Journal of Mathematical Inequalities
Volume 18, Number 4 (2024), 1289–1301 doi:10.7153/jmi-2024-18-74

INEQUALITIES INVOLVING BEREZIN NORM AND BEREZIN NUMBER OF HILBERT SPACE OPERATORS

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(*Communicated by M. Sababheh*)

Abstract. This paper presents several Berezin number and norm inequalities for Hilbert space operators. These inequalities improve some earlier related inequalities. Among other inequalities, it is shown that if A is a bounded linear operator on a Hilbert space, then

$$\mathbf{ber}^2(A) \leqslant \left\| \frac{A^*A+AA^*}{2} - \frac{1}{2R}\left((1-t)A^*A+tAA^* - \left((1-t)(A^*A)^{\frac{1}{2}} + t(AA^*)^{\frac{1}{2}}\right)^2\right)\right\|_{\mathbf{ber}}$$

where $R = \max\{t, 1-t\}$ and $0 \leqslant t \leqslant 1$.

1. Introduction

Let $(\mathbb{H}; \langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The numerical range of an operator A is the subset of the complex numbers $\mathbb{C}$ given by:

$$W(A) = \{\langle Ax, x\rangle : x \in \mathbb{H}, \|x\| = 1\}.$$

The numerical radius of an operator A on $\mathbb{H}$ is shown by:

$$\omega(A) = \sup\{|\langle Ax, x\rangle| : x \in \mathbb{H}, \|x\| = 1\}.$$

It is well-known that $\omega(\cdot)$ is a norm on the Banach algebra $\mathbb{B}(\mathbb{H})$ of all bounded linear operators $A : \mathbb{H} \to \mathbb{H}$. This norm is equivalent to the operator norm. The following more precise result holds:

$$\frac{1}{2}\|A\| \leqslant \omega(A) \leqslant \|A\|. \tag{1.1}$$

Kittaneh has established in [20] that if $A \in \mathbb{B}(\mathbb{H})$,

$$\omega^2(A) \leqslant \frac{1}{2}\left\| |A|^2 + |A^*|^2 \right\| \tag{1.2}$$

Mathematics subject classification (2020): Primary 47A12; Secondary 47A30, 47A63.
Keywords and phrases: Berezin norm, Berezin number, bounded linear operators, numerical radius, operator norm, inequality.

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where $|A| = (A^*A)^{1/2}$. The inequality (1.2) is stronger than the second inequality in (1.1). This can be noticed by using the fact that

$$\frac{1}{2}\left\| |A|^2 + |A^*|^2 \right\| \leqslant \frac{1}{2}\left\| |A|^2 \right\| + \frac{1}{2}\left\| |A^*|^2 \right\| = \|A\|^2.$$

Interestingly, inequality (1.2) has been refined in [21] and [26] by using the operator Hermite-Hadamard inequality. Moreover, In [8], El-Haddad and Kittaneh generalize (1.2) in the following form

$$\omega^{2r}(A) \leqslant \left\| (1-t)|A|^{2r} + t|A^*|^{2r} \right\|, \ 0 \leqslant t \leqslant 1, \ r \geqslant 1.$$

For recent and interesting results regarding inequalities for the numerical radius, see [17, 22, 23, 27, 28].

A functional Hilbert space $\mathbb{H} = \mathbb{H}(\Omega)$ is a Hilbert space of complex-valued functions on a (nonempty) set Ω, which has the property that point evaluations are continuous, i.e., for each $\lambda \in \Omega$ the map $f \mapsto f(\lambda)$ is a continuous linear functional on $\mathbb{H}$. The Riesz representation theorem ensure that for each $\lambda \in \Omega$ there is a unique element $k_\lambda \in \mathbb{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathbb{H}$. The collection $\{k_\lambda : \lambda \in \Omega\}$ is called the reproducing kernel of $\mathbb{H}$. If $\{e_n\}$ is an orthonormal basis for a functional Hilbert space $\mathbb{H}$, then the reproducing kernel of $\mathbb{H}$ is given by $k_\lambda(z) = \sum_n \overline{e_n(\lambda)} e_n(z)$; (see [14, problem 37]). For $\lambda \in \Omega$, let $\hat{k_\lambda} = \frac{k_\lambda}{\|k_\lambda\|}$ be the normalized reproducing kernel of $\mathbb{H}$. For a bounded linear operator T on $\mathbb{H}$, the function $\widetilde{T}$ defined on Ω by $\widetilde{T}(\lambda) = \langle T\hat{k_\lambda}, \hat{k_\lambda} \rangle$ is the Berezin symbol of T, which firstly have been introduced by Berezin [3, 4]. Berezin set and Berezin number of the operator, T, are defined by

$$\mathbf{Ber}(T) := \{\widetilde{T}(\lambda) : \lambda \in \Omega\} \qquad \text{and} \qquad \mathbf{ber}(T) := \sup\{|\widetilde{T}(\lambda)| : \lambda \in \Omega\},$$

respectively, (see [16]). Of course, the Berezin norm of T can also be defined as follows:

$$\|T\|_{\mathbf{ber}} = \sup\left\{ \left| \left\langle T\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle \right| : \ \lambda, \mu \in \Omega \right\}.$$

