Journal of Mathematical Inequalities
Volume 18, Number 4 (2024), 1313–1326
doi:10.7153/jmi-2024-18-76

BERNSTEIN TYPE INEQUALITIES FOR SCHUR–SZEGÖ COMPOSITION OF POLYNOMIALS

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(*Communicated by V. Rao Allu*)

Abstract. In this paper we prove some inequalities for Schur-Szegö composition of polynomials, which inter-alia include classical Bernstein type inequalities for polynomials with restricted zeros.

1. Introduction

If $P(z)$ is a polynomial of degree n, then concerning the estimate of $|P'(z)|$ and $|P(Rz)|$, $R \geqslant 1$ on the unit disk $|z| = 1$, we have

$$\max_{|z|=1} |P'(z)| \leqslant n \max_{|z|=1} |P(z)| \tag{1}$$

and

$$\max_{|z|=R} |P(z)| = \max_{|z|=1} |P(Rz)| \leqslant R^n \max_{|z|=1} |P(z)|. \tag{2}$$

Inequality (1) is an immediate consequence of Bernstein's inequality [3] for the derivative of a trigonometric polynomial. Inequality (2) is a simple consequence of maximum modulus principle (for reference see [6]), for every $R \geqslant 1$.

If we restrict ourselves to the class of polynomials having no zero in $|z| < 1$, then inequalities (1) and (2) gets sharpened and can be respectively replaced by

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{2} \max_{|z|=1} |P(z)| \tag{3}$$

and for $R \geqslant 1$

$$\max_{|z|=R} |P(z)| \leqslant \frac{R^n + 1}{2} \max_{|z|=1} |P(z)|. \tag{4}$$

Inequality (3) was conjectured by Erdös and later proved by Lax [4], whereas inequality (4) was proved by Ankeny and Rivlin [1].

Mathematics subject classification (2020): 12D10, 30C10, 30C15.
Keywords and phrases: Polynomial, composition, Hadmard's product, convolution, inequalities, Bernstein, Schur-Szegö.

Paper JMI-18-76

Given two polynomials $P(z)=\sum_{j=0}^{n}A_jz^j$ and $Q(z)=\sum_{j=0}^{n}B_jz^j$, both of degree n, their convolution or Schur-Szegö composition or Hadmard's product is defined by

$$(P*Q)(z)=\sum_{j=0}^{n}\frac{A_jB_j}{\binom{n}{j}}z^j. \tag{5}$$

Also for any two polynomials $P(z)$ and $Q(z)$, we define the composite polynomial $(P\circ Q)$ by

$$(P\circ Q)(z)=P(Q(z)).$$

For such compositions of polynomials, we have the following:

PREPOSITION 1.1. *Let $P(z)$ be a polynomial of degree n and $Q(z):=\sum_{j=0}^{n}\binom{n}{j}jz^j$, then*

$$(P*Q)(z)=zP'(z).$$

This result follows by simple calculations.

PREPOSITION 1.2. *Let $P(z)$ be a polynomial of degree n, then for any polynomial $g(z):=\sum_{j=0}^{n}\binom{n}{j}z^j$ and a linear polynomial $f(z)$, we have*

$$\{(P\circ f)*g\}(z)=(P\circ f)(z).$$

Proof. Let $f(z)=Rz+S$ be a linear polynomial and $P(z)=\sum_{j=0}^{n}A_jz^j$, therefore

$$\begin{aligned}(P\circ f)(z)&=P(Rz+S)\\&=\sum_{j=0}^{n}A_j(Rz+S)^j\\&=\sum_{j=0}^{n}\sum_{i=j}^{n}\binom{i}{i-j}A_iS^{i-j}R^jz^j.\end{aligned}$$

This gives by using (5)

$$\begin{aligned}\{(P\circ f)*g\}(z)&=\sum_{j=0}^{n}\frac{\binom{n}{j}\sum_{i=j}^{n}\binom{i}{i-j}A_iS^{i-j}R^j}{\binom{n}{j}}z^j\\&=\sum_{j=0}^{n}\sum_{i=j}^{n}\binom{i}{i-j}A_iS^{i-j}R^jz^j\\&=(P\circ f)(z).\quad\Box\end{aligned}$$

Using these observations, we now prove some Bernstein type inequalities for Schur-Szegö composition of polynomials. In this direction, we first prove:

THEOREM 1.3. *Let $P(z)$ be a polynomial of degree n and let $h(z) := \sum_{j=0}^{n} l_j z^j$ be a polynomial of degree n having all its zeros in the disk $|z| \leqslant 1$, then for $f(z) = Rz + S$, such that $|R| > 1 + |S|$, we have*

$$\max_{|z|=1} |((P \circ f) * h)(z)| \leqslant \sum_{j=0}^{n} |l_j||R^j||S^{n-j}| \max_{|z|=1} |P(z)|. \tag{6}$$

The result is sharp and equality holds for the polynomial $P(z) = az^n$, $a \neq 0$.

