(B^*A)-\Phi^*(B)\Phi(A)|^2\\&\leqslant \inf_{\mu\in\mathbb{C}}\left\|\Phi\left(|B-\mu I|^2\right)-m^2(\Phi(B-\mu I))I\right\|\cdot\inf_{\lambda\in\mathbb{C}}\left\|\Phi\left(|A-\lambda I|^2\right)-m^2(\Phi(A-\lambda I))I\right\|.\end{aligned}$$

By taking the square root, we get the desired result. □

2.2. Schwarz inequalities involving the quadratic mean

In this subsection, we present several Schwarz-type inequalities that involve the quadratic mean. We remark that in [9, 11], Schwarz inequalities were given via the geometric mean. Some results in [9, 11] are being used here to accomplish our results.

THEOREM 2.2. *Let $A,B,C,D\in\mathscr{B}(\mathscr{H})$ with invertible A,B. If $DCBA=U|DCBA|$ is the polar decomposition of $DCBA$, then*

$$|DCBA|\leqslant BA\circledS U^*|C^*D^*|U.$$

If $(ABCD)^*=U|(ABCD)^*|$ is the polar decomposition of $(ABCD)^*$, then

$$|ABCD|\leqslant U(B^*A^*)U^*\circledS|CD|.$$

Proof. Let A,B be two operators and let $B^*A=U|B^*A|$ be the polar decomposition of B^*A. Then, from [9, Theorem 2.4], we have

$$|B^*A|\leqslant A\circledS U^*|B|U, \tag{2.1}$$

and

$$|A^*B|\leqslant UAU^*\circledS|B|. \tag{2.2}$$

Substituting A and B by BA and C^*D^*, respectively, in (2.1), we get the desired result. In addition, by substituting $A:=B^*A^*$ and $B:=CD$ in (2.2), we get the second result. □

REMARK 2.2. The following particular cases are of interest.

(i) Let $T=V|T|$ be the polar decomposition of T and let $\alpha,\beta\geqslant0$ with $\alpha+\beta\geqslant1$. If we take $D=V$, $C=|T|^\beta$, $B=I$, and $A=|T|^\alpha$, in Theorem 2.2, then we have

$$\left|V|T|^\beta|T|^\alpha\right|\leqslant|T|^\alpha\circledS U^*\left||T|^\beta V^*\right|U.$$

Notice that

$$\left|V|T|^\beta|T|^\alpha\right|=\left|V|T||T|^{\beta-1}|T|^\alpha\right|=\left|T|T|^{\alpha+\beta-1}\right|,$$

and

$$\left||T|^\beta V^*\right|=\left(V|T|^{2\beta}V^*\right)^{\frac12}=|T^*|^\beta.$$

Thus,

$$\left|T|T|^{\alpha+\beta-1}\right| \leqslant |T|^{\alpha} \circledS U^*|T^*|^{\beta}U,$$

provided that $T|T|^{\alpha+\beta-1} = U\left|T|T|^{\alpha+\beta-1}\right|$ is the polar decomposition of $T|T|^{\alpha+\beta-1}$.

(ii) Let $T = V|T|$ be the polar decomposition of T and let $\alpha,\beta \geqslant 1$. If we take $D = V$, $C = |T|^{\beta}$, $B = V$, and $A = |T|^{\alpha}$, in Theorem 2.2, then we have

$$\left|V|T|^{\beta}V|T|^{\alpha}\right| \leqslant V|T|^{\alpha} \circledS U^*\left(\left||T|^{\beta}V^*\right|\right)U.$$

Since

$$\left|V|T|^{\beta}V|T|^{\alpha}\right| = \left|V|T||T|^{\beta-1}V|T||T|^{\alpha-1}\right| = \left|T|T|^{\beta-1}T|T|^{\alpha-1}\right|,$$

$$V|T|^{\alpha} = V|T||T|^{\alpha-1} = T|T|^{\alpha-1},$$

we obtain

$$\left|T|T|^{\beta-1}T|T|^{\alpha-1}\right| \leqslant T|T|^{\alpha-1} \circledS U^*|T^*|^{\beta}U$$

where $T|T|^{\beta-1}T|T|^{\alpha-1} = U\left|T|T|^{\beta-1}T|T|^{\alpha-1}\right|$ is the polar decomposition of $T|T|^{\beta-1}T|T|^{\alpha-1}$.

(iii) Let $T = V|T|$ be the polar decomposition of T and let $\alpha,\beta \geqslant 0$ with $\alpha+\beta \geqslant 2$. If we take $D = V$, $C = |T|^{\beta}$, $B = |T|^{\alpha}$, and $A = V^*$, in Theorem 2.2, then we have

$$\left|V|T|^{\beta}|T|^{\alpha}V^*\right| \leqslant |T|^{\alpha}V^* \circledS U^*\left||T|^{\beta}V^*\right|U.$$

Notice

$$\left|V|T|^{\beta}|T|^{\alpha}V^*\right| = \left|V|T||T|^{\beta-1}|T|^{\alpha-1}|T|V^*\right| = \left|T|T|^{\alpha+\beta-2}T^*\right|,$$

$$|T|^{\alpha}V^* = |T|^{\alpha-1}|T|V^* = |T|^{\alpha-1}T^*,$$

so

$$\left|T|T|^{\alpha+\beta-2}T^*\right| \leqslant |T|^{\alpha-1}T^* \circledS U^*|T^*|^{\beta}U,$$

where $T|T|^{\alpha+\beta-2}T^* = U\left|T|T|^{\alpha+\beta-2}T^*\right|$ is the polar decomposition of $T|T|^{\alpha+\beta-2}T^*$.

