Journal of
Mathematical
Inequalities
Volume 18, Number 4 (2024), 1303–1312 doi:10.7153/jmi-2024-18-75

HYERS–ULAM TYPE STABILITY OF JENSEN OPERATORS IN LOCALLY CONVEX CONES

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(*Communicated by T. Burić*)

Abstract. In the theory of locally convex cones, some methods are rather technical. This is the consequence of including infinity-type unbounded elements and the general non-availability of the cancellation law. In this paper, we consider the Hyers-Ulam stability problem in locally convex cones and prove the stability of Jensen operators in locally convex cones.

1. Introduction

Suppose that E_1 and E_2 are Banach spaces and $f : E_1 \to E_2$ is a mapping for which there exists $\varepsilon > 0$ such that $\|f(x+y) - f(x) - f(y)\| < \varepsilon$ for all $x, y \in E_1$. Then there is a unique additive mapping $A : E_1 \to E_2$ defined by

$$A(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

such that $\|f(x) - A(x)\| < \varepsilon$ for $x \in E_1$. Moreover, if f is continuous at least in one point $x \in E_1$, then A is continuous everywhere in E_1. Furthermore, if for each $x \in E_1$ the function $t \to f(tx)$ from $\mathbb{R}$ to E_2 is continuous for each fixed x, then A is linear. This theorem was proved in 1941 by Hyers [12] which is a partial solution of the problem was raised in 1940 by Ulam [36].

Hyers's Theorem was generalized by Aoki [1] for additive mappings in 1950, and independently, by Th. M. Rassias [32] in 1978 for linear mappings considering the Cauchy difference controlled by the sum of powers of norms. This type of stability is called Hyers-Ulam-Rassias stability. On the other hand, J. M. Rassias [33] considered the Cauchy difference controlled by the product of different powers of norms. A generalization of the Hyers-Ulam-Rassias stability was obtained by Găvruta [11] in 1994, who replaced the general control function instead of the sum of powers of norms and the product of powers of norms.

The Hyers-Ulam-Rassias stability has been considered in various spaces, e.g. in C^*-algebras [30], quasi-$(2,\beta)$-Banach spaces [6], non-Archimedean Banach algebras [9], C^*-ternary algebras [10], fuzzy Banach spaces [16], see also [8, 13, 18, 23, 24]. Stability of some functional equations was also verified in Lipschitz spaces [4,5,26–28].

Mathematics subject classification (2020): 46A03, 39B82, 39B52.
Keywords and phrases: Locally convex cone, Jensen operator, stability.
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The theory of locally convex cones was introduced and developed in [17] and [34]. A nonempty set $\mathscr{P}$ endowed with an addition and a scalar multiplication for non-negative real numbers is called a *cone* whenever the addition is associative and commutative, there is a neutral element $0 \in \mathscr{P}$ and for the scalar multiplication the usual associative and distributive properties hold, that is $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha+\beta)a = \alpha a + \beta a$, $\alpha(a+b) = \alpha a + \alpha b$, $1a = a$ and $0a = 0$ for all $a, b \in \mathscr{P}$ and non-negative reals α and β.

A *preordered cone* is a cone $\mathscr{P}$ endowed with a preorder (reflexive transitive relation) $\leqslant$ which is compatible with the addition and scalar multiplication, that is $x \leqslant y$ implies $x+z \leqslant y+z$ and $r \cdot x \leqslant r \cdot y$ for all $x, y, z \in \mathscr{P}$ and $r \in \mathbb{R}_+ = \{r \in \mathbb{R} \,:\, r \geqslant 0\}$. Every ordered vector space is an ordered cone. The cones $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ and $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$, with the usual order and algebraic operations (specially $0 \cdot (+\infty) = 0$), are ordered cones that are not embeddable in vector spaces.

A subset $\mathscr{V}$ of a preordered cone $\mathscr{P}$ is called an *(abstract) 0-neighborhood system*, if

(v_1) $0 < v$ for all $v \in \mathscr{V}$;

(v_2) for all $u, v \in \mathscr{V}$ there is a $w \in \mathscr{V}$ with $w \leqslant u$ and $w \leqslant v$;

(v_3) $u + v \in \mathscr{V}$ and $\alpha v \in \mathscr{V}$ whenever $u, v \in \mathscr{V}$ and $\alpha > 0$.