If we substitute $S = 0$ in (6), we get the following:

COROLLARY 1.4. *Let $P(z)$ be a polynomial of degree n and let $h(z) := \sum_{j=0}^{n} l_j z^j$ be a polynomial of degree n having all its zeros in the disk $|z| \leqslant 1$, then for $f(z) = Rz$, such that $|R| > 1$, we have*

$$\max_{|z|=1} |((P \circ f) * h)(z)| \leqslant |l_n||R^n| \max_{|z|=1} |P(z)|. \tag{7}$$

The result is sharp and equality holds for the polynomial $P(z) = az^n$, $a \neq 0$.

REMARK 1.5. Theorem 1.1 of Gulzar and Rather [7] follows from Corollary 1.4 on choosing $R > 1$ and Corollary 1.1 of the same paper follows from Corollary 1.4 on letting $R \to 1$.

Considering $h(z) = \sum_{j=0}^{n} \binom{n}{j} z^j$ in Theorem 1.3, we get the following:

COROLLARY 1.6. *Let $P(z)$ be a polynomial of degree n, then for $f(z) = Rz + S$, such that $|R| > 1 + |S|$, we have*

$$\max_{|z|=1} |(P \circ f)(z)| \leqslant \sum_{j=0}^{n} \binom{n}{j} |R^j||S^{n-j}| \max_{|z|=1} |P(z)|. \tag{8}$$

The result is sharp and equality holds for the polynomial $P(z) = az^n$, $a \neq 0$.

On choosing $h(z) = \sum_{j=0}^{n} \binom{n}{j} j z^j$ in Theorem 1.3, we get the following:

COROLLARY 1.7. *Let $P(z)$ be a polynomial of degree n, then for $f(z) = Rz + S$, such that $|R| > 1 + |S|$, we have*

$$\max_{|z|=1} |(P \circ f)'(z)| \leqslant \sum_{j=1}^{n} j \binom{n}{j} |R^j||S^{n-j}| \max_{|z|=1} |P(z)|. \tag{9}$$

The result is sharp and equality holds for the polynomial $P(z) = az^n$, $a \neq 0$.

REMARK 1.8. Inequality (2) follows from Corollary 1.6 if we take $S = 0$ and choose $R > 1$. Inequality (1) is a special case of Corollary 1.7, when we take $S = 0$ and let $R \to 1$.

Further for $h(z) = z^n + z^k$ in Theorem 1.3, where $k = 0, 1, 2, \dots, n-1$, we have the following:

COROLLARY 1.9. *Let $P(z)$ be a polynomial of degree n, then for any R, S, such that $|R| > 1 + |S|$*

$$|R^n||a_n| + \frac{|R^k|}{\binom{n}{k}} \sum_{j=k}^{n} \binom{j}{j-k} |S^{j-k}||a_j| \leqslant (|R^k||S^{n-k}| + |R^n|) \max_{|z|=1} |P(z)|. \tag{10}$$

REMARK 1.10. On substituting $S = 0$ and letting $R \to 1$, we obtain an extension of Visser's inequality [8] due to Gulzar and Rather [7].

Next, we prove a result concerning the minimum modulus of a polynomial $P(z)$ on $|z| = 1$, with the restriction on the zeros of $P(z)$. In this case, we have the following result.

THEOREM 1.11. *Let $P(z)$ and $h(z) := \sum_{j=0}^{n} l_j z^j$ be polynomials of degree n having all their zeros in the disk $|z| \leqslant 1$, then for $f(z) = Rz + S$, such that $|R| > 1 + |S|$, we have*

$$|((P \circ f) * h)(z)| \geqslant |\sum_{j=0}^{n} R^j S^{n-j} l_j z^j| \min_{|z|=1} |P(z)|. \tag{11}$$

The result is sharp and equality holds for the polynomial $P(z) = az^n$, $a \neq 0$.