If $T^* = T \geqslant O$, then we have $T^{\alpha+\beta} \leqslant T^{\alpha} \circledS U^*T^{\beta}U$ for all cases (i), (ii) and (iii).

The following is a Schwarz inequality for the quadratic mean.

THEOREM 2.3. *Let $A,B,X,Y \in \mathscr{B}(\mathscr{H})$ with invertible A,B, and let Φ be a 2-positive linear map. Then for any $0 \leqslant \alpha,\beta \leqslant 1$ with $\alpha+\beta=1$,*

$$|\Phi(Y^*X)| \leqslant \Phi(X^*(A \circledS_\alpha B)X) \sharp U^*\Phi\left(Y^*(B \circledS_\beta A)^{-1}Y\right)U,$$

where $\Phi(Y^*X) = U|\Phi(Y^*X)|$ is the polar decomposition of $\Phi(Y^*X)$.

Proof. It has been shown in [11, Theorem 2.1] that

$$|\Phi(Y^*TX)| \leqslant \Phi\left(X^*|T|^{2\alpha}X\right) \sharp U^*\Phi\left(Y^*|T^*|^{2\beta}Y\right)U, \tag{2.3}$$

where Φ is a 2-positive linear map, $0 \leqslant \alpha,\beta \leqslant 1$ with $\alpha+\beta=1$, and $\Phi(Y^*TX) = U|\Phi(Y^*TX)|$ is the polar decomposition of $\Phi(Y^*TX)$. If we substitute T by BA^{-1}, in (2.3), we get

$$\begin{aligned}
&\left|\Phi\left(Y^*BA^{-1}X\right)\right| \\
&\leqslant \Phi\left(X^*\left|BA^{-1}\right|^{2\alpha}X\right) \sharp U^*\Phi\left(Y^*\left|\left(BA^{-1}\right)^*\right|^{2\beta}Y\right)U \\
&= \Phi\left(X^*\left|BA^{-1}\right|^{2\alpha}X\right) \sharp U^*\Phi\left(Y^*\left|\left(A^{-1}\right)^*B^*\right|^{2\beta}Y\right)U \\
&= \Phi\left(X^*\left|BA^{-1}\right|^{2\alpha}X\right) \sharp U^*\Phi\left(Y^*\left|\left(A^{-1}\right)^*\left(\left(B^*\right)^{-1}\right)^{-1}\right|^{2\beta}Y\right)U,
\end{aligned} \tag{2.4}$$

where $\Phi\left(Y^*BA^{-1}X\right) = U\left|\Phi\left(Y^*BA^{-1}X\right)\right|$ is the polar decomposition of $\Phi\left(Y^*BA^{-1}X\right)$. Now, by letting $X := AX$ and $Y := (B^*)^{-1}Y$, in (2.4), we infer that

$$\begin{aligned}
&|\Phi(Y^*X)| \\
&\leqslant \Phi\left(X^*A^*\left|BA^{-1}\right|^{2\alpha}AX\right) \sharp U^*\Phi\left(Y^*\left(\left(B^*\right)^{-1}\right)^*\left|\left(A^{-1}\right)^*\left(\left(B^*\right)^{-1}\right)^{-1}\right|^{2\beta}\left(B^*\right)^{-1}Y\right)U \\
&= \Phi(X^*(A \circledS_\alpha B)X) \sharp U^*\Phi\left(Y^*(B^*)^{-1} \circledS_\beta (A^*)^{-1}Y\right)U \\
&= \Phi(X^*(A \circledS_\alpha B)X) \sharp U^*\Phi\left(Y^*(B \circledS_\beta A)^{-1}Y\right)U,
\end{aligned}$$

where $\Phi(Y^*X) = U|\Phi(Y^*X)|$ is the polar decomposition of $\Phi(Y^*X)$. Notice, in the last equality, we employed the fact that

$$(A \circledS_\nu B)^{-1} = (A^*)^{-1} \circledS_\nu (B^*)^{-1}; \ (0 \leqslant \nu \leqslant 1).$$

This completes the proof. □

A more elaborated Schwarz inequality for the quadratic mean can be stated as follows.

THEOREM 2.4. *Let $A,B,X,Y \in \mathscr{B}(\mathscr{H})$ with invertible A and let Φ be a 2-positive linear map, Then for any $0 \leqslant \alpha,\beta \leqslant 1$ with $\alpha+\beta=1$,*

$$|\Phi(Y^*(A \circledS B)X)| \leqslant \Phi(X^*(A \circledS_\alpha B)X) \sharp U^*\Phi\left(Y^*\left(A \circledS_\beta B\right)Y\right)U$$

where $\Phi(Y^*(A\Phi B)X) = U|\Phi(Y^*(A\Phi B)X)|$ is the polar decomposition of $\Phi(Y^*(A\Phi B)X)$.

