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# SOME INEQUALITIES FOR WEIGHTED POWER MEAN

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*Abstract.* In this paper, we mainly present an inequality for weighted power mean, which extend a key result of I. H. Gümüş, S. Furuichi, H. R. Moradi and M. Sababheh. To be more precise,

$$A\sharp_{p,v}B \leqslant \frac{(m^p\nabla_\lambda M^p)^{\frac{1}{p}}}{m\sharp_\lambda M} A\sharp_v B,$$

where $p>0$, $v\in[0,1]$, $\lambda=\min\{v,1-v\}$ and $0<mI\leqslant A,B\leqslant MI$ for some scalars $m<M$. As applications, we obtain some inequalities for Hilbert-Schmidt norms.

## 1. Introduction

Let $(\mathbb{H},\langle\cdot,\cdot\rangle)$ be a complex Hilbert space and let $\mathbb{B}(\mathbb{H})$ denote the algebra of all bounded linear operators acting on $\mathbb{H}$. A self adjoint operator $A$ is said to be positive if $\langle Ax,x\rangle\geqslant 0$ for all $x\in\mathbb{H}$, while it is said to be strictly positive if $A$ is positive and invertible, denoted by $A\geqslant 0$ and $A>0$ respectively. In this paper, $A-B\geqslant 0$ means $A\geqslant B$. Moreover, we identify the matrix algebra $\mathbb{M}_n(\mathbb{C})$ of all $n\times n$ complex matrices with entries in the complex field $\mathbb{C}$ with the space of $\mathbb{B}(\mathbb{C}^n)$, and by positive definite matrices we mean the strictly positive operators on $\mathbb{B}(\mathbb{C}^n)$.

As usual, we define $v$-weighted arithmetic-geometric-harmonic means (AM-GM-HM) by

$$a\nabla_v b=(1-v)a+vb,\ \ a\sharp_v b=a^{1-v}b^v \ \text{ and } \ a!_v b=\left((1-v)a^{-1}+vb^{-1}\right)^{-1}$$

for $a,b>0$ and $v\in[0,1]$. Similarly, we denote the corresponding $v$-weighted operator AM-GM-HM as

$$A\nabla_v B=(1-v)A+vB,\ A\sharp_v B=A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v A^{\frac{1}{2}} \text{ and } A!_v B=\left((1-v)A^{-1}+vB^{-1}\right)^{-1}$$

for $A,B>0$ and $v\in[0,1]$. A more generalized $v$-weighted means is the weighted power mean defined by

$$a\sharp_{p,v}b=\left((1-v)a^p+vb^p\right)^{\frac{1}{p}}$$

for $a,b>0$, $p\neq 0$ and $v\in[0,1]$. The following proposition explained the weighted power mean is an increasing function:

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PROPOSITION 1.1. ([3] p. 26) *For* $a,b>0$, $v\in[0,1]$, *and* $p\neq 0$, *let* $M_p(a,b,v)=\left((1-v)a^p+vb^p\right)^{\frac{1}{p}}$ *and* $M_0(a,b,v)=a^{1-v}b^v$. *Then*

$$M_p(a,b,v)\leqslant M_s(a,b,v) \ \text{ for } \ p\leqslant s.$$

In this paper, we define the weighted operator power mean as follows: if $A,B>0$ and $v\in[0,1]$, then

$$A\sharp_{p,v}B=A^{\frac{1}{2}}\left((1-v)I+v(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^p\right)^{\frac{1}{p}}A^{\frac{1}{2}}$$

for $p\neq 0$; and

$$A\sharp_{0,v}B=A\sharp_v B.$$

It is easy to see that $A\sharp_{1,v}B=A\nabla_v B$ and $A\sharp_{-1,v}B=A!_vB$. Moreover, $A\sharp_{p,v}B=B\sharp_{p,1-v}A$ is consistent with the properties of $v$-weighted operator arithmetic-geometric-harmonic means.