If we substitute $S = 0$ in (11), we get the following:

COROLLARY 1.12. *Let $P(z)$ and $h(z) := \sum_{j=0}^{n} l_j z^j$ be polynomials of degree n having all their zeros in the disk $|z| \leqslant 1$, then for $f(z) = Rz$, such that $|R| > 1$, we have*

$$\min_{|z|=1} |((P \circ f) * h)(z)| \geqslant |R^n||l_n| \min_{|z|=1} |P(z)|. \tag{12}$$

REMARK 1.13. Theorem 1.2 and Corollary 1.3 of Gulzar and Rather [7] are the special cases of the above result for choice of R as $R > 1$ and $R \to 1$ respectively.

On choosing $h(z) = \sum_{j=0}^{n} \binom{n}{j} z^j$ in Theorem 1.11, we get the following:

COROLLARY 1.14. *Let $P(z)$ be a polynomial of degree n, then for $f(z) = Rz + S$, such that $|R| > 1 + |S|$, we have*

$$|(P \circ f)(z)| \geqslant |\sum_{j=0}^{n} \binom{n}{j} R^j S^{n-j} z^j| \min_{|z|=1} |P(z)|. \tag{13}$$

The result is sharp and equality holds for the polynomial $P(z) = az^n$, $a \neq 0$.

Considering $h(z) = \sum_{j=0}^{n} \binom{n}{j} j z^j$ in Theorem 1.11, we get the following:

COROLLARY 1.15. *Let $P(z)$ be a polynomial of degree n, then for $f(z) = Rz + S$, such that $|R| > 1 + |S|$, we have*

$$|z(P \circ f)'(z)| \geqslant |\sum_{j=0}^{n} j \binom{n}{j} R^j S^{n-j} z^j| \min_{|z|=1} |P(z)|. \tag{14}$$

The result is sharp and equality holds for the polynomial $P(z) = az^n$, $a \neq 0$.

REMARK 1.16. Substituting $S = 0$ and with the suitable choice of R, Theorem 1 due to Aziz and Dawood [2] follows from above results.

Also choosing $h(z) = z^n + z^k$ in Theorem 1.11, where $k = 0, 1, 2, \dots, n-1$, we have the following result.

COROLLARY 1.17. *Let $P(z)$ be a polynomial of degree n, then for any R, S, such that $|R| > 1 + |S|$, we have*

$$\left||R^n||a_n| - \frac{|R^k|}{\binom{n}{k}} \sum_{j=k}^{n} \binom{j}{j-k} |S^{j-k}||a_j|\right| \geqslant |\sum_{j=0}^{n} R^j S^{n-j} z^j| \min_{|z|=1} |P(z)|. \tag{15}$$

REMARK 1.18. On substituting $S = 0$ and letting $R \to 1$, we obtain Corollary 1.4 due to Gulzar and Rather [7].

We also prove the following results for the class of polynomials having no zeros in $|z| < 1$.

THEOREM 1.19. *Let $P(z)$ be a polynomial of degree n having no zero in $|z| < 1$ and let $h(z) = \sum_{j=0}^{n} l_j z^j$ be a polynomial of degree n having all its zero in $|z| \leqslant 1$, then for $f(z) = Rz + S$, such that $|R| > 1 + |S|$, we have*

$$\max_{|z|=1} |((P \circ f) * h)(z)| \leqslant \frac{1}{2} \{ \sum_{j=0}^{n} |l_j||R^j||S^{n-j}| + |l_0| \} \max_{|z|=1} |P(z)|. \tag{16}$$

The result is sharp and equality holds for the polynomial $P(z) = az^n + b$, $|a| = |b| \neq 0$.

If we take $S = 0$ in (1.19), we get the following:

COROLLARY 1.20. *Let $P(z)$ be a polynomial of degree n having no zero in $|z| < 1$ and let $h(z) = \sum_{j=0}^{n} l_j z^j$ be a polynomial of degree n having all its zeros in $|z| \leqslant 1$, then for $f(z) = Rz$, such that $|R| > 1$, we have*

$$\max_{|z|=1} |((P \circ f) * h)(z)| \leqslant \frac{1}{2} \{ |l_n||R^n| + |l_0| \} \max_{|z|=1} |P(z)|. \tag{17}$$

The result is sharp and equality holds for the polynomial $P(z) = az^n + b$, $|a| = |b| \neq 0$.

REMARK 1.21. Theorem 1.3 and Corollary 1.5 of Gulzar and Rather [7] are the special cases of the above results if we choose $R>1$ and $R\to 1$ respectively.