Let $a \in \mathscr{P}$ and $v \in \mathscr{V}$. We define $v(a) = \{b \in \mathscr{P} \mid b \leqslant a + v\}$, resp. $(a)v = \{b \in \mathscr{P} \mid a \leqslant b + v\}$, to be a neighborhood of a in the *upper*, resp. *lower* topologies on $\mathscr{P}$. The common refinement of the upper and lower topologies is called *symmetric* topology. We denote the neighborhoods of a in the symmetric topology by $v(a)v$. The pair $(\mathscr{P}, \mathscr{V})$ is called a *full locally convex cone* if the elements of $\mathscr{P}$ are *bounded below*, i.e. for every $a \in \mathscr{P}$ and $v \in \mathscr{V}$ we have $0 \leqslant a + \rho v$ for some $\rho > 0$. Each subcone of $\mathscr{P}$, not necessarily containing $\mathscr{V}$, is called a *locally convex cone*. An element $a \in \mathscr{P}$ is upper bounded if for each $v \in \mathscr{V}$ there exists $\lambda > 0$ such that $a \leqslant \lambda v$. The element a is bounded if it is lower and upper bounded. By considering $\xi = \{\varepsilon > 0 \,:\, \varepsilon \in \mathbb{R}\}$, $(\overline{\mathbb{R}}, \xi)$ and $(\overline{\mathbb{R}}_+, \xi)$ are full locally convex cones.

A sequence $\{x_n\}$ in the locally convex cone $(\mathscr{P}, \mathscr{V})$ is *(symmetric) Cauchy* if for each $v \in \mathscr{V}$ one can fined $n_0 \in \mathbb{N}$ such that $x_n \leqslant x_m + v$ for all $n, m \geqslant n_0$.

The locally convex cone $(\mathscr{P}, \mathscr{V})$ is a uc-cone, whenever $\mathscr{V} = \{\alpha v \,:\, \alpha > 0\}$ for some $v \in \mathscr{V}$. In this situation, we call the element v the generating element of $\mathscr{V}$. If $\mathscr{P}$ is a vector space over $\mathbb{R}$ and $(\mathscr{P}, \mathscr{V})$ is a uc-cone, then $\mathscr{P}$ is a seminormed space endowed with the symmetric topology of $(\mathscr{P}, \mathscr{V})$. If $\mathscr{V} = \{\alpha v \,:\, \alpha > 0\}$, then this seminorm is given by

$$p(a) = \inf\{\lambda > 0 \,:\, a \in \lambda v(0)v\} \tag{1}$$

for $a \in \mathscr{P}$. If the symmetric topology on $\mathscr{P}$ is Hausdorff, then p is a norm on $\mathscr{P}$ (see [2] for more details).

Let $(\mathscr{P}, \mathscr{V})$ be a locally convex cone. For $a \in (\mathscr{P}, \mathscr{V})$, we define $\bar{a} = \cap\{v(a) : v \in \mathscr{V}\}$, and we call $\mathscr{P}$ *separated* if $\bar{a} = \bar{b}$ implies $a = b$ for all $a, b \in \mathscr{P}$. Note that the locally convex cone $\mathscr{P}$ is separated if and only if the symmetric topology on $\mathscr{P}$ is

Hausdorff (see [17], I.3.9.). For cones $\mathscr{P}$ and $\mathscr{Q}$ a mapping $g:\mathscr{P}\to\mathscr{Q}$ is called a *Jensen operator* if

$$2g\left(\frac{a+b}{2}\right)=g(a)+g(b)$$

holds for $a,b\in\mathscr{P}$.

The first result on the stability of the classical Jensen functional equation was given by Kominek [19], see also [14]. The stability of the Jensen equation and of its generalizations was studied by numerous researchers (cf., e.g., [3, 7, 20–22, 35]. The stability problem in the sense of Ulam has been received an attention in [29] by the authors in the theory of locally convex cones. In this paper, we verify the Hyers-Ulam type stability in locally convex cones and prove the stability of the Jensen operator in locally convex cones under certain assumptions.

2. Main results

The following lemma has essential role in our main results. This lemma is an especial case of [31, Proposition 2.3].