We understand that

$$\mathbf{ber}(T) \leqslant \omega(T).$$

Moreover, the Berezin number and the Berezing norm of an operator T satisfies the following properties:

(i) $\mathbf{ber}(\alpha T) = |\alpha|\mathbf{ber}(T)$ for all $\alpha \in \mathbb{C}$.

(ii) $\mathbf{ber}(S+T) \leqslant \mathbf{ber}(S) + \mathbf{ber}(T)$.

(iii) $\|S+T\|_{\mathbf{ber}} \leqslant \|S\|_{\mathbf{ber}} + \|T\|_{\mathbf{ber}}$.

(iv) [6, Proposition 2.11] $\|T\|_{\mathbf{ber}} = \mathbf{ber}(T)$, whenever T is positive.

The Berezin symbol has been thoroughly examined for the Toeplitz and Hankel operators on the Hardy and Bergman spaces; it is extensively employed in various analytical inquiries and exclusively characterizes the operator (i.e., for all $\lambda \in \Omega, \widetilde{T}(\lambda) =$

$\widetilde{S}(\lambda)$ implies $T = S$). For some recent articles, including Berezin number inequalities, we refer the interested reader to [10, 11, 12, 13, 29].

This article aims to present further generalizations of Berezin number inequalities. We will employ certain approaches to achieve this objective, as suggested in [21, 26].

In order to achieve the goal of this section, we need the following lemmas.

LEMMA 1.1. [18] *If $A, B \in \mathbb{B}(\mathbb{H})$ are positive operators, then*

$$\|A+B\| \leqslant \frac{1}{2}\left(\|A\|+\|B\|+\sqrt{(\|A\|-\|B\|)^2+4\left\|A^{\frac{1}{2}}B^{\frac{1}{2}}\right\|^2}\right).$$

LEMMA 1.2. (Buzano's inequality [7]) *If a, b, x are vectors in an inner product space, then*

$$|\langle a,x\rangle|\,|\langle x,b\rangle| \leqslant \frac{\|a\|\,\|b\|+|\langle a,b\rangle|}{2}\|x\|^2.$$

LEMMA 1.3. [19] *Let $A \in \mathbb{B}(\mathbb{H})$ and let $x, y \in \mathbb{H}$ be any vectors. If $0 \leqslant t \leqslant 1$,*

$$|\langle Ax,y\rangle|^2 \leqslant \left\langle |A|^{2(1-t)}x,x\right\rangle \left\langle |A^*|^{2t}y,y\right\rangle.$$

LEMMA 1.4. *Let $A \in \mathbb{B}(\mathbb{H})$ be a self-adjoint operator, and let $x \in \mathbb{H}$ be a unit vector. Then,*

$$\langle Ax,x\rangle^2 \leqslant \left\langle A^2x,x\right\rangle.$$

Proof. Utilizing the Cauchy-Schwarz inequality, we have

$$\langle Ax,x\rangle^2 \leqslant \|Ax\|^2 = \langle Ax,Ax\rangle = \left\langle A^2x,x\right\rangle. \quad \square$$

2. Results

Now, we can prove our general Berezin number inequality, which includes several inequalities as special cases.

THEOREM 2.1. *If $A, B \in \mathbb{B}(\mathbb{H}(\Omega))$, then*

$$\begin{aligned}\mathbf{ber}^2(A+B) \leqslant{}& \frac{1}{4}\left\|3\left(|A|^2+|B^*|^2\right)+|A^*|^2+|B|^2\right\|_{\mathbf{ber}}\\&+\frac{1}{2}\left(\mathbf{ber}\left(A^2\right)+\mathbf{ber}\left(B^2\right)\right)+\mathbf{ber}(BA).\end{aligned}$$