Considering $h(z)=\sum_{j=0}^{n}\binom{n}{j}z^j$ in Theorem 1.19, we get the following:

COROLLARY 1.22. *Let $P(z)$ be a polynomial of degree n, then for $f(z)=Rz+S$, such that $|R|>1+|S|$, we have*

$$\max_{|z|=1}|(P\circ f)(z)| \leqslant \frac{1}{2}\Big\{\sum_{j=0}^{n}\binom{n}{j}|R^j||S^{n-j}|+1\Big\}\max_{|z|=1}|P(z)|. \tag{18}$$

The result is sharp and equality holds for the polynomial $P(z)=az^n+b,\ |a|=|b|\neq 0$.

Taking $h(z)=\sum_{j=0}^{n}\binom{n}{j}jz^j$ in Theorem 1.3, we have the following:

COROLLARY 1.23. *Let $P(z)$ be a polynomial of degree n, then for $f(z)=Rz+S$, such that $|R|>1+|S|$, we have*

$$\max_{|z|=1}|(P\circ f)'(z)| \leqslant \frac{1}{2}\Big\{\sum_{j=1}^{n}j\binom{n}{j}|R^j||S^{n-j}|\Big\}\max_{|z|=1}|P(z)|. \tag{19}$$

The result is sharp and equality holds for the polynomial $P(z)=az^n+b,\ |a|=|b|\neq 0$.

REMARK 1.24. Substituting $S=0$ in Corollary 1.22 and choosing $R>1$, we get inequality (4). Also taking $S=0$ and letting $R\to 1$ in Corollary 1.23, we get inequality (3).

By taking $h(z)=z^n+z^k,\ 0\leqslant k\leqslant n-1$ in Theorem 1.19, the following result follows.

COROLLARY 1.25. *Let $P(z)$ be a polynomial of degree n, then for any R,S, such that $|R|>1+|S|$*

$$|R^n||a_n|+\frac{|R^k|}{\binom{n}{k}}\sum_{j=k}^{n}\binom{j}{j-k}|S^{n-k}||a_j|| \leqslant \frac{\Lambda}{2}\max_{|z|=1}|P(z)|, \tag{20}$$

where

$$\Lambda=\begin{cases}|R^n|+|R^k||S^{n-k}|, & \text{if } 1\leqslant k\leqslant n-1\\ |R^n|+|S^n|+1, & \text{if } k=0\end{cases}$$

REMARK 1.26. On substituting $S=0$ and letting $R\to 1$, we obtain Corollary 1.6 due to Gulzar and Rather [7].

A polynomial $P(z)$ of degree n is said to be self-inversive if $P(z)\equiv uP^*(z)$, where $|u|=1$ and $P^*(z)$ is the conjugate polynomial of $P(z)$, that is, $P^*(z)=z^n\overline{P(\frac{1}{\bar{z}})}$. We next present the following result for self-inversive polynomials.

THEOREM 1.27. *Let $P(z)$ be a self-inversive polynomial of degree n and let $h(z)=\sum_{j=0}^{n} l_j z^j$ be a polynomial of degree n having all its zero in $|z|\leqslant 1$, then for $f(z)=Rz+S$, such that $|R|>1+|S|$, we have*

$$\max_{|z|=1}|((P\circ f)*h)(z)|\leqslant \frac{1}{2}\{\sum_{j=0}^{n}|l_j||R^j||S^{n-j}|+|l_0|\}\max_{|z|=1}|P(z)|. \tag{21}$$

The result is sharp and equality holds for the polynomial $P(z)=az^n+b$, $|a|=|b|\neq 0$.

REMARK 1.28. If we substitute $S=0$ in Theorem 1.27 and choose $R>1$, Theorem 1.4 due to Gulzar and Rather [7] follows.

2. Lemmas

To prove these results, we need the following lemmas. The first lemma is a consequence of the Schur-Szegö theorem (for refrence see [5]).

LEMMA 2.1. *Let f and g be polynomials of degree n. If all the zeros of f are of modulus at most r and all the zeros of g are of modulus at most s, then all the zeros of $f*g$ are of modulus at most rs.*

LEMMA 2.2. *Let $F(z)$ and $h(z)$ be polynomials of degree n having all their zeros in $|z|\leqslant 1$, and let $P(z)$ be a polynomial of degree n, such that $|P(z)|\leqslant|F(z)|$ for $|z|=1$. Then*

$$|\{(P\circ f)*h\}(z)|\leqslant|\{(F\circ f)*h\}(z)| \text{ for } |z|=1, \tag{22}$$

where $f(z)=Rz+S$, with R,S, such that $|R|>1+|S|$.