LEMMA 1. ([29], Lemma 1) *Let $(\mathscr{P},\mathscr{V})$ be a locally convex cone, $x\in\mathscr{P}$ and let $\{\alpha_n\}$ be a sequence of non-negative scalars such that $\alpha_n\to 0$ as $n\to\infty$. Then x is bounded if and only if $\alpha_n x\to 0$ as $n\to\infty$ with respect to the symmetric topology.*

Now, we provide our main result.

THEOREM 1. *Let $(\mathscr{P}_1,\mathscr{V}_1)$ be a locally convex cone and $(\mathscr{P}_2,\mathscr{V}_2)$ a separated full locally convex cone. Let $(\mathscr{P}_2,\mathscr{V}_2)$ be complete under the symmetric topology and the preorder in $\mathscr{P}_2$ is antisymmetric. If the mapping $f:\mathscr{P}_1\to\mathscr{P}_2$ satifies*

$$2f\left(\frac{x+y}{2}\right)\in v(f(x)+f(y))v \tag{2}$$

for some bounded element $v\in\mathscr{V}_2$ and for all $x,y\in\mathscr{P}_1$ where $f(0)$ is bounded, then there exist a unique additive Jensen operator $g:\mathscr{P}_1\to\mathscr{P}_2$ and a positive real number γ such that

$$g(x)\in(\gamma v)(f(x))(\gamma v) \tag{3}$$

for all $x\in\mathscr{P}_1$.

Proof. Let $v\in\mathscr{V}_2$ be the bounded element for which (2) holds. Considering $y=0$ in (2), we get

$$2f\left(\frac{x}{2}\right)\in v(f(x)+f(0))v$$

for all $x\in\mathscr{P}_1$, that is

$$\begin{aligned}2f\left(\frac{x}{2}\right)&\leqslant f(x)+f(0)+v,\\ f(x)+f(0)&\leqslant 2f\left(\frac{x}{2}\right)+v\end{aligned} \tag{4}$$

for all $x \in \mathscr{P}_1$. By boundedness of $f(0)$, there exists a real number $\lambda > 0$ such that

$$f(0) + \lambda v \geqslant 0 \text{ and } f(0) \leqslant \lambda v. \tag{5}$$

The relations (4) and (5) imply that

$$\begin{aligned} 2f\left(\frac{x}{2}\right) &\leqslant f(x) + (\lambda+1)v, \\ f(x) &\leqslant f(x) + f(0) + \lambda v \leqslant 2f\left(\frac{x}{2}\right) + (\lambda+1)v \end{aligned} \tag{6}$$

for all $x \in \mathscr{P}_1$. Replacing x by $2x$ in (6), we deduce

$$\begin{aligned} 2f(x) &\leqslant f(2x) + (\lambda+1)v, \\ f(2x) &\leqslant 2f(x) + (\lambda+1)v. \end{aligned} \tag{7}$$

Multiplying both sides by $\frac{1}{2}$ in (7) and replacing x by $2x$ in the resulting inequalities, one has

$$\begin{aligned} f(2x) &\leqslant \frac{1}{2}f(2^2x) + \frac{1}{2}(\lambda+1)v, \\ \frac{1}{2}f(2^2x) &\leqslant f(2x) + \frac{1}{2}(\lambda+1)v. \end{aligned} \tag{8}$$

It follows from (7) and (8) that

$$\begin{aligned} 2f(x) &\leqslant \frac{1}{2}f(2^2x) + (\lambda+1)v + \frac{1}{2}(\lambda+1)v, \\ \frac{1}{2}f(2^2x) &\leqslant 2f(x) + (\lambda+1)v + \frac{1}{2}(\lambda+1)v. \end{aligned} \tag{9}$$

Multiplying both sides by $\frac{1}{2}$ in (9), we have

$$\begin{aligned} f(x) &\leqslant \frac{1}{2^2}f(2^2x) + \frac{1}{2}(\lambda+1)v + \frac{1}{2^2}(\lambda+1)v, \\ \frac{1}{2^2}f(2^2x) &\leqslant f(x) + \frac{1}{2}(\lambda+1)v + \frac{1}{2^2}(\lambda+1)v. \end{aligned}$$

By induction, it follows that

$$\begin{aligned} f(x) &\leqslant \frac{1}{2^n}f(2^nx) + \left(1 - \frac{1}{2^n}\right)(\lambda+1)v, \\ \frac{1}{2^n}f(2^nx) &\leqslant f(x) + \left(1 - \frac{1}{2^n}\right)(\lambda+1)v. \end{aligned} \tag{10}$$