Proof. Let $\widehat{k}_\lambda$ be a normalized reproducing kernel. Replacing $a = S\widehat{k}_\lambda$, $b = T\widehat{k}_\lambda$, in Lemma 1.2, then

$$\begin{aligned}2\left|\left\langle S\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\left|\left\langle \widehat{k}_\lambda,T\widehat{k}_\lambda\right\rangle\right| &= 2\left|\left\langle S\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\left|\left\langle T^*\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\\&\leqslant \left\|S\widehat{k}_\lambda\right\|\left\|T\widehat{k}_\lambda\right\|+\left|\left\langle S\widehat{k}_\lambda,T\widehat{k}_\lambda\right\rangle\right|\end{aligned}$$

$$= \left\| S\widehat{k}_\lambda \right\| \left\| T\widehat{k}_\lambda \right\| + \left| \left\langle T^* S\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|$$
$$= \sqrt{\left\langle S\widehat{k}_\lambda, S\widehat{k}_\lambda \right\rangle \left\langle T\widehat{k}_\lambda, T\widehat{k}_\lambda \right\rangle} + \left| \left\langle T^* S\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|$$
$$= \sqrt{\left\langle |S|^2\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \left\langle |T|^2\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle} + \left| \left\langle T^* S\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|$$
$$\leqslant \frac{1}{2} \left(\left\langle |S|^2\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \left\langle |T|^2\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right) + \left| \left\langle T^* S\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|$$
(by the arithmetic-geometric mean inequality)
$$= \frac{1}{2} \left\langle \left(|S|^2 + |T|^2 \right) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \left| \left\langle T^* S\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|.$$

Observe that

$$\left| \left\langle S\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| \left| \left\langle T^*\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| \leqslant \frac{1}{4} \left\langle \left(|S|^2 + |T|^2 \right) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \frac{1}{2} \left| \left\langle T^* S\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|. \tag{2.1}$$

In particular,

$$\left| \left\langle S\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^2 \leqslant \frac{1}{4} \left\langle \left(|S|^2 + |S^*|^2 \right) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \frac{1}{2} \left| \left\langle S^2\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|. \tag{2.2}$$

So, we have

$$\left| \left\langle (A+B)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^2$$
$$\leqslant \left(\left| \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| + \left| \left\langle B\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| \right)^2 \quad \text{(by the triangle inequality)}$$
$$= \left| \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^2 + \left| \left\langle B\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^2 + 2 \left| \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| \left| \left\langle B\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|$$
$$\leqslant \frac{1}{4} \left\langle \left(|A|^2 + |A^*|^2 \right) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \frac{1}{2} \left| \left\langle A^2\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|$$
$$+ \frac{1}{4} \left\langle \left(|B|^2 + |B^*|^2 \right) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \frac{1}{2} \left| \left\langle B^2\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|$$
$$+ \frac{1}{2} \left\langle \left(|A|^2 + |B^*|^2 \right) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \left| \left\langle BA\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| \quad \text{(by (2.1) and (2.2))}$$
$$= \left\langle \left(\frac{3\left(|A|^2 + |B^*|^2 \right) + |A^*|^2 + |B|^2}{4} \right) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle$$
$$+ \frac{1}{2} \left(\left| \left\langle A^2\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| + \left| \left\langle B^2\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| \right) + \left| \left\langle BA\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|$$
$$\leqslant \frac{1}{4} \left\| 3\left(|A|^2 + |B^*|^2 \right) + |A^*|^2 + |B|^2 \right\|_{\mathbf{ber}} + \frac{1}{2} \left(\mathbf{ber}\left(A^2\right) + \mathbf{ber}\left(B^2\right) \right) + \mathbf{ber}(BA).$$

Thus,

$$\left| \left\langle (A+B)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^2$$
$$\leqslant \frac{1}{4} \left\| 3\left(|A|^2 + |B^*|^2 \right) + |A^*|^2 + |B|^2 \right\|_{\mathbf{ber}} + \frac{1}{2} \left(\mathbf{ber}\left(A^2\right) + \mathbf{ber}\left(B^2\right) \right) + \mathbf{ber}(BA),$$

and so

$$\mathbf{ber}^2(A+B) \leqslant \frac{1}{4}\left\|3\left(|A|^2+|B^*|^2\right)+|A^*|^2+|B|^2\right\|_{\mathbf{ber}} + \frac{1}{2}\left(\mathbf{ber}\left(A^2\right)+\mathbf{ber}\left(B^2\right)\right)+\mathbf{ber}(BA),$$

as required. □

As an immediate consequence of the preceding theorem, we improve the second inequality in (1.1).

COROLLARY 2.1. If $A \in \mathbb{B}(\mathbb{H}(\Omega))$, then

$$\mathbf{ber}^2(A) \leqslant \frac{1}{4}\left\||A|^2+|A^*|^2\right\|_{\mathbf{ber}} + \frac{1}{2}\mathbf{ber}\left(A^2\right).$$

Proof. Letting $A = B$ in Theorem 2.1. □

Also, the following result is of attraction in itself.