Proof. If we substitute T by $\left|BA^{-1}\right|$, in (2.3), we reach

$$\left|\Phi\left(Y^*\left|BA^{-1}\right|X\right)\right| \leqslant \Phi\left(X^*\left|BA^{-1}\right|^{2\alpha}X\right)\sharp U^*\Phi\left(Y^*\left|BA^{-1}\right|^{2\beta}Y\right)U \tag{2.5}$$

where $\Phi\left(Y^*\left|BA^{-1}\right|X\right) = U\left|\Phi\left(Y^*\left|BA^{-1}\right|X\right)\right|$ is the polar decomposition of $\Phi\left(Y^*\left|BA^{-1}\right|X\right)$. Now, by letting $X = AX$ and $Y = AY$, in (2.5), we conclude that

$$\begin{aligned}
&\left|\Phi\left(Y^*A^*\left|BA^{-1}\right|AX\right)\right| \\
&= |\Phi(Y^*(A \circledS B)X)| \\
&\leqslant \Phi\left(X^*A^*\left|BA^{-1}\right|^{2\alpha}AX\right)\sharp U^*\Phi\left(Y^*A^*\left|BA^{-1}\right|^{2\beta}AY\right)U \\
&= \Phi(X^*(A \circledS_\alpha B)X)\sharp U^*\Phi\left(Y^*\left(A \circledS_\beta B\right)Y\right)U
\end{aligned}$$

as desired. □

2.3. Schwarz inequalities involving the tensor product

In [9], some Schwarz-type inequalities were given in terms of the tensor product. In this subsection, the quadratic mean presents more elaborated forms of such results.

LEMMA 2.1. *Let $X,Y,T \in \mathscr{B}(\mathscr{H})$. If $0 \leqslant \alpha,\beta \leqslant 1$ with $\alpha+\beta=1$, then*

$$\begin{aligned}
&(Y^*TX)\otimes(X^*T^*Y)+(X^*T^*Y)\otimes(Y^*TX) \\
&\leqslant Y^*|T^*|^{2\alpha}Y\otimes X^*|T|^{2\beta}X + X^*|T|^{2\beta}X\otimes Y^*|T^*|^{2\alpha}Y.
\end{aligned}$$

Proof. From [9, Theorem 2.2], for any $A,B \in \mathscr{B}(\mathscr{H})$, we have

$$(A^*B)\otimes(B^*A)+(B^*A)\otimes(A^*B) \leqslant |A|^2\otimes|B|^2+|B|^2\otimes|A|^2. \tag{2.6}$$

Let $T := U|T|$ be the polar decomposition of T. Let $A := |T|^\alpha U^*Y$ and $B := |T|^\beta X$. Then

$$\begin{aligned}
A^*B &= Y^*U|T|^{\alpha+\beta}X = Y^*TX, \\
B^*A &= X^*|T|^{\alpha+\beta}U^*Y = X^*T^*Y, \\
|A|^2 &= Y^*U|T|^{2\alpha}U^*Y = Y^*|T^*|^{2\alpha}Y, \\
|B|^2 &= X^*|T|^{2\beta}X.
\end{aligned}$$

This, together with (2.6), completes the proof. □

Now, we use Lemma 2.1 to present the following two forms of the Schwarz inequality via the quadratic mean and the tensor product of operators.

THEOREM 2.5. *Let $A,B,X,Y \in \mathscr{B}(\mathscr{H})$ with invertible A. If $0 \leqslant \alpha,\beta \leqslant 1$ with $\alpha+\beta=1$, then*

$$\begin{aligned}&(Y^*(A \circledS B)X) \otimes (X^*(A \circledS B)Y) + (X^*(A \circledS B)Y) \otimes (Y^*(A \circledS B)X)\\ &\leqslant Y^*(A \circledS_\alpha B)Y \otimes X^*\left(A \circledS_\beta B\right)X + X^*\left(A \circledS_\beta B\right)X \otimes Y^*(A \circledS_\alpha B)Y.\end{aligned}$$

Proof. Take $T = \left|BA^{-1}\right|$, in Lemma 2.1, we infer that

$$\begin{aligned}&\left(Y^*\left|BA^{-1}\right|X\right) \otimes \left(X^*\left|BA^{-1}\right|Y\right) + \left(X^*\left|BA^{-1}\right|Y\right) \otimes \left(Y^*\left|BA^{-1}\right|X\right)\\ &\leqslant Y^*\left|BA^{-1}\right|^{2\alpha}Y \otimes X^*\left|BA^{-1}\right|^{2\beta}X \\ &\quad + X^*\left|BA^{-1}\right|^{2\beta}X \otimes Y^*\left|BA^{-1}\right|^{2\alpha}Y.\end{aligned} \tag{2.7}$$