In addition, the Kantorovich constant and the Specht's ratio are defined by

$$K(h)=\frac{(h+1)^2}{4h} \ \text{ for } h>0 \quad \text{ and } \quad S(h)=\begin{cases}\dfrac{h^{\frac{1}{h-1}}}{e\log\left(h^{\frac{1}{h-1}}\right)} & \text{if } \ h\in(0,1)\cup(1,\infty),\\ 1 & \text{if } \ h=1.\end{cases}$$

The $v$-weighted operator AM-GM inequality reads

$$A\sharp_v B\leqslant A\nabla_v B \tag{1.1}$$

for $A,B>0$ and $v\in[0,1]$. Tominaga [6] obtained a reverse of (1.1) with Specht's ratio

$$A\nabla_v B\leqslant S(h)A\sharp_v B, \tag{1.2}$$

where $0<mI\leqslant A,B\leqslant MI$, $h=\frac{M}{m}$, and $v\in[0,1]$. In 2015, Liao et al. [5] showed another reverse of (1.1) with Kantorovich constant

$$A\nabla_v B\leqslant K(h)^R A\sharp_v B, \tag{1.3}$$

where $0<mI\leqslant A\leqslant m'I<M'I\leqslant B\leqslant MI$ or $0<mI\leqslant B\leqslant m'I<M'I\leqslant A\leqslant MI$, $h=\frac{M}{m}$, $R=\max\{v,1-v\}$ and $v\in[0,1]$.

Recently, Furuichi et al. [1] and Gümüş et al. [2] showed

$$A\nabla_v B\leqslant \frac{m\nabla_\lambda M}{m\sharp_\lambda M}A\sharp_v B, \tag{1.4}$$

where $mI\leqslant A,B\leqslant MI$ for some scalars $0<m<M$, $\lambda=\min\{v,1-v\}$ and $v\in[0,1]$. Furthermore, they [2] also explained that (1.4) is better than the results of (1.2) and (1.3), and the constant $\frac{m\nabla_\lambda M}{m\sharp_\lambda M}$ is best possible.

In this paper, we shall present some weighted operator power mean inequalities, which extend the inequality (1.4). As applications, we obtain some inequalities for Hilbert-Schmidt norms.

## 2. Main results

We firstly give the weighted power mean inequalities as promised.

THEOREM 2.1. *Let* $A,B \in \mathbb{B}(\mathbb{H})$ *be such that* $0 < mI \leqslant A,B \leqslant MI$ *for some scalars* $m < M$. *For* $\lambda = \min\{v, 1-v\}$ *and* $v \in [0,1]$,
*if* $p > 0$, *then*

$$A\sharp_{p,v}B \leqslant \frac{\left(m^p \nabla_\lambda M^p\right)^{\frac{1}{p}}}{m\sharp_\lambda M} A\sharp_v B; \tag{2.1}$$

*if* $p < 0$, *then*

$$A\sharp_{p,v}B \geqslant \frac{\left(M^p \nabla_\lambda m^p\right)^{\frac{1}{p}}}{M\sharp_\lambda m} A\sharp_v B. \tag{2.2}$$

*Moreover, the inequalities are sharp.*

*Proof.* To proof the results, we define

$$f(x) = \frac{(1-v)+vx^p}{x^{pv}} \quad \text{for } x \in \left[\frac{m}{M}, \frac{M}{m}\right].$$

Then

$$f'(x) = pv(1-v)(x^p-1)x^{-pv-1}.$$

(1) if $p > 0$: then $f'(x) \geqslant 0$ when $x \geqslant 1$, which implies $f\left(\frac{M}{m}\right) \geqslant f(x)$; and $f'(x) \leqslant 0$ when $0 < x \leqslant 1$, which implies $f\left(\frac{m}{M}\right) \geqslant f(x)$. Therefore, $f(x) \leqslant \max\left\{f\left(\frac{m}{M}\right), f\left(\frac{M}{m}\right)\right\}$.