COROLLARY 2.2. If $A, B \in \mathbb{B}(\mathbb{H}(\Omega))$ are two positive operators, then

$$\|A+B\|_{\mathbf{ber}}^2 \leqslant \left\|A^2+B^2\right\|_{\mathbf{ber}} + \frac{1}{2}\left(\|A\|_{\mathbf{ber}}^2+\|B\|_{\mathbf{ber}}^2\right)+\mathbf{ber}(BA).$$

We derive the following new bound employing Lemmas 1.3 and 1.4.

THEOREM 2.2. *Let* $A \in \mathbb{B}(\mathbb{H}(\Omega))$ *and let* $0 \leqslant t \leqslant 1$. *Then*

$$\mathbf{ber}^4(A) \leqslant \left\|(1-t)|A|^4+t|A^*|^4-2r\left(\frac{|A|^4+|A^*|^4}{2}-\left(\frac{|A|^2+|A^*|^2}{2}\right)^2\right)\right\|_{\mathbf{ber}},$$

where $r = \min\{t, 1-t\}$.

Proof. If $0 \leqslant t \leqslant 1/2$, then we have

$$\begin{aligned}
&\left((1-t)|A|^2+t|A^*|^2\right)^2 \\
&= \left((1-2t)|A|^2+2t\left(\frac{|A|^2+|A^*|^2}{2}\right)\right)^2 \\
&\leqslant (1-2t)|A|^4+2t\left(\frac{|A|^2+|A^*|^2}{2}\right)^2 \\
&\qquad \text{(since } f(t)=t^2 \text{ is operator convex; see Theorem 1.5.8 in [5])} \\
&= (1-t)|A|^4+t|A^*|^4-2r\left(\frac{|A|^4+|A^*|^4}{2}-\left(\frac{|A|^2+|A^*|^2}{2}\right)^2\right).
\end{aligned}$$

A similar discussion holds for $1/2 \leqslant t \leqslant 1$. So we obtain

$$\left((1-t)|A|^2+t|A^*|^2\right)^2 \leqslant (1-t)|A|^4+t|A^*|^4-2r\left(\frac{|A|^4+|A^*|^4}{2}-\left(\frac{|A|^2+|A^*|^2}{2}\right)^2\right).$$

Now, let $\widehat{k}_\lambda$ be a normalized reproducing kernel. We can write

$$\begin{aligned}
\left|\left\langle A\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|^4 &\leqslant \left(\left\langle |A|^{2(1-t)}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\left\langle |A^*|^{2t}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right)^2 \quad \text{(by Lemma 1.3)}\\
&\leqslant \left(\left\langle |A|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle^{1-t}\left\langle |A^*|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle^{t}\right)^2 \quad \text{(by [8, Lemma 3])}\\
&\leqslant \left((1-t)\left\langle |A|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+t\left\langle |A^*|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right)^2\\
&\qquad \text{(by the weighted arithmetic-geometric mean inequality)}\\
&= \left\langle\left((1-t)|A|^2+t|A^*|^2\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle^2\\
&\leqslant \left\langle\left((1-t)|A|^2+t|A^*|^2\right)^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle \quad \text{(by Lemma 1.4)}
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left|\left\langle A\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|^4\\
&\leqslant \left\langle\left((1-t)|A|^4+t|A^*|^4-2r\left(\frac{|A|^4+|A^*|^4}{2}-\left(\frac{|A|^2+|A^*|^2}{2}\right)^2\right)\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle \qquad (2.3)\\
&\leqslant \left\|(1-t)|A|^4+t|A^*|^4-2r\left(\frac{|A|^4+|A^*|^4}{2}-\left(\frac{|A|^2+|A^*|^2}{2}\right)^2\right)\right\|_{\mathbf{ber}}.
\end{aligned}$$

Therefore, from (2.3) it follows that

$$\mathbf{ber}^4(A) \leqslant \left\|(1-t)|A|^4+t|A^*|^4-2r\left(\frac{|A|^4+|A^*|^4}{2}-\left(\frac{|A|^2+|A^*|^2}{2}\right)^2\right)\right\|_{\mathbf{ber}}. \quad \square$$

REMARK 2.1. Since $f(t)=t^2$ is an operator convex on $(0,\infty)$, we have

$$\frac{|A|^4+|A^*|^4}{2}-\left(\frac{|A|^2+|A^*|^2}{2}\right)^2 \geqslant 0,$$

therefore, we have

$$\begin{aligned}
&\left\|(1-t)|A|^4+t|A^*|^4-2r\left(\frac{|A|^4+|A^*|^4}{2}-\left(\frac{|A|^2+|A^*|^2}{2}\right)^2\right)\right\|_{\mathbf{ber}}\\
&\leqslant \left\|(1-t)|A|^4+t|A^*|^4\right\|_{\mathbf{ber}}.
\end{aligned}$$

We conclude this section with the following result. The reader may compare it to [28, Theorem 2.3].