Replacing x by 2^mx in (10) and multiplying $\frac{1}{2^m}$ on its both sides, we conclude that

$$\begin{aligned} \frac{1}{2^m}f(2^mx) &\leqslant \frac{1}{2^{n+m}}f(2^{n+m}x) + \frac{1}{2^m}\left(1 - \frac{1}{2^n}\right)(\lambda+1)v, \\ \frac{1}{2^{n+m}}f(2^{n+m}x) &\leqslant \frac{1}{2^m}f(2^mx) + \frac{1}{2^m}(1 - \frac{1}{2^n})(\lambda+1)v. \end{aligned}$$

Now, let $u \in \mathscr{V}$ be arbitrary. Since v is bounded, there exists $\eta > 0$ such that $(\lambda + 1)v \leqslant \eta u$. For sufficiently large m, n, we have $\frac{1}{2^m}(1 - \frac{1}{2^n}) \leqslant \frac{1}{\eta}$ and then

$$\frac{1}{2^m} f(2^m x) \leqslant \frac{1}{2^{n+m}} f(2^{n+m} x) + u,$$
$$\frac{1}{2^{n+m}} f(2^{n+m} x) \leqslant \frac{1}{2^m} f(2^m x) + u.$$

So, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is Cauchy in $(\mathscr{P}_2, \mathscr{V}_2)$ under the symmetric topology. This sequence is convergent in $(\mathscr{P}_2, \mathscr{V}_2)$ under the symmetric topology. Since $(\mathscr{P}_2, \mathscr{V}_2)$ is separated, the symmetric topology is Hausdorff and the limit of this sequence is unique. Let

$$g(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x).$$

Set $\gamma = \lambda + 1$. By using Lemma 1, $\lim_{n\to\infty} \frac{1}{2^n} v = 0$, so taking the limit as $n \to \infty$ in (10), we find that $g(x) \in (\gamma v)(f(x)(\gamma v)$ which entails (3). Replacing $2^n x$ and $2^n y$, respectively by x and y in (2) and multiplying $\frac{1}{2^n}$ on its both sides, we get

$$2\frac{1}{2^n} f\left(2^n\left(\frac{x+y}{2}\right)\right) \leqslant \frac{1}{2^n} f(2^n x) + \frac{1}{2^n} f(2^n y) + \frac{1}{2^n} v,$$
$$\frac{1}{2^n} f(2^n x) + \frac{1}{2^n} f(2^n y) \leqslant 2\frac{1}{2^n} f\left(2^n\left(\frac{x+y}{2}\right)\right) + \frac{1}{2^n} v.$$

By Lemma 1 and letting $n \to \infty$, one can find that

$$2g\left(\frac{x+y}{2}\right) \leqslant g(x) + g(y) \quad \text{and} \quad g(x) + g(y) \leqslant 2g\left(\frac{x+y}{2}\right).$$

Since the order in $\mathscr{P}_2$ is antisymmetric, we realize that

$$2g\left(\frac{x+y}{2}\right) = g(x) + g(y). \tag{11}$$

Since $f(0)$ is bounded, by Lemma 1, we have

$$g(0) = \lim_{n\to\infty} \frac{1}{2^n} f(0) = 0.$$

Since g is Jensen operator and $g(0) = 0$, the operator g is also additive, by [25, Lemma 2.1]. Putting $y = 0$ in (11), we observe that

$$2g\left(\frac{x}{2}\right) = g(x) + g(0) = g(x) \tag{12}$$

and replacing x by $2x$ in (12), one has

$$g(x) = \frac{1}{2} g(2x).$$

So, by induction, it follows that

$$g(x) = \frac{1}{2^n} g(2^n x). \tag{13}$$

We demonstrate the operator g is unique. Let h be another Jensen operator satisfying (3). So, we have

$$f(x) \leqslant h(x) + \gamma v \quad \text{and} \quad h(x) \leqslant f(x) + \gamma v. \tag{14}$$

Since g and h are Jensen operators, replacing $2^n x$ by x in (14), multiplying both sides by $\frac{1}{2^n}$, and using (13) for the Jensen operator h, we achieve

$$\frac{1}{2^n} f(2^n x) \leqslant h(x) + \frac{1}{2^n}\gamma v \quad \text{and} \quad h(x) \leqslant \frac{1}{2^n} f(2^n x) + \frac{1}{2^n}\gamma v. \tag{15}$$