Letting $X = AX$ and $Y = AY$, in (2.7), we obtain

$$\begin{aligned}&(Y^*(A \circledS B)X) \otimes (X^*(A \circledS B)Y) + (X^*(A \circledS B)Y) \otimes (Y^*(A \circledS B)X)\\ &= \left(Y^*A^*\left|BA^{-1}\right|AX\right) \otimes \left(X^*A^*\left|BA^{-1}\right|AY\right) + \left(X^*A^*\left|BA^{-1}\right|AY\right) \otimes \left(Y^*A^*\left|BA^{-1}\right|AX\right)\\ &\leqslant Y^*A^*\left|BA^{-1}\right|^{2\alpha}AY \otimes X^*A^*\left|BA^{-1}\right|^{2\beta}AX + X^*A^*\left|BA^{-1}\right|^{2\beta}AX \otimes Y^*A^*\left|BA^{-1}\right|^{2\alpha}AY\\ &= Y^*(A \circledS_\alpha B)Y \otimes X^*\left(A \circledS_\beta B\right)X + X^*\left(A \circledS_\beta B\right)X \otimes Y^*(A \circledS_\alpha B)Y.\end{aligned}$$

This completes the proof. □

THEOREM 2.6. *Let $A,B,X,Y \in \mathscr{B}(\mathscr{H})$ with invertible A,B. If $0 \leqslant \alpha,\beta \leqslant 1$ with $\alpha+\beta=1$, then*

$$\begin{aligned}&(Y^*X) \otimes (X^*Y) + (X^*Y) \otimes (Y^*X)\\ &\leqslant Y^*(B \circledS_\alpha A)^{-1}Y \otimes X^*\left(A \circledS_\beta B\right)X + X^*\left(A \circledS_\beta B\right)X \otimes Y^*(B \circledS_\alpha A)^{-1}Y.\end{aligned}$$

Proof. If we put BA^{-1} instead of T, in Lemma 2.1, we obtain

$$\begin{aligned}&\left(Y^*BA^{-1}X\right) \otimes \left(X^*\left(A^{-1}\right)^*\left(\left(B^*\right)^{-1}\right)^{-1}Y\right) + \left(X^*\left(A^{-1}\right)^*\left(\left(B^*\right)^{-1}\right)^{-1}Y\right) \otimes \left(Y^*BA^{-1}X\right)\\ &\leqslant Y^*\left|\left(A^{-1}\right)^*\left(\left(B^*\right)^{-1}\right)^{-1}\right|^{2\alpha}Y \otimes X^*\left|BA^{-1}\right|^{2\beta}X + X^*\left|BA^{-1}\right|^{2\beta}X \otimes Y^*\left|\left(A^{-1}\right)^*\left(\left(B^*\right)^{-1}\right)^{-1}\right|^{2\alpha}Y.\end{aligned} \tag{2.8}$$

Letting $X := AX$ and $Y := \left(B^*\right)^{-1}Y$, in (2.8), we get the desired result. □

2.4. A refinement and reverse of Dragomir's result

The Hölder–McCarthy inequality was improved by Fujii and Nakamoto in [10, Theorem 2.3]. We borrow their result:

LEMMA 2.2. *For positive operator $A \in \mathscr{B}(\mathscr{H})$ and unit vector $u \in \mathscr{H}$ and $\lambda \geqslant 1$, we have*

$$M(\nu,\mu)\left(1-\left(\frac{\langle A^{\mu}u,u\rangle}{\langle Au,u\rangle^{\mu}}\right)^{\lambda}\right) \geqslant 1-\frac{\langle A^{\nu}u,u\rangle}{\langle Au,u\rangle^{\nu}} \geqslant m(\nu,\mu)\left(1-\left(\frac{\langle A^{\mu}u,u\rangle}{\langle Au,u\rangle^{\mu}}\right)^{\lambda}\right) \geqslant 0$$

for $0<\nu,\mu<1$ with $M(\nu,\mu):=\max\left\{\frac{1-\nu}{1-\mu},\frac{\nu}{\mu}\right\}$ and $m(\nu,\mu):=\min\left\{\frac{1-\nu}{1-\mu},\frac{\nu}{\mu}\right\}$.

Applying Lemma 2.2, we obtain the following result.

THEOREM 2.7. *Let $\lambda \geqslant 1$ and $T,V \in \mathscr{B}(\mathscr{H})$ with T invertible. For $p,q,p',q' > 1$ with $\frac{1}{p}+\frac{1}{q}=1=\frac{1}{p'}+\frac{1}{q'}$, $M(p,p'):=\max\left\{\frac{p'}{p},\frac{q'}{q}\right\}$, $m(p,p'):=\min\left\{\frac{p'}{p},\frac{q'}{q}\right\}$ and any $x \in \mathscr{H}$, we have*

$$M(p,p')\left(1-\left(\frac{\langle T \circledS_{\frac{1}{p'}} Vx,x\rangle}{\langle |V|^2x,x\rangle^{\frac{1}{p'}}\langle |T|^2x,x\rangle^{\frac{1}{q'}}}\right)^{\lambda}\right) \geqslant 1-\left(\frac{\langle T \circledS_{\frac{1}{p}} Vx,x\rangle}{\langle |V|^2x,x\rangle^{\frac{1}{p}}\langle |T|^2x,x\rangle^{\frac{1}{q}}}\right)^{\lambda}$$
$$\geqslant m(p,p')\left(1-\left(\frac{\langle T \circledS_{\frac{1}{p'}} Vx,x\rangle}{\langle |V|^2x,x\rangle^{\frac{1}{p'}}\langle |T|^2x,x\rangle^{\frac{1}{q'}}}\right)^{\lambda}\right) \geqslant 0.$$