Proof. Let $P(z)=\sum_{j=0}^{n}a_jz^j$, $h(z)=\sum_{j=0}^{n}l_jz^j$, $F(z)=\sum_{j=0}^{n}b_jz^j$. If $P^*(z)=z^n\overline{P(\frac{1}{\bar{z}})}$ and $F^*(z)=z^n\overline{F(\frac{1}{\bar{z}})}$, then for $|z|=1$, we have

$$|P^*(z)|=|P(z)| \text{ and } |F^*(z)|=|F(z)|.$$

Since $|P(z)|\leqslant|F(z)|$ for $|z|=1$. Therefore we have

$$|P^*(z)|\leqslant|F^*(z)| \text{ for } |z|=1. \tag{23}$$

By hypothesis all the zeros of $F(z)$ lie in $|z|\leqslant 1$, therefore all the zeros of $F^*(z)$ lie in $|z|\geqslant 1$. Also noting that by (23), on $|z|=1$, zeros of $F^*(z)$ are also zeros of $P^*(z)$. We conclude that the function $H(z)=\frac{P^*(z)}{F^*(z)}$ is analytic in $|z|\leqslant 1$ and $|H(z)|\leqslant 1$ for $|z|=1$. Therefore by maximum modulus principle,

$$|H(z)|\leqslant 1, \text{ for } |z|\leqslant 1.$$

This implies

$$|P^*(z)|\leqslant|F^*(z)|, \text{ for } |z|\leqslant 1.$$

That is

$$|P(z)| \leqslant |F(z)|, \text{ for } |z| \geqslant 1. \tag{24}$$

Substitute $Re^{i\theta}+S$, $0 \leqslant \theta < 2\pi$, for z and note that by assumption

$$\begin{aligned}|z| &= |Re^{i\theta}+S| \\ &\geqslant |Re^{i\theta}|-|S| \\ &= |R|-|S| \\ &> 1,\end{aligned}$$

we get

$$|P(Rz+S)| \leqslant |F(Rz+S)|, \text{ for } |z| = 1.$$

Therefore for any $\alpha \in \mathbb{C}$, $|\alpha| > 1$

$$|P(Rz+S)| < |\alpha F(Rz+S)|, \text{ for } |z| = 1.$$

Also, all the zeros of the polynomial $F(Rz+S)$ lie in $|z| \leqslant \frac{1+|S|}{|R|} < 1$. Therefore by Rouche's theorem, it follows that all the zeros of the polynomial

$$\begin{aligned}(P\circ f)(z)-\alpha(F\circ f)(z) &= P(Rz+S)-\alpha F(Rz+S) \\ &= \sum_{j=0}^{n}\sum_{i=j}^{n}\binom{i}{i-j}S^{i-j}R^{j}(a_i-\alpha b_i)z^{j}\end{aligned}$$

lie in $|z| < 1$. Also by hypothesis, all the zeros of $h(z)$ lie in $|z| \leqslant 1$. This implies with the help of Lemma 2.1 that all the zeros of the polynomial

$$\begin{aligned}\{((P\circ f)(z)-\alpha(F\circ f))*h\}(z) &= \sum_{j=0}^{n}\frac{l_j\sum_{i=j}^{n}\binom{i}{i-j}S^{i-j}R^{j}(a_i-\alpha b_i)}{\binom{n}{j}}z^{j} \\ &= \sum_{j=0}^{n}\frac{l_j\sum_{i=j}^{n}\binom{i}{i-j}S^{i-j}R^{j}a_i}{\binom{n}{j}}z^{j}- \\ &\qquad \alpha\sum_{j=0}^{n}\frac{l_j\sum_{i=j}^{n}\binom{i}{i-j}S^{i-j}R^{j}b_i}{\binom{n}{j}}z^{j} \\ &= \{(P\circ f)*h\}(z)-\alpha\{(F\circ f)*h\}(z)\end{aligned} \tag{25}$$

has all its zeros in $|z| < 1$. This implies for $|z| \geqslant 1$ and $|R| > 1+|S|$

$$|\{(P\circ f)*h\}(z)| \leqslant |\{(F\circ f)*h\}(z)|. \tag{26}$$

If inequality (26) is not true, then there exists a point z_0 with $|z_0| \geqslant 1$, such that