To compare $f\left(\frac{m}{M}\right)$ and $f\left(\frac{M}{m}\right)$, we let $h = \frac{M}{m} > 1$ and put

$$g(h) = \frac{(1-v)+vh^p}{h^{pv}} - \frac{(1-v)+v(\frac{1}{h})^p}{(\frac{1}{h})^{pv}} = \frac{(1-v)+vh^p}{h^{pv}} - \frac{(1-v)h^p+v}{h^{p(1-v)}}.$$

Direct calculations show that

$$g'(h) = pv(1-v)\frac{\left(h^{(2v-1)p}-1\right)(1-h^p)}{h^{pv+1}}.$$

So $g'(h) \geqslant 0$ if $v \in [0, \frac{1}{2}]$, which means $g(h) \geqslant g(1) = 0$; and $g'(h) \leqslant 0$ if $v \in [\frac{1}{2}, 1]$, which means $g(h) \leqslant g(1) = 0$. That is

$$\max_{x\in[\frac{m}{M},\frac{M}{m}]} f(x) = \begin{cases} f\left(\frac{M}{m}\right) = \frac{m^p \nabla_v M^p}{m^p \sharp_v M^p} & \text{for } 0 \leqslant v \leqslant \frac{1}{2}, \\ f\left(\frac{m}{M}\right) = \frac{M^p \nabla_v m^p}{M^p \sharp_v m^p} & \text{for } \frac{1}{2} \leqslant v \leqslant 1. \end{cases}$$

Then we have

$$\begin{cases} (1-v)+vx^p \leqslant \frac{m^p\nabla_v M^p}{m^p\sharp_v M^p}x^{pv} & \text{for } 0\leqslant v\leqslant \frac{1}{2}, \\ (1-v)+vx^p \leqslant \frac{M^p\nabla_v m^p}{M^p\sharp_v m^p}x^{pv} & \text{for } \frac{1}{2}\leqslant v\leqslant 1. \end{cases}$$

This is equivalent to

$$(1-v)+vx^p \leqslant \frac{m^p\nabla_\lambda M^p}{m^p\sharp_\lambda M^p}x^{pv}. \tag{2.3}$$

That is

$$\left((1-v)+vx^p\right)^{\frac{1}{p}} \leqslant \frac{\left(m^p\nabla_\lambda M^p\right)^{\frac{1}{p}}}{m\sharp_\lambda M}x^{v}. \tag{2.4}$$

By a standard functional calculus in the inequality (2.4) with $x = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, we obtain

$$\left((1-v)I+v\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^p\right)^{\frac{1}{p}} \leqslant \frac{\left(m^p\nabla_\lambda M^p\right)^{\frac{1}{p}}}{m\sharp_\lambda M}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{v}. \tag{2.5}$$

Multiplying $A^{\frac{1}{2}}$ to both sides of (2.5), we can obtain

$$A\sharp_{p,v}B \leqslant \frac{\left(m^p\nabla_\lambda M^p\right)^{\frac{1}{p}}}{m\sharp_\lambda M}A\sharp_v B.$$

(2) we use the same calculations as above to discuss the case of $p<0$: it is not difficult to find $f'(x)\geqslant 0$ when $x\geqslant 1$, and $f'(x)\leqslant 0$ when $0<x\leqslant 1$, respectively. Therefore, $f(x)\leqslant \max\left\{f\left(\frac{m}{M}\right), f\left(\frac{M}{m}\right)\right\}$.

Meanwhile, if $v\in[0,\frac{1}{2}]$, then $g'(h)\leqslant 0 \Rightarrow g(h)\leqslant g(1)=0$; if $v\in[\frac{1}{2},1]$, then $g'(h)\geqslant 0 \Rightarrow g(h)\geqslant g(1)=0$. That is

$$\max_{x\in[\frac{m}{M},\frac{M}{m}]} f(x) = \begin{cases} f\left(\frac{m}{M}\right) = \frac{M^p\nabla_v m^p}{M^p\sharp_v m^p} & \text{for } 0\leqslant v\leqslant\frac{1}{2}, \\ f\left(\frac{M}{m}\right) = \frac{m^p\nabla_v M^p}{m^p\sharp_v M^p} & \text{for } \frac{1}{2}\leqslant v\leqslant 1. \end{cases}$$

This is equivalent to

$$(1-v)+vx^p \leqslant \frac{M^p\nabla_\lambda m^p}{M^p\sharp_\lambda m^p}x^{pv}.$$

That is

$$\left((1-v)+vx^p\right)^{\frac{1}{p}} \geqslant \frac{\left(M^p\nabla_\lambda m^p\right)^{\frac{1}{p}}}{M\sharp_\lambda m}x^{v}.$$

Using the same technique as (2.4), we can get (2.2).