Proof. The proof can be done in a similar way to the proof in [8, Theorem 1] with Lemma 2.2. If we take $u:=\frac{y}{\|y\|}$ in Lemma 2.2, then we have

$$M(\nu,\mu)\left(1-\left(\frac{\langle A^{\mu}y,y\rangle}{\langle Ay,y\rangle^{\mu}\langle y,y\rangle^{1-\mu}}\right)^{\lambda}\right) \geqslant 1-\left(\frac{\langle A^{\nu}y,y\rangle}{\langle Ay,y\rangle^{\nu}\langle y,y\rangle^{1-\nu}}\right)^{\lambda}$$
$$\geqslant m(\nu,\mu)\left(1-\left(\frac{\langle A^{\mu}y,y\rangle}{\langle Ay,y\rangle^{\mu}\langle y,y\rangle^{1-\mu}}\right)^{\lambda}\right) \geqslant 0.$$

If we put $A:=(T^*)^{-1}V^*VT$, $\nu:=\frac{1}{p}$ and $\mu:=\frac{1}{p'}$, then we have

$$M(p,p')\left(1-\left(\frac{\langle |VT^{-1}|^{\frac{2}{p'}}y,y\rangle}{\langle (T^*)^{-1}V^*VT^{-1}y,y\rangle^{\frac{1}{p'}}\langle y,y\rangle^{\frac{1}{q'}}}\right)^{\lambda}\right)$$
$$\geqslant \left(1-\left(\frac{\langle |VT^{-1}|^{\frac{2}{p}}y,y\rangle}{\langle (T^*)^{-1}V^*VT^{-1}y,y\rangle^{\frac{1}{p}}\langle y,y\rangle^{\frac{1}{q}}}\right)^{\lambda}\right)$$
$$\geqslant m(p,p')\left(1-\left(\frac{\langle |VT^{-1}|^{\frac{2}{p'}}y,y\rangle}{\langle (T^*)^{-1}V^*VT^{-1}y,y\rangle^{\frac{1}{p'}}\langle y,y\rangle^{\frac{1}{q'}}}\right)^{\lambda}\right) \geqslant 0.$$

Finally, taking $y := Tx$ and some calculations, we get the result. □

The second inequality and the first inequality in Theorem 2.7 with $\lambda = 1$ give the refinement and reverse for the result by Dragomir in [8, Theorem 1], respectively. For the case of $0 < \lambda \leqslant 1$, Fujii and Nakamoto obtained the inequality in [10, Theorem 2.4]. We can obtain the corresponding inequality by the use of it. However, we omit it.

3. Applications related to the numerical radius

This section presents some Schwarz inequalities for numerical radius via the quadratic mean.

It is well known that, for $X,Y \in \mathscr{B}(\mathscr{H})$, one has the following arithmetic-geometric mean inequality [3, Corollary IX. 4. 4]

$$\|Y^*X\| \leqslant \frac{1}{2}\|YY^* + XX^*\|. \tag{3.1}$$

Noting that, for any $T \in \mathscr{B}(\mathscr{H})$, we have $\omega(T) \leqslant \|T\|$. Thus it is evident that, for $X,Y \in \mathscr{B}(\mathscr{H})$,

$$\omega\left(Y^*X\right) \leqslant \frac{1}{2}\|YY^* + XX^*\|. \tag{3.2}$$

We remark that the inequality $\omega\left(Y^*X\right) \leqslant \frac{1}{2}\omega\left(YY^* + XX^*\right)$ does not hold in general. We refer the reader to [25, 27] for some related discussion. In the following result, we present a mixed form of (3.2) via the quadratic mean.

COROLLARY 3.1. Let $A,B,X,Y \in \mathscr{B}(\mathscr{H})$ with invertible A,B. Then for any $0 \leqslant \alpha,\beta \leqslant 1$ with $\alpha+\beta = 1$,

$$\omega\left(Y^*X\right) \leqslant \frac{1}{2}\left\|X^*\left(A \circledS_\alpha B\right)X + Y^*\left(A \circledS_\beta B\right)^{-1}Y\right\|,$$

and

$$\omega\left(Y^*X\right) \leqslant \left\|\left(A \circledS_\alpha B\right)^{\frac{1}{2}}X\right\|\left\|\left(A \circledS_\beta B\right)^{-\frac{1}{2}}Y\right\|.$$

Proof. Let $\Phi(X) = \langle Xx,x\rangle$ where $x \in \mathscr{H}$ is a unit vector, in Theorem 2.3. We infer from that

$$|\langle Y^*Xx,x\rangle| \leqslant \sqrt{\langle X^*\left(A \circledS_\alpha B\right)Xx,x\rangle\left\langle Y^*\left(A \circledS_\beta B\right)^{-1}Yx,x\right\rangle}. \tag{3.3}$$