THEOREM 2.3. *Let* $A \in \mathbb{B}(\mathbb{H}(\Omega))$ *and let* $0 \leqslant t \leqslant 1$. *Then*

$$\mathbf{ber}^2(A) \leqslant \left\| \frac{|A|^2+|A^*|^2}{2} - \frac{1}{2R}\left((1-t)|A|^2+t|A^*|^2-((1-t)|A|+t|A^*|)^2\right) \right\|_{\mathbf{ber}},$$

where $R = \max\{t, 1-t\}$.

Proof. By Lemma 3.12 in [24], we have

$$(1-t)|A|^2+t|A^*|^2 \leqslant ((1-t)|A|+t|A^*|)^2 + 2R\left(\frac{|A|^2+|A^*|^2}{2} - \left(\frac{|A|+|A^*|}{2}\right)^2\right),$$

where $0 \leqslant t \leqslant 1$ and $R = \max\{t, 1-t\}$. This inequality gives

$$\left(\frac{|A|+|A^*|}{2}\right)^2 \leqslant \frac{|A|^2+|A^*|^2}{2} - \frac{1}{2R}\left((1-t)|A|^2+t|A^*|^2-((1-t)|A|+t|A^*|)^2\right). \tag{2.4}$$

Now, let $\widehat{k}_\lambda$ be a normalized reproducing kernel. We obtain

$$\begin{aligned}
\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right|^2 &\leqslant \left\langle |A|\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \left\langle |A^*|\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \quad \text{(by Lemma 1.3)} \\
&\leqslant \left(\frac{\left\langle |A|\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \left\langle |A^*|\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle}{2}\right)^2 \\
&\quad \text{(by the arithmetic-geometric mean inequality)} \\
&= \left\langle \left(\frac{|A|+|A^*|}{2}\right)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^2 \\
&\leqslant \left\langle \left(\frac{|A|+|A^*|}{2}\right)^2\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \quad \text{(by Lemma 1.4).}
\end{aligned}$$

Utilizing (2.4) we get

$$\begin{aligned}
&\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right|^2 \\
&\leqslant \left\langle \left(\frac{|A|^2+|A^*|^2}{2} - \frac{1}{2R}\left((1-t)|A|^2+t|A^*|^2-((1-t)|A|+t|A^*|)^2\right)\right)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \\
&\leqslant \left\| \frac{|A|^2+|A^*|^2}{2} - \frac{1}{2R}\left((1-t)|A|^2+t|A^*|^2-((1-t)|A|+t|A^*|)^2\right) \right\|_{\mathbf{ber}}.
\end{aligned} \tag{2.5}$$

We deduce the desired result by taking the supremum in (2.5) over $\lambda \in \Omega$. $\square$

REMARK 2.2. Since

$$(1-t)|A|^2+t|A^*|^2-((1-t)|A|+t|A^*|)^2 \geqslant 0,$$

we have

$$\begin{aligned}&\left\|\frac{|A|^2+|A^*|^2}{2}-\frac{1}{2R}\left((1-t)|A|^2+t|A^*|^2-((1-t)|A|+t|A^*|)^2\right)\right\|_{\mathbf{ber}}\\ &\leqslant \frac{1}{2}\left\||A|^2+|A^*|^2\right\|_{\mathbf{ber}}.\end{aligned}$$

It is well-known that for any $A,B \in \mathbb{B}(\mathbb{H}(\Omega))$ (see [2, Lemma 2.1 (b)])

$$\mathbf{ber}\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix}\right) \leqslant \frac{1}{2}\left(\|A\|_{\mathbf{ber}}+\|B\|_{\mathbf{ber}}\right). \tag{2.6}$$

To obtain the following result, which contains a refinement of (2.6), we mimic some ideas from [25, Corollary 2.1].

THEOREM 2.4. *Let $A,B \in \mathbb{B}(\mathbb{H}(\Omega))$. Then for any $t \in \mathbb{R}$,*

$$\mathbf{ber}\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix}\right) \leqslant \frac{1}{2}\left(\mathbf{ber}\left(\begin{bmatrix} O & (1+t)A \\ (1-t)B & O \end{bmatrix}\right)+\mathbf{ber}\left(\begin{bmatrix} O & (1-t)A \\ (1+t)B & O \end{bmatrix}\right)\right).$$