The sharpness of (2.1) comes from $A=m$, $B=M$ when $v\in[0,\frac{1}{2}]$; and $A=M$, $B=m$ when $v\in[\frac{1}{2},1]$. On the other hand, the sharpness of (2.2) due to $A=M$, $B=m$ when $v\in[0,\frac{1}{2}]$; and $A=m$, $B=M$ when $v\in[\frac{1}{2},1]$. □

REMARK 2.2. We can get the inequality (1.4) by (2.1) when $p=1$.

Khosravi [4] presented

$$A\sharp_{p,v}B \leqslant A\sharp_{q,v}B \quad \text{for} \quad -1 \leqslant p \leqslant q \leqslant 1. \tag{2.6}$$

So, when $0<p\leqslant 1$, it is easy to see that $A\sharp_{p,v}B \leqslant A\nabla_v B$ and $\frac{\left(m^p\nabla_\lambda M^p\right)^{\frac{1}{p}}}{m\sharp_\lambda M} \leqslant \frac{m\nabla_\lambda M}{m\sharp_\lambda M}$ (by Proposition 1.1), which implies that neither (2.1) nor (1.4) is uniformly better than the other under some conditions. On the other hand, when $-1\leqslant p<0$, then $\frac{\left(M^p\nabla_\lambda m^p\right)^{\frac{1}{p}}}{M\sharp_\lambda m} < 1$ and $A\sharp_{p,v}B \leqslant A\sharp_v B$, that is to say the inequality (2.2) is a refinement about operator geometric mean to harmonic mean. Especially, if $p=-1$ in (2.2), then we obtain a reverse of operator geometric-harmonic mean inequality. Therefore, Theorem 2.1 is a new generalized $v$-weighted operator means inequality.

Some reverses of Theorem 2.1 are as follows:

COROLLARY 2.3. *Let $A,B\in\mathbb{B}(\mathbb{H})$ be such that $0<mI\leqslant A,B\leqslant MI$ for some scalars $m<M$. For $\lambda=\min\{v,1-v\}$ and $v\in[0,1]$,*
*if $p>0$, then*

$$A\sharp_v B \leqslant \frac{\left(m^p\nabla_\lambda M^p\right)^{\frac{1}{p}}}{m\sharp_\lambda M} A\sharp_{-p,v}B; \tag{2.7}$$

*if $p<0$, then*

$$\frac{\left(M^p\nabla_\lambda m^p\right)^{\frac{1}{p}}}{M\sharp_\lambda m} A\sharp_{-p,v}B \leqslant A\sharp_v B. \tag{2.8}$$

*Proof.* Let $A=A^{-1}$ and $B=B^{-1}$ in (2.1). Then

$$A^{-1}\sharp_{p,v}B^{-1} \leqslant \frac{\left[\left(\frac{1}{M}\right)^p\nabla_\lambda\left(\frac{1}{m}\right)^p\right]^{\frac{1}{p}}}{\left(\frac{1}{M}\right)\sharp_\lambda\left(\frac{1}{m}\right)} A^{-1}\sharp_v B^{-1},$$

that is

$$\frac{\left[\left(\frac{1}{M}\right)^p\nabla_\lambda\left(\frac{1}{m}\right)^p\right]^{\frac{1}{p}}}{\left(\frac{1}{M}\right)\sharp_\lambda\left(\frac{1}{m}\right)} \left(A^{-1}\sharp_{p,v}B^{-1}\right)^{-1} \geqslant \left(A^{-1}\sharp_v B^{-1}\right)^{-1},$$

which is equivalent to (2.7). We can similarly obtain (2.8) by (2.2). □

To avoid repetition of the article, the rest of this paper only provides results related to factor $\frac{\left(m^p\nabla_\lambda M^p\right)^{\frac{1}{p}}}{m\sharp_\lambda M}$.