Applying the arithmetic-geometric mean inequality implies that

$$\begin{aligned}
&\sqrt{\langle X^*\left(A \circledS_\alpha B\right)Xx,x\rangle\left\langle Y^*\left(A\Phi_\beta B\right)^{-1}Yx,x\right\rangle} \\
&\leqslant \frac{1}{2}\left\langle\left(X^*\left(A \circledS_\alpha B\right)X + Y^*\left(A\circledS_\beta B\right)^{-1}Y\right)x,x\right\rangle \\
&\leqslant \frac{1}{2}\left\|X^*\left(A \circledS_\alpha B\right)X + Y^*\left(A \circledS_\beta B\right)^{-1}Y\right\|.
\end{aligned}$$

Consequently,

$$|\langle Y^*Xx,x\rangle| \leqslant \frac{1}{2}\left\|X^*(A \circledS_\alpha B)X + Y^*\left(A \circledS_\beta B\right)^{-1}Y\right\|$$

for any unit vector $x \in \mathscr{H}$. By taking supremum over $x \in \mathscr{H}$ with $\|x\| = 1$, we obtain

$$\omega(Y^*X) \leqslant \frac{1}{2}\left\|X^*(A \circledS_\alpha B)X + Y^*\left(A \circledS_\beta B\right)^{-1}Y\right\|$$

as desired.

By taking supremum over $x \in \mathscr{H}$ with $\|x\| = 1$, in (3.3), we have

$$\omega(Y^*X) \leqslant \sqrt{\|X^*(A \circledS_\alpha B)X\|\left\|Y^*\left(A \circledS_\beta B\right)^{-1}Y\right\|}. \tag{3.4}$$

On the other hand,

$$\begin{aligned}\|X^*(A \circledS_\alpha B)X\| &= \left\|X^*(A \circledS_\alpha B)^{\frac{1}{2}}(A \circledS_\alpha B)^{\frac{1}{2}}X\right\| \\ &= \left\|X^*(A \circledS_\alpha B)^{\frac{1}{2}}\left(X^*(A \circledS_\alpha B)^{\frac{1}{2}}\right)^*\right\| \\ &= \left\|(A \circledS_\alpha B)^{\frac{1}{2}}X\right\|^2.\end{aligned} \tag{3.5}$$

Similarly,

$$\left\|Y^*\left(A \circledS_\beta B\right)^{-1}Y\right\| = \left\|\left(A \circledS_\beta B\right)^{-\frac{1}{2}}Y\right\|^2. \tag{3.6}$$

We reach the second inequality by combining the relations (3.4), (3.5), and (3.6). □

REMARK 3.1. If we chose $A, B = I$ in Corollary 3.1, we obtain

$$\omega(Y^*X) \leqslant \frac{1}{2}\|Y^*Y + X^*X\|. \tag{3.7}$$

Notice that this form complements the form presented in (3.2), which we found via (3.1).

Utilizing the same approach as in the proof of Corollary 3.1, we deduce the following result as an application of Theorem 2.4.

COROLLARY 3.2. Let $A, B, X, Y \in \mathscr{B}(\mathscr{H})$ with invertible A. Then for any $0 \leqslant \alpha, \beta \leqslant 1$ with $\alpha + \beta = 1$,

$$\omega(Y^*(A \circledS B)X) \leqslant \frac{1}{2}\left\|X^*(A \circledS_\alpha B)X + Y^*\left(A \circledS_\beta B\right)Y\right\|,$$

and

$$\omega(Y^*(A \circledS B)X) \leqslant \left\|(A \circledS_\alpha B)^{\frac{1}{2}}X\right\|\left\|\left(A \circledS_\beta B\right)^{\frac{1}{2}}Y\right\|.$$

COROLLARY 3.3. Let $A,B,X,Y \in \mathscr{B}(\mathscr{H})$ with invertible A. Then for any $0 \leqslant \alpha,\beta \leqslant 1$ with $\alpha+\beta=1$,

$$\omega\left(Y^*\left(A \circledS B\right)X\right) \leqslant \frac{\sqrt{\|A \circledS_\alpha B\| \left\|A \circledS_\beta B\right\|}}{2}\left\||X|^2+|Y|^2\right\|.$$

Proof. Let $x \in \mathscr{H}$ be a unit vector. We have by Theorem 2.4 that

$$\begin{aligned}
\left|\left\langle Y^*\left(A \circledS B\right)Xx,x\right\rangle\right| &\leqslant \sqrt{\left\langle X^*\left(A \circledS_\alpha B\right)Xx,x\right\rangle\left\langle Y^*\left(A \circledS_\beta B\right)Yx,x\right\rangle} \\
&= \sqrt{\left\langle\left(A \circledS_\alpha B\right)Xx,Xx\right\rangle\left\langle\left(A \circledS_\beta B\right)Yx,Yx\right\rangle} \\
&\leqslant \sqrt{\left\|\left(A \circledS_\alpha B\right)Xx\right\|\|Xx\|\left\|\left(A \circledS_\beta B\right)Yx\right\|\|Yx\|} \\
&\leqslant \sqrt{\|A \circledS_\alpha B\|\left\|A \circledS_\beta B\right\|}\,\|Xx\|\,\|Yx\| \\
&\leqslant \frac{\sqrt{\|A \circledS_\alpha B\|\left\|A \circledS_\beta B\right\|}}{2}\left\langle\left(|X|^2+|Y|^2\right)x,x\right\rangle \\
&\qquad \text{(by the arithmetic-geometric mean inequality)} \\
&\leqslant \frac{\sqrt{\|A \circledS_\alpha B\|\left\|A \circledS_\beta B\right\|}}{2}\left\||X|^2+|Y|^2\right\|
\end{aligned}$$

i.e.,

$$\left|\left\langle Y^*\left(A \circledS B\right)Xx,x\right\rangle\right| \leqslant \frac{\sqrt{\|A \circledS_\alpha B\|\left\|A \circledS_\beta B\right\|}}{2}\left\||X|^2+|Y|^2\right\|.$$