Proof. It has been established in [9, Corollary 2.4] that

$$\|A+B\|_{\mathbf{ber}} \leqslant \left\|tA+(1-t)\frac{A+B}{2}\right\|_{\mathbf{ber}}+\left\|tB+(1-t)\frac{A+B}{2}\right\|_{\mathbf{ber}} \leqslant \|A\|_{\mathbf{ber}}+\|B\|_{\mathbf{ber}}$$

for any $t \in \mathbb{R}$. If we substitute B by $e^{i\theta}B$, we get

$$\begin{aligned}&\left\|A+e^{i\theta}B\right\|_{\mathbf{ber}}\\ &\leqslant \left\|tA+(1-t)\frac{A+e^{i\theta}B}{2}\right\|_{\mathbf{ber}}+\left\|te^{i\theta}B+(1-t)\frac{A+e^{i\theta}B}{2}\right\|_{\mathbf{ber}}\\ &=\frac{1}{2}\left(\left\|(1+t)A+(1-t)e^{i\theta}B\right\|_{\mathbf{ber}}+\left\|(1-t)A+(1+t)e^{i\theta}B\right\|_{\mathbf{ber}}\right).\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{1}{2}\left\|A+e^{i\theta}B\right\|_{\mathbf{ber}} &\leqslant \frac{1}{4}\left(\left\|(1+t)A+(1-t)e^{i\theta}B\right\|_{\mathbf{ber}}+\left\|(1-t)A+(1+t)e^{i\theta}B\right\|_{\mathbf{ber}}\right)\\ &\leqslant \frac{1}{2}\left(\mathbf{ber}\left(\begin{bmatrix} O & (1+t)A \\ (1-t)B^* & O \end{bmatrix}\right)+\mathbf{ber}\left(\begin{bmatrix} O & (1-t)A \\ (1+t)B^* & O \end{bmatrix}\right)\right).\end{aligned}$$

Namely,

$$\frac{1}{2}\left\|A+e^{i\theta}B\right\|_{\mathbf{ber}} \leqslant \frac{1}{2}\left(\mathbf{ber}\left(\begin{bmatrix} O & (1+t)A \\ (1-t)B^* & O \end{bmatrix}\right)+\mathbf{ber}\left(\begin{bmatrix} O & (1-t)A \\ (1+t)B^* & O \end{bmatrix}\right)\right)$$

for any $t \in \mathbb{R}$. If we take supremum over $\theta \in \mathbb{R}$, we have

$$\mathbf{ber}\left(\begin{bmatrix} O & A \\ B^* & O \end{bmatrix}\right) \leqslant \frac{1}{2}\left(\mathbf{ber}\left(\begin{bmatrix} O & (1+t)A \\ (1-t)B^* & O \end{bmatrix}\right) + \mathbf{ber}\left(\begin{bmatrix} O & (1-t)A \\ (1+t)B^* & O \end{bmatrix}\right)\right).$$

We conclude the desired result by substituting B by B^*. □

COROLLARY 2.3. Let $A, B \in \mathbb{B}(\mathbb{H}(\Omega))$. Then

$$\begin{aligned} \mathbf{ber}\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix}\right) &\leqslant \frac{1}{2}\left(\int_0^1 \mathbf{ber}\left(\begin{bmatrix} O & (1+t)A \\ (1-t)B & O \end{bmatrix}\right) dt \right. \\ &\qquad \left. + \int_0^1 \mathbf{ber}\left(\begin{bmatrix} O & (1-t)A \\ (1+t)B & O \end{bmatrix}\right) dt\right) \\ &\leqslant \frac{1}{2}\left(\|A\|_{\mathbf{ber}} + \|B\|_{\mathbf{ber}}\right). \end{aligned}$$

Proof. Assume that $0 \leqslant t \leqslant 1$. We can write by (2.6) that

$$\begin{aligned} &\mathbf{ber}\left(\begin{bmatrix} O & (1+t)A \\ (1-t)B & O \end{bmatrix}\right) + \mathbf{ber}\left(\begin{bmatrix} O & (1-t)A \\ (1+t)B & O \end{bmatrix}\right) \\ &\leqslant \frac{(1+t)\|A\|_{\mathbf{ber}} + (1-t)\|B\|_{\mathbf{ber}}}{2} + \frac{(1-t)\|A\|_{\mathbf{ber}} + (1+t)\|B\|_{\mathbf{ber}}}{2} \\ &= \|A\|_{\mathbf{ber}} + \|B\|_{\mathbf{ber}}. \end{aligned}$$

After taking the integral over $0 \leqslant t \leqslant 1$, we reach the desired result, thanks to Theorem 2.4. □

We conclude this section with the following two theorems.