Next, we show the double-sided inequality involving the operator weighted power mean and $v$-weighted geometric mean.

COROLLARY 2.4. *Let* $\lambda=\min\{v,1-v\}$ *for* $v\in[0,1]$ *and* $A,B\in\mathbb{B}(\mathbb{H})$ *be such that* $0<mI\leqslant A,B\leqslant MI$ *for some scalars* $m<M$.
*If* $p>0$, *then*

$$A\sharp_{p,v}B-M\left(\frac{\left(m^p\nabla_\lambda M^p\right)^{\frac{1}{p}}}{m\sharp_\lambda M}-1\right)I\leqslant A\sharp_v B;$$

*If* $0<p\leqslant 1$, *then*

$$A\sharp_v B\leqslant A\sharp_{-p,v}B+M\left(\frac{\left(m^p\nabla_\lambda M^p\right)^{\frac{1}{p}}}{m\sharp_\lambda M}-1\right)I.$$

*Proof.* Notice that if $0<mI\leqslant A,B\leqslant MI$, then

$$mI=m(I\sharp_v I)=\big((mI)\sharp_v(mI)\big)\leqslant A\sharp_v B\leqslant\big((MI)\sharp_v(MI)\big)=MI. \tag{2.9}$$

If $p>0$, we have

$$\begin{aligned}A\sharp_{p,v}B-A\sharp_v B&\leqslant\left(\frac{\left(m^p\nabla_\lambda M^p\right)^{\frac{1}{p}}}{m\sharp_\lambda M}-1\right)A\sharp_v B\quad(\text{by } 2.1)\\&\leqslant M\left(\frac{\left(m^p\nabla_\lambda M^p\right)^{\frac{1}{p}}}{m\sharp_\lambda M}-1\right)I\quad\big(\text{by } (2.9)\big).\end{aligned}$$

If $0<p\leqslant 1$, then

$$\begin{aligned}A\sharp_v B-A\sharp_{-p,v}B&\leqslant\left(\frac{\left(m^p\nabla_\lambda M^p\right)^{\frac{1}{p}}}{m\sharp_\lambda M}-1\right)A\sharp_{-p,v}B\quad(\text{by } (2.7))\\&\leqslant\left(\frac{\left(m^p\nabla_\lambda M^p\right)^{\frac{1}{p}}}{m\sharp_\lambda M}-1\right)A\sharp_v B\quad(\text{by } 2.6)\\&\leqslant M\left(\frac{\left(m^p\nabla_\lambda M^p\right)^{\frac{1}{p}}}{m\sharp_\lambda M}-1\right)I\quad\big(\text{by } (2.9)\big).\quad\Box\end{aligned}$$

THEOREM 2.5. *Let* $A,B,X\in\mathbb{M}_n(\mathbb{C})$ *and* $A$, $B$ *be positive definite matrices such that* $0<mI\leqslant A,B\leqslant MI$ *for some scalars* $m<M$. *Then*

$$\left|\left|(1-v)A^pX+vXB^p\right|\right|_2\leqslant\frac{m^p\nabla_\lambda M^p}{m^p\sharp_\lambda M^p}\left|\left|A^{p(1-v)}XB^{pv}\right|\right|_2,$$

*where* $p>0$ *and* $\lambda=\min\{v,1-v\}$ *for* $v\in[0,1]$.