We get the desired result by taking supremum over $x \in \mathscr{H}$ with $\|x\|=1$. □

COROLLARY 3.4. Let $A,B,X,Y \in \mathscr{B}(\mathscr{H})$ with invertible A. Then for any $0 \leqslant \alpha,\beta \leqslant 1$ with $\alpha+\beta=1$,

$$\omega\left(Y^*\left(A \circledS B\right)X\right) \leqslant \frac{1}{2}\sqrt{\left\|X^*\left(A \circledS_\alpha B\right)^2X+Y^*\left(A \circledS_\beta B\right)^2Y\right\|\left\||X|^2+|Y|^2\right\|}.$$

Proof. Let $x \in \mathscr{H}$ be a unit vector. We have by Theorem 2.4 that

$$\begin{aligned}
&\left|\left\langle Y^*\left(A \circledS B\right)Xx,x\right\rangle\right| \\
&\leqslant \sqrt{\left\langle X^*\left(A \circledS_\alpha B\right)Xx,x\right\rangle\left\langle Y^*\left(A \circledS_\beta B\right)Yx,x\right\rangle} \\
&\leqslant \frac{1}{2}\left(\left\langle\left(A \circledS_\alpha B\right)Xx,Xx\right\rangle+\left\langle\left(A \circledS_\beta B\right)Yx,Yx\right\rangle\right) \\
&\qquad \text{(by the arithmetic-geometric mean inequality)} \\
&\leqslant \frac{1}{2}\left(\left\|\left(A \circledS_\alpha B\right)Xx\right\|\|Xx\|+\left\|\left(A \circledS_\beta B\right)Yx\right\|\|Yx\|\right)
\end{aligned}$$

$$= \frac{1}{2}\left(\sqrt{\left\langle X^*(A \circledS_\alpha B)^2 Xx,x\right\rangle \left\langle |X|^2 x,x\right\rangle} + \sqrt{\left\langle Y^*\left(A \circledS_\beta B\right)^2 Yx,x\right\rangle \left\langle |Y|^2 x,x\right\rangle}\right)$$

$$\leqslant \frac{1}{2}\left(\sqrt{\left\langle X^*(A \circledS_\alpha B)^2 Xx,x\right\rangle + \left\langle Y^*\left(A \circledS_\beta B\right)^2 Yx,x\right\rangle}\sqrt{\left\langle |X|^2 x,x\right\rangle + \left\langle |Y|^2 x,x\right\rangle}\right)$$

$$\leqslant \frac{1}{2}\sqrt{\left\|X^*(A \circledS_\alpha B)^2 X + Y^*\left(A \circledS_\beta B\right)^2 Y\right\| \left\||X|^2 + |Y|^2\right\|}.$$

In the second last inequality above, we used $\sqrt{ac}+\sqrt{bd} \leqslant \sqrt{(a+b)(c+d)}$ for $a,b,c,d > 0$. □

4. Concluding remarks

It is known the following famous theory by Kubo–Ando [15]. For any operator connection σ, there uniquely exists an operator monotone function $f : [0,\infty) \to [0,\infty)$ such that $f(x)I = I\sigma(xI)$ for $x \geqslant 0$. In addition, if $A \in \mathscr{B}(\mathscr{H})$ is invertible, then we have

$$A\sigma B = A^{1/2} f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

Furthermore, σ is an operator mean if and only if $f(1) = 1$.

For any operator connection σ and $A,B \geqslant 0$, we have

$$\Phi(A\sigma B) \leqslant \Phi(A)\sigma\Phi(B),$$

where $\Phi : \mathscr{H} \to \mathscr{K}$ is a positive linear map.

Considering whether a similar inequality holds for a generalized quadratic type mean may be interesting, which will be defined below.

For operator monotone function $f : [0,\infty) \to [0,\infty)$, we define the following

$$A\sigma^{\circledS} B := A^* f\left((A^*)^{-1}B^*BA^{-1}\right)A$$

for $A,B \in \mathscr{B}(\mathscr{H})$ with invertible A. Note that $\sigma^{\circledS}$ is not an operator connection, so it is not operator mean. Since $\sigma^{\circledS}$ is defined for all operators (including non-Hermitian), we can not consider the ordering so that it does not satisfy the joint monotonicity.

Then, we have the following relations.