By a similar reason and using (13) for the Jensen operator g, one can say that

$$\frac{1}{2^n} f(2^n x) \leqslant g(x) + \frac{1}{2^n}\gamma v \quad \text{and} \quad g(x) \leqslant \frac{1}{2^n} f(2^n x) + \frac{1}{2^n}\gamma v. \tag{16}$$

The inequalities (15) and (16) imply that

$$h(x) \leqslant \frac{1}{2^n} f(2^n x) + \frac{1}{2^n}\gamma v \leqslant g(x) + \frac{1}{2^n}\gamma v + \frac{1}{2^n}\gamma v, \tag{17}$$

$$g(x) \leqslant \frac{1}{2^n} f(2^n x) + \frac{1}{2^n}\gamma v \leqslant h(x) + \frac{1}{2^n}\gamma v + \frac{1}{2^n}\gamma v. \tag{18}$$

Letting $n \to \infty$ in (17) and (18), respectively, we ensure that $h(x) \leqslant g(x)$ and $g(x) \leqslant h(x)$. Since the preorder in $\mathscr{P}_2$ is antisymmetric, we explore

$$g(x) = h(x),$$

which completes the proof. $\square$

In particular, one can deduce the following corollary and remark.

COROLLARY 2. *Let $(\mathscr{P}, \mathscr{V})$ be a locally convex cone. If the mapping $f : (\mathscr{P}, \mathscr{V}) \to (\overline{\mathbb{R}}, \xi)$ (or $f : (\mathscr{P}, \mathscr{V}) \to (\overline{\mathbb{R}}_+, \xi)$) satisfies*

$$2f\left(\frac{x+y}{2}\right) \in \delta(f(x) + f(y))\delta$$

for some $\delta > 0$ and for all $x, y \in \mathscr{P}$ where $f(0) < \infty$, then there exist a unique additive Jensen operator $g : (\mathscr{P}, \mathscr{V}) \to (\overline{\mathbb{R}}, \xi)$ (or $g : (\mathscr{P}, \mathscr{V}) \to (\overline{\mathbb{R}}_+, \xi)$) and a positive real number ε such that

$$g(x) \in \varepsilon(f(x))\varepsilon$$

for all $x \in \mathscr{P}$.

Proof. Consider the locally convex cone $(\overline{\mathbb{R}},\xi)$, where $\xi=\{\varepsilon>0\ :\ \varepsilon\in\mathbb{R}\}$. This locally convex cone is a full separated locally convex cone. It is complete under the symmetric topology. Furthermore, the usual order in $\overline{\mathbb{R}}$ is antisymmetric. So, the desired result now follows from Theorem 1. □

REMARK 1. Let $(\mathscr{P}_1,\mathscr{V}_1)$ be a locally convex cone and $(\mathscr{P}_2,\mathscr{V}_2)$ a separated full uc-cone with $\mathscr{V}_2=\{\gamma v:\gamma\in\mathbb{R} \text{ and } \gamma>0\}$. Let $(\mathscr{P}_2,\mathscr{V}_2)$ be complete under the symmetric topology and the preorder on $\mathscr{P}_2$ is antisymmetric. If the mapping $f:\mathscr{P}_1\to\mathscr{P}_2$ satisfies

$$2f\left(\frac{x+y}{2}\right)\in v(f(x)+f(y))v$$

for some bounded $v\in\mathscr{V}_2$ and for all $x,y\in\mathscr{P}_1$ where $f(0)$ is bounded, then the conditions of Theorem 1 fulfill, so there exist a unique additive Jensen operator $g:\mathscr{P}_1\to\mathscr{P}_2$ and a positive real number γ such that

$$g(x)\in(\gamma v)(f(x))(\gamma v)$$

for all $x\in\mathscr{P}_1$.

At the end, we provide a Hyers type stability of Jensen operators in locally convex cones. Although the following corollary followed from Theorem 1, we bring a classic proof.