THEOREM 2.5. *Let $A \in \mathbb{B}(\mathbb{H}(\Omega))$ and let $0 \leqslant \nu \leqslant 1$. Then*

$$\begin{aligned} \mathbf{ber}^2(A) \leqslant \frac{1}{2}\Bigg(&\mathbf{ber}\left(|A^*|^{2(1-\nu)}|A|^{2\nu}\right) \\ &+ \sqrt{\frac{1}{2}\mathbf{ber}\left(|A^*|^{4(1-\nu)}|A|^{4\nu}\right) + \frac{1}{4}\left\||A|^{8\nu} + |A^*|^{8(1-\nu)}\right\|_{\mathbf{ber}}}\Bigg). \end{aligned}$$

In particular,

$$\mathbf{ber}^2(A) \leqslant \frac{1}{2}\left(\mathbf{ber}(|A^*|\,|A|) + \sqrt{\frac{1}{2}\mathbf{ber}\left(|A^*|^2|A|^2\right) + \frac{1}{4}\left\||A|^4 + |A^*|^4\right\|_{\mathbf{ber}}}\right).$$

Proof. We prove the first inequality. Let $\widehat{k}_\lambda$ be a normalized reproducing kernel. Let $x = |A|^{2\nu}\widehat{k}_\lambda$, $y = |A^*|^{2(1-\nu)}\widehat{k}_\lambda$, and $z = \widehat{k}_\lambda$, in Lemma 1.2, we get

$$\left\langle |A|^{2\nu}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \left\langle |A^*|^{2(1-\nu)}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \leqslant \frac{1}{2}\left(\left| \left\langle |A|^{2\nu}\widehat{k}_\lambda, |A^*|^{2(1-\nu)}\widehat{k}_\lambda \right\rangle \right| + \left\| |A|^{2\nu}\widehat{k}_\lambda \right\| \left\| |A^*|^{2(1-\nu)}\widehat{k}_\lambda \right\| \right),$$

i.e.,

$$\left| \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^2 \leqslant \frac{1}{2}\left(\left| \left\langle |A^*|^{2(1-\nu)}|A|^{2\nu}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| + \sqrt{\left\langle |A|^{4\nu}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \left\langle |A^*|^{4(1-\nu)}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle} \right).$$

In the same manner, we can show that

$$\sqrt{\left\langle |A|^{4\nu}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \left\langle |A^*|^{4(1-\nu)}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle} \leqslant \sqrt{\frac{1}{2}\left(\left| \left\langle |A|^{4\nu}\widehat{k}_\lambda, |A^*|^{4(1-\nu)}\widehat{k}_\lambda \right\rangle \right| + \left\| |A|^{4\nu}\widehat{k}_\lambda \right\| \left\| |A^*|^{4(1-\nu)}\widehat{k}_\lambda \right\| \right)}.$$

Therefore,

$$\begin{aligned}
&\left| \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^2 \\
&\leqslant \frac{1}{2}\left(\left| \left\langle |A^*|^{2(1-\nu)}|A|^{2\nu}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| + \sqrt{\left\langle |A|^{4\nu}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \left\langle |A^*|^{4(1-\nu)}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle} \right) \\
&\leqslant \frac{1}{2}\Bigg(\left| \left\langle |A^*|^{2(1-\nu)}|A|^{2\nu}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| \\
&\quad + \sqrt{\frac{1}{2}\left(\left| \left\langle |A^*|^{4(1-\nu)}|A|^{4\nu}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| + \sqrt{\left\langle |A|^{8\nu}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \left\langle |A^*|^{8(1-\nu)}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle} \right)} \Bigg) \\
&\leqslant \frac{1}{2}\Bigg(\left| \left\langle |A^*|^{2(1-\nu)}|A|^{2\nu}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| \\
&\quad + \sqrt{\frac{1}{2}\left(\left| \left\langle |A^*|^{4(1-\nu)}|A|^{4\nu}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| + \frac{1}{2}\left\langle \left(|A|^{8\nu} + |A^*|^{8(1-\nu)} \right)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right)} \Bigg) \\
&\leqslant \frac{1}{2}\left(\mathbf{ber}\left(|A^*|^{2(1-\nu)}|A|^{2\nu} \right) + \sqrt{\frac{1}{2}\mathbf{ber}\left(|A^*|^{4(1-\nu)}|A|^{4\nu} \right) + \frac{1}{4}\left\| |A|^{8\nu} + |A^*|^{8(1-\nu)} \right\|_{\mathbf{ber}}} \right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbf{ber}^2(A) \leqslant \frac{1}{2}\Bigg(&\mathbf{ber}\left(|A^*|^{2(1-\nu)}|A|^{2\nu} \right) \\
&+ \sqrt{\frac{1}{2}\mathbf{ber}\left(|A^*|^{4(1-\nu)}|A|^{4\nu} \right) + \frac{1}{4}\left\| |A|^{8\nu} + |A^*|^{8(1-\nu)} \right\|_{\mathbf{ber}}} \Bigg),
\end{aligned}$$

as desired.