$$|\{(P\circ f)*h\}(z_0)| > |\{(F\circ f)*h\}(z_0)|.$$

But all the zeros of the polynomial $(F\circ f)(z) = F(Rz+S)$, $|R| > 1+|S|$ lie in $|z| \leqslant \frac{1+|S|}{|R|} < 1$ and all the zeros of $h(z)$ lie in $|z| \leqslant 1$. Therefore by Lemma 2.1, all the zeros of $((F\circ f)*h)(z)$ lie in $|z| < 1$ and hence

$$((F\circ f)*h)(z_0) \neq 0, \text{ as } |z_0| \geqslant 1.$$

We take

$$\alpha = \frac{((P\circ f)*h)(z_0)}{((F\circ f)*h)(z_0)},$$

so that α is well defined real or complex number with $|\alpha| > 1$. With this choice of α, we obtain

$$((P\circ f)*h)(z_0) - \alpha((F\circ f)*h)(z_0) = 0 \quad \text{for } |z_0| \geqslant 1.$$

This is contradiction to the fact that all the zeros of $\{\big((P\circ f)-\alpha(F\circ f)\big)*h\}(z)$ lie in $|z| < 1$.

Hence the proof is complete. □

As a consequence of Lemma 2.2, we have the following:

LEMMA 2.3. *If $P(z)$ is a polynomial of degree n, not vanishing in $|z| < 1$ and $P^*(z) = z^n\overline{P(\frac{1}{\bar z})}$, then*

$$|((P\circ f)*h)(z)| \leqslant |((P^*\circ f)*h)(z)| \text{ for } |z| = 1,$$

where $f(z) = Rz+S$, $|R| > 1+|S|$ and $h(z) := \sum_{j=0}^{n} l_j z^j$ is a polynomial of degree n with all zeros in $|z| \leqslant 1$.

LEMMA 2.4. *Let $P(z)$ be a polynomial of degree n and let $h(z) = \sum_{j=0}^{n} l_j z^j$ be a polynomial of degree n having all zeros in the disk $|z| \leqslant 1$, then for $f(z) = Rz+S$, such that, $|R| > 1+|S|$ and $|z| = 1$, we have*

$$|((P\circ f)*h)(z)| + |((P^*\circ f)*h)(z)| \leqslant \{\sum_{j=0}^{n} |l_j||R^j||S^{n-j}| + |l_0|\} \max_{|z|=1} |P(z)|, \tag{27}$$

where $P^*(z) = z^n\overline{P(\frac{1}{\bar z})}$.

Proof. Let $M = \max_{|z|=1}|P(z)|$. Since $P(z)$ is a polynomial of degree n and $|P(z)| \leqslant M$, for $|z| = 1$. Therefore, by Rouche's theorem $G(z) = P(z) - \gamma M = P(z) - I_1(z)$ doesn't vanish in $|z| < 1$, for every complex number γ with $|\gamma| > 1$ and $I_1(z) = \gamma M$. Also, let

$$\begin{aligned} G^*(z) &= z^n\overline{G\Big(\frac{1}{\bar z}\Big)} \\ &= z^n\overline{P\Big(\frac{1}{\bar z}\Big)} - z^n\bar\gamma\bar M \\ &= P^*(z) - z^n\bar\gamma M \\ &= P^*(z) - I_2(z), \end{aligned}$$

where $I_2(z) = \bar{\gamma}Mz^n$. Therefore, by Lemma 2.3, we have for $|z| = 1$

$$|((G \circ f) * h)(z)| \leqslant |((G^* \circ f) * h)(z)|. \tag{28}$$

Now

$$\begin{aligned}((Gof) * h)(z) &= (((P - I_1) \circ f) * h)(z) \\ &= (((P \circ f) - (I_1 \circ f)) * h)(z) \\ &= ((Pof) * h)(z) - ((I_1 \circ f) * h)(z) \\ &= ((Pof) * h)(z) - (I_1 * h)(z) \\ &= ((Pof) * h)(z) - \gamma M l_0. \end{aligned} \tag{29}$$