*Proof.* Set $x=\frac{b}{a}$ in (2.3), we get

$$(1-v)a^p+vb^p\leqslant\frac{m^p\nabla_\lambda M^p}{m^p\sharp_\lambda M^p}a^{p(1-v)}b^{pv}. \tag{2.10}$$

Let $U$ and $V$ be unitary matrices such that $A = U\mathrm{diag}(\lambda_i)U^*$ and $B = V\mathrm{diag}(\mu_i)V^*$ are spectral decompositions of $A$ and $B$. Furthermore, let $Y = U^*XV$. Then we have

$$\begin{aligned}
\left|\left|(1-v)A^pX + vXB^p\right|\right|_2^2 &= \left|\left|U\left((1-v)\mathrm{diag}(\lambda_i^p)Y + vY\mathrm{diag}(\mu_i^p)\right)V^*\right|\right|_2^2 \\
&= \left|\left|\left[(1-v)\lambda_i^p + v\mu_j^p\right] \circ [y_{ij}]\right|\right|_2^2 \\
&= \sum_{i,j=1}^{n} \left((1-v)\lambda_i^p + v\mu_j^p\right)^2 |y_{ij}|^2 \\
&\leqslant \left(\frac{m^p\nabla_\lambda M^p}{m^p\sharp_\lambda M^p}\right)^2 \sum_{i,j=1}^{n} \left(\lambda_i^{p(1-v)}\mu_j^{pv}\right)^2 |y_{ij}|^2 \qquad \text{(by (2.10))} \\
&= \left(\frac{m^p\nabla_\lambda M^p}{m^p\sharp_\lambda M^p}\right)^2 \left|\left|A^{p(1-v)}XB^{pv}\right|\right|_2^2. \quad \square
\end{aligned}$$

REMARK 2.6. When $p = 1$ in Theorem 2.5, we get

$$\left|\left|(1-v)AX + vXB\right|\right|_2 \leqslant \frac{m\nabla_\lambda M}{m\sharp_\lambda M}\left|\left|A^{1-v}XB^v\right|\right|_2,$$

which is a reverse Young-type inequality for Hilbert-Schmidt norms.

A generalized reverse of the Heinz inequality for Hilbert-Schmidt norms is as follows.

COROLLARY 2.7. *Under the same conditions as in Theorem 2.5, we have*

$$\left|\left|A^pX + XB^p\right|\right|_2 \leqslant \frac{m^p\nabla_\lambda M^p}{m^p\sharp_\lambda M^p}\left|\left|A^{p(1-v)}XB^{pv} + A^{pv}XB^{p(1-v)}\right|\right|_2.$$

*Proof.* Replace $a$ with $b$ and $b$ with $a$ in (2.10) respectively, then

$$va^p + (1-v)b^p \leqslant \frac{m^p\nabla_\lambda M^p}{m^p\sharp_\lambda M^p}a^{pv}b^{p(1-v)}. \tag{2.11}$$

Combination (2.10) and (2.11), we have

$$a^p + b^p \leqslant \frac{m^p\nabla_\lambda M^p}{m^p\sharp_\lambda M^p}\left(a^{p(1-v)}b^{pv} + a^{pv}b^{p(1-v)}\right).$$

Using the same technique as in Theorem 2.5, we complete the proof. $\square$

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## REFERENCES


[1] S. FURUICHI, H. R. MORADI, M. SABABHEH, *New sharp inequalities for operator means*, Linear Multilinear Algebra, **67** (2019) 1567–1578.

[2] I. H. GÜMÜŞ, H. R. MORADI, M. SABABHEH, *More accurate operator means inequalities*, J. Math. Anal. Appl., **465** (2018) 267–280.

[3] G. H. HARDY, J. E. LITTLEWOOD, G. PÓLYA, *Inequalities, 2nd ed.*, Cambridge Univ. Press, Cambridge, 1988.

[4] M. KHOSRAVI, *Some martix inequalities for weighted power mean*, Ann. Funct., Anal., **7** (2016) 348–357.

[5] W. LIAO, J. WU, J. ZHAO, *New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant*, Taiwanese J. Math., **19** (2015) 467–479.

[6] M. TOMINAGA, *Specht's ratio in the Young inequality*, Sci. Math. Japon., **55** (2002) 583–588.


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