PROPOSITION 4.1. *Let $\Phi : \mathscr{H} \to \mathscr{K}$ be a unital positive linear map, and let $A,B \in \mathscr{B}(\mathscr{H})$ with invertible A. Then we have*

$$\Phi(A\sigma^{\circledS} B) \leqslant \Phi(|A|^2)\sigma\Phi(|B|^2).$$

If we impose the condition on Φ such that Φ preserves product i.e., $\Phi(XY) = \Phi(X)\Phi(Y)$ for any $X,Y \in \mathscr{B}(\mathscr{H})$, then we have

$$\Phi(A\sigma^{\circledS} B) \leqslant \Phi(A)\sigma^{\circledS}\Phi(B).$$

Proof. Note that $\Phi(X^*) = \Phi(X)^*$ for any $X \in \mathscr{B}(\mathscr{H})$ and every positive linear map [4, Lemma 2.3.1]. Also considering $A_\varepsilon := A + \varepsilon I_{\mathscr{H}}$, $B_\varepsilon := B + \varepsilon I_{\mathscr{H}}$ and $\Phi_\varepsilon(X) := \Phi(X) + \varepsilon\langle Xx, x\rangle I_{\mathscr{K}}$ for a fixed $x \in \mathscr{H}$ with $x \neq 0$ and taking a limit $\varepsilon \searrow 0$, we may regard A, B and $\Phi(A), \Phi(B)$ as invertible. We define a unital positive linear map Φ_0 as

$$\Phi_0(X) := \Phi(|A|^2)^{-1/2}\Phi(A^*XA)\Phi(|A|^2)^{-1/2}.$$

Then we have

$$\Phi_0\left(f\left((A^*)^{-1}B^*BA^{-1}\right)\right) = \Phi(|A|^2)^{-1/2}\Phi\left(A\sigma^{\circledS}B\right)\Phi(|A|^2)^{-1/2}.$$

Using the opposite signed inequality of (1.2) since $-f$ is operator convex, we have

$$\begin{aligned}
\Phi\left(A\sigma^{\circledS}B\right) &= \Phi(|A|^2)^{1/2}\Phi_0\left(f\left((A^*)^{-1}B^*BA^{-1}\right)\right)\Phi(|A|^2)^{1/2} \\
&\leqslant \Phi(|A|^2)^{1/2}f\left(\Phi_0\left((A^*)^{-1}B^*BA^{-1}\right)\right)\Phi(|A|^2)^{1/2} \\
&= \Phi(|A|^2)^{1/2}f\left(\Phi(|A|^2)^{-1/2}\Phi\left(|B|^2\right)\Phi(|A|^2)^{-1/2}\right)\Phi(|A|^2)^{1/2} \\
&= \Phi(|A|^2)\sigma\Phi(|B|^2),
\end{aligned}$$

which shows the first statement. To show the second statement, we set a unital positive linear map Φ_1 as

$$\Phi_1(X) := \Phi(A^*)^{-1}\Phi(A^*XA)\Phi(A)^{-1}.$$

The map Φ_1 is assured to be unital by the condition that Φ preserves the product. Then we have

$$\Phi_1\left(f\left((A^*)^{-1}B^*BA^{-1}\right)\right) = \Phi(A^*)^{-1}\Phi(A\sigma^{\circledS}B)\Phi(A)^{-1}.$$

By the use of the preserving product, we thus have

$$\begin{aligned}
\Phi\left(A\sigma^{\circledS}B\right) &= \Phi(A)^*\Phi_1\left(f\left((A^*)^{-1}B^*BA^{-1}\right)\right)\Phi(A) \\
&\leqslant \Phi(A)^*f\left(\Phi_1\left((A^*)^{-1}B^*BA^{-1}\right)\right)\Phi(A) \\
&= \Phi(A)^*f\left(\Phi(A^*)^{-1}\Phi(B^*B)\Phi(A)^{-1}\right)\Phi(A) \\
&= \Phi(A)^*f\left(\Phi(A^*)^{-1}\Phi(B)^*\Phi(B)\Phi(A)^{-1}\right)\Phi(A) \\
&= \Phi(A)\sigma^{\circledS}\Phi(B). \quad \square
\end{aligned}$$

It is also natural to define the Tsallis relative operator entropy [12, Chapter 7] with the quadratic mean $\circledS_\nu$ instead of the geometric mean $\sharp_\nu$ as

$$D^{\circledS_\nu}(A|B) := \frac{A\circledS_\nu B - |A|^2}{\nu}, \quad (0 < \nu \leqslant 1)$$

for $A, B \in \mathscr{B}(\mathscr{H})$ with invertible A. It may be a considerable topic, but we do not state it here.

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Hamid Reza Moradi
Department of Mathematics
Mashhad Branch, Islamic Azad University
Mashhad, Iran
e-mail: hrmoradi@mshdiau.ac.ir

Shigeru Furuichi
Department of Information Science
College of Humanities and Sciences, Nihon University
Setagaya-ku, Tokyo, Japan
and
Department of Mathematics
Saveetha School of Engineering
SIMATS, Thandalam, Chennai – 602105, Tamilnadu, India
e-mail: furuichi.shigeru@nihon-u.ac.jp

Mohammad Sababheh
Department of Basic Sciences
Princess Sumaya University for Technology
Amman, Jordan
e-mail: sababheh@psut.edu.jo