Also

$$\begin{aligned}((G^* \circ f) * h)(z) &= \{((P^* - I_2) \circ f) * h\}(z) \\ &= ((P^* \circ f) * h)(z) - ((I_2 \circ f) * h)(z) \\ &= ((P^* \circ f) * h)(z) - (I_2 \circ f)(z) * h(z) \\ &= ((P^* \circ f) * h)(z) - I_2(Rz + S) * h(z) \\ &= ((P^* \circ f) * h)(z) - \bar{\gamma}M(Rz + S)^n * h(z) \\ &= ((P^* \circ f) * h)(z) - \bar{\gamma}M \sum_{j=0}^{n} \binom{n}{n-j} R^j S^{n-j} z^j * \sum_{j=0}^{n} l_j z^j \\ &= ((P^* \circ f) * h)(z) - M\bar{\gamma} \sum_{j=0}^{n} l_j R^j S^{n-j} z^j. \end{aligned} \tag{30}$$

Using (29) and (30) in (28), we get for $|z| = 1$

$$|((P \circ f) * h)(z) - \gamma M l_0| \leqslant |((P^* \circ f) * h)(z) - M\bar{\gamma} \sum_{j=0}^{n} l_j R^j S^{n-j} z^j|. \tag{31}$$

Since

$$|P^*(z)| = |P(z)| \leqslant M \ \text{ for } \ |z| = 1,$$

therefore, by Theorem 1.3, we have for $|z| = 1$,

$$|((P^* \circ f) * h)(z)| \leqslant M \sum_{j=0}^{n} |l_j||R^j||S^{n-j}|. \tag{32}$$

Choose argument of γ in (31), which is possible by (32), such that for $|z| = 1$,

$$|((P^* \circ f) * h)(z) - M\bar{\gamma} \sum_{j=0}^{n} l_j R^j S^{n-j}| = |\bar{\gamma}M \sum_{j=0}^{n} l_j R^j S^{n-j}| - |((P^* \circ f) * h)(z)|.$$

Using this in (31), we get for $|z| = 1$,

$$|((P \circ f) * h)(z) - \gamma M l_0| \leqslant |\bar{\gamma}M \sum_{j=0}^{n} l_j R^j S^{n-j}| - |((P^* \circ f) * h)(z)|.$$

This gives for $|z|=1$,

$$|((P\circ f)*h)(z)|-|\gamma Ml_0|\leqslant|\bar{\gamma}M\sum_{j=0}^{n}l_jR^jS^{n-j}|-|((P^*\circ f)*h)(z)|.$$

Hence for $|z|=1$, after letting $|\gamma|\to 1$, we get

$$\begin{aligned}|((P\circ f)*h)(z)|+|((P^*\circ f)*h)(z)|&\leqslant|M\sum_{j=0}^{n}l_jR^jS^{n-j}|+|Ml_0|\\&\leqslant M\{\sum_{j=0}^{n}|l_j||R^j||S^{n-j}|+|l_0|\}.\end{aligned}$$

This proves Lemma 2.4. □

3. Proof of theorems

Proof of Theorem 1.3. Consider the polynomial $F(z)=Mz^n$, where $M=\max_{|z|=1}|P(z)|$. Since, $|P(z)|\leqslant M$ for $|z|=1$, therefore $|P(z)|\leqslant|Mz^n|$ for $|z|=1$. This in particular gives

$$|P(z)|\leqslant|F(z)|\ \text{ for }\ |z|=1.$$

This shows that $P(z)$ and $F(z)$ satisfy the conditions of Lemma 2.2 and therefore

$$|((P\circ f)*h)(z)|\leqslant|((F\circ f)*h)(z)|\ \text{ for }\ |z|=1, \tag{33}$$

where $h(z)=\sum_{j=0}^{n}l_jz^j$ is a polynomial of degree n having all its zeros in the disk $|z|\leqslant 1$. Now,

$$\begin{aligned}(F\circ f)(z)&=F(f(z))\\&=F(Rz+S)\\&=M(Rz+S)^n\\&=M\sum_{j=0}^{n}\binom{n}{n-j}R^jS^{n-j}z^j.\end{aligned}$$

Therefore from Definition of Hadmard's product, we have

$$\begin{aligned}((F\circ f)*h)(z)&=M\sum_{j=0}^{n}\frac{\binom{n}{n-j}l_jR^jS^{n-j}}{\binom{n}{j}}z^j\\&=M\sum_{j=0}^{n}l_jR^jS^{n-j}z^j.\end{aligned}$$

This implies for $|z|=1$,

$$\begin{aligned}|((F\circ f)*h)(z)| &= |M\sum_{j=0}^{n} l_jR^jS^{n-j}z^j| \\ &\leqslant M\sum_{j=0}^{n}|l_jR^jS^{n-j}z^j| \\ &= M\sum_{j=0}^{n}|l_j||R^j||S^{n-j}|.\end{aligned}$$