Letting $\nu = \frac{1}{2}$ in the first inequality, we obtain the second inequality. □

THEOREM 2.6. *Let* $A \in \mathbb{B}(\mathbb{H}(\Omega))$ *and let* $0 \leqslant \nu \leqslant 1$. *Then*

$$\mathbf{ber}^2(A) \leqslant \frac{1}{2}\left(\mathbf{ber}\left(|A^*|^{2(1-\nu)}|A|^{2\nu}\right) + \left(\int_0^1 \sqrt{\left\|(1-t)|A|^{4\nu} + t|A^*|^{4(1-\nu)}\right\|_{\mathbf{ber}}}\, dt\right)^2\right).$$

Proof. Let $0 \leqslant \nu \leqslant 1$ and let $\widehat{k}_\lambda$ be a normalized reproducing kernel. By taking $x = |A|^{2\nu}\widehat{k}_\lambda$, $y = |A^*|^{2(1-\nu)}\widehat{k}_\lambda$, and $z = \widehat{k}_\lambda$, in Lemma 1.2, we get

$$\begin{aligned} &\left\langle |A|^{2\nu}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \left\langle |A^*|^{2(1-\nu)}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \\ &\leqslant \frac{1}{2}\left(\left|\left\langle |A|^{2\nu}\widehat{k}_\lambda, |A^*|^{2(1-\nu)}\widehat{k}_\lambda\right\rangle\right| + \left\||A|^{2\nu}\widehat{k}_\lambda\right\| \left\||A^*|^{2(1-\nu)}\widehat{k}_\lambda\right\|\right). \end{aligned}$$

On the other hand, it observes from the mixed Schwarz inequality that

$$\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^2 \leqslant \frac{1}{2}\left(\left|\left\langle |A|^{2\nu}\widehat{k}_\lambda, |A^*|^{2(1-\nu)}\widehat{k}_\lambda\right\rangle\right| + \left\||A|^{2\nu}\widehat{k}_\lambda\right\| \left\||A^*|^{2(1-\nu)}\widehat{k}_\lambda\right\|\right).$$

Thus, by the logarithmic-geometric mean inequality, we can write

$$\begin{aligned} &\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^2 \\ &\leqslant \frac{1}{2}\left(\left|\left\langle |A^*|^{2(1-\nu)}|A|^{2\nu}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| + \left\||A|^{2\nu}\widehat{k}_\lambda\right\| \left\||A^*|^{2(1-\nu)}\widehat{k}_\lambda\right\|\right) \\ &\leqslant \frac{1}{2}\left(\left|\left\langle |A^*|^{2(1-\nu)}|A|^{2\nu}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| + \left(\int_0^1 \left\||A|^{2\nu}\widehat{k}_\lambda\right\|^{1-t} \left\||A^*|^{2(1-\nu)}\widehat{k}_\lambda\right\|^t dt\right)^2\right) \\ &= \frac{1}{2}\left(\left|\left\langle |A^*|^{2(1-\nu)}|A|^{2\nu}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| + \left(\int_0^1 \sqrt{\left\langle |A|^{4\nu}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^{1-t} \left\langle |A^*|^{4(1-\nu)}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^t}\, dt\right)^2\right) \\ &\leqslant \frac{1}{2}\left(\left|\left\langle |A^*|^{2(1-\nu)}|A|^{2\nu}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| + \left(\int_0^1 \sqrt{\left\langle \left((1-t)|A|^{4\nu} + t|A^*|^{4(1-\nu)}\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle}\, dt\right)^2\right) \end{aligned}$$

(by the weighted arithmetic-geometric mean inequality)

$$\leqslant \frac{1}{2}\left(\mathbf{ber}\left(|A^*|^{2(1-t)}|A|^{2\nu}\right) + \left(\int_0^1 \sqrt{\left\|(1-t)|A|^{4\nu} + t|A^*|^{4(1-\nu)}\right\|_{\mathbf{ber}}}\, dt\right)^2\right).$$

This implies that

$$\left|\left\langle A\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|^2$$
$$\leqslant \frac{1}{2}\left(\mathbf{ber}\left(|A^*|^{2(1-\nu)}|A|^{2\nu}\right)+\left(\int_0^1\sqrt{\left\|(1-t)\,|A|^{4\nu}+t|A^*|^{4(1-\nu)}\right\|_{\mathbf{ber}}}dt\right)^2\right).$$

We reach the desired result by taking the supremum over $\lambda\in\Omega$. □

REMARK 2.3. Putting $\nu=\frac{1}{2}$ in Theorem 2.6, we obtain

$$\mathbf{ber}^2(A)\leqslant\frac{1}{2}\left(\mathbf{ber}\left(|A^*|\,|A|\right)+\left(\int_0^1\sqrt{\left\|(1-t)\,|A|^2+t|A^*|^2\right\|_{\mathbf{ber}}}dt\right)^2\right).$$

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(Received July 1, 2024)

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