COROLLARY 3. *Let $(\mathscr{P}_1,\mathscr{V}_1)$ be a locally convex cone and let $(\mathscr{P}_2,\mathscr{V}_2)$ be a separated full uc-cone for which at the same time $\mathscr{P}_2$ is a vector space over $\mathbb{R}$. If $(\mathscr{P}_2,\mathscr{V}_2)$ is complete under the symmetric topology and there exists $\varepsilon>0$ such that the mapping $f:\mathscr{P}_1\to\mathscr{P}_2$ satisfies*

$$2f\left(\frac{x+y}{2}\right)\in(\varepsilon v)(f(x)+f(y))(\varepsilon v)$$

for all $x,y\in\mathscr{P}_1$ where v is the generating element of $\mathscr{V}_2$, then there exists a unique additive Jensen operator $g:\mathscr{P}_1\to\mathscr{P}_2$ such that

$$g(x)\in(\varepsilon v)(f(x))(\varepsilon v)$$

for all $x\in\mathscr{P}_1$.

Proof. Since the locally convex cone $(\mathscr{P}_2,\mathscr{V}_2)$ is separated, the symmetric topology on $\mathscr{P}_2$ is Hausdorff (see [17], I.3.9), so $(\mathscr{P}_2,\mathscr{V}_2)$ is a Banach space with the norm p defined in (1). By the hypothesis there exists $\varepsilon>0$ such that the mapping $f:\mathscr{P}_1\to\mathscr{P}_2$ satisfies

$$p\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\leqslant\varepsilon \tag{19}$$

for all $x,y \in \mathscr{P}_1$. By using Hyers's method [12, 14, 15] and letting $y=0$ in (19), one can find

$$p\left(2f\left(\frac{x}{2}\right)-f(x)-f(0)\right) \leqslant \varepsilon. \tag{20}$$

Replacing x by $2x$ in (20) and multiplying both sides by $\frac{1}{2}$, one has

$$p\left(f(x)-\frac{1}{2}f(2x)-\frac{1}{2}f(0)\right) \leqslant \frac{1}{2}\varepsilon.$$

By induction, it follows that

$$p\left(f(x)-\frac{1}{2^n}f(2^n x)-\frac{1}{2^n}f(0)\right) \leqslant \varepsilon\sum_{i=1}^{n}\frac{1}{2^i}.$$

Consider $g:\mathscr{P}_1 \to \mathscr{P}_2$ by

$$g(x)=\lim_{n\to\infty}\frac{1}{2^n}f(2^n x).$$

So, g is the unique Jensen operator for which we have

$$p(g(x)-f(x)) \leqslant \varepsilon$$

for all $x \in \mathscr{P}_1$. We show that $f(0)$ is bounded. As we know each element in a locally convex cone is bounded below. So, $f(0)$ is bounded below and hence there exists a positive real number λ_1 such that $f(0)+\lambda_1 v \geqslant 0$ for the generating element $v \in \mathscr{V}_2$. On the other hand, $\mathscr{P}_2$ is a vector space at the same time. So, $-f(0) \in \mathscr{P}_2$ and $-f(0)$ is also bounded below and then there exists a positive real number λ_2 sucth that $-f(0)+\lambda_2 v \geqslant 0$. This entails that $f(0) \leqslant \lambda_2 v$ and hence $f(0)$ is also upper bounded, i.e., $f(0)$ is bounded. Using Lemma 1, one concludes that $\lim_{n\to\infty}\frac{1}{2^n}f(0)=0$. Applying (1) we see that

$$g(x)-f(x) \in \lambda\,(v(0)v)$$

for some $0<\lambda \leqslant \varepsilon$. Since the symmetric neighborhood $v(0)v$ is convex and containing 0, we have $\lambda\,(v(0)v) \subseteq \varepsilon\,(v(0)v)$. This entails that

$$g(x)-f(x) \in \varepsilon\,(v(0)v)=\varepsilon v(0)\varepsilon v$$

for all $x \in \mathscr{P}_1$. Then

$$g(x) \in \varepsilon v(f(x))\varepsilon v$$

for all $x \in \mathscr{P}_1$. □

Acknowledgements. The authors thank the anonymous reviewers for their constructive comments and suggestions which improved the quality of the paper.

Declaration. The authors declare that they have no known conflict of interest, data sharing is not applicable to this manuscript, and no data sets were generated during the current study. We declare that I. Nikoufar and A. Ranjbari are corresponding authors and have an equal share in writing the manuscript and the order of authors listed in the manuscript has been approved by all of us. We would like to confirm that there has not been any significant financial support for this work that could have influenced its outcome.

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(Received August 12, 2024)

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