Using this, we get from inequality (33)

$$|((P\circ f)*h)(z)| \leqslant M\sum_{j=0}^{n}|l_j||R^j||S^{n-j}|.$$

From this the required result follows. □

Proof of Theorem 1.11. Consider the polynomial $F(z)=mz^n$, where $m=\min_{|z|=1}|P(z)|$. If $P(z)$ has a zero on $|z|=1$, then the result is trivial. Therefore, assume $P(z)$ has all the zeros in $|z|<1$, so that $m>0$. Also $|P(z)|\geqslant m$ for $|z|=1$ gives $|P(z)|\geqslant|mz^n|$ for $|z|=1$. That is,

$$|P(z)|\geqslant|F(z)| \text{ for } |z|=1.$$

This shows that $P(z)$ and $F(z)$ satisfy the conditions of Lemma 2.3 and therefore

$$|((P\circ f)*h)(z)|\geqslant|((F\circ f)*h)(z)| \text{ for } |z|=1. \tag{34}$$

Now, as in the case of above theorem, we have

$$(F\circ f)(z)=m\sum_{j=0}^{n}\binom{n}{n-j}R^jS^{n-j}z^j.$$

This gives by the convolution of $(P\circ f)$ and h

$$\begin{aligned}((P\circ f)*h)(z) &= m\sum_{j=0}^{n}\frac{\binom{n}{n-j}l_jR^jS^{n-j}}{\binom{n}{j}}z^j \\ &= m\sum_{j=0}^{n}l_jR^jS^{n-j}z^j.\end{aligned}$$

That is,

$$|((F\circ f)*h)(z)| = m|\sum_{j=0}^{n}l_jR^jS^{n-j}z^j|.$$

Therefore inequality (34) implies,

$$|((P\circ f)*h)(z)| \geqslant m|\sum_{j=0}^{n}l_jR^jS^{n-j}z^j|.$$

That is

$$|((P\circ f)*h)(z)|\geqslant\Big|\sum_{j=0}^{n}l_jR^jS^{n-j}z^j\Big|\min_{|z|=1}|P(z)|.$$

This completes the proof. □

Proof of Theorem 1.19. We know from Lemma 2.3, that if $P(z)$ is a polynomial of degree n, not vanishing in $|z|<1$, $h(z)=\sum_{j=0}^{n}l_jz^j$, $f(z)=Rz+S$ and $P^*(z)=z^n\overline{P(\frac{1}{\bar z})}$, then

$$|((P\circ f)*h)(z)|\leqslant|((P^*\circ f)*h)(z)|\ \text{ for }\ |z|=1. \tag{35}$$

Also by Lemma 2.4, for every $|R|>1+|S|$ and $|z|=1$,

$$|((P\circ f)*h)(z)|+|((P^*\circ f)*h)(z)|\leqslant M\{\sum_{j=0}^{n}|l_j||R^j||S^{n-j}|+|l_0|\}. \tag{36}$$

where $f(z)=Rz+S$, such that $|R|>1+|S|$.

Combining (35) and (36), we have for $|R|>1+|S|$ and $|z|=1$

$$2|((P\circ f)*h)(z)|\leqslant M\{\sum_{j=0}^{n}|l_j||R^j||S^{n-j}|+|l_0|\}.$$

This implies for $|R|>1+|S|$ and $|z|=1$

$$|((P\circ f)*h)(z)|\leqslant\frac{1}{2}\{\sum_{j=0}^{n}|l_j||R^j||S^{n-j}|+|l_0|\}\max_{|z|=1}|P(z)|.$$

This completes the proof. □

*Proof of Theorem*1.27. Since $P(z)$ is a self-inverse polynomial of degree n, therefore for some $u\in\mathbb{C}$ with $|u|=1$, we have $P(z)=uP^*(z)$ for all $z\in\mathbb{C}$, where $P^*(z)=z^p\overline{P(\frac{1}{\bar z})}$. This gives, for $|z|=1$,

$$|((P\circ f)*h)(z)|=|((P^*\circ f)*h)(z)|. \tag{37}$$

By Lemma 2.4, we have for $|z|=1$,

$$|((P\circ f)*h)(z)|+|((P^*\circ f)*h)(z)|\leqslant\{\sum_{j=0}^{n}|l_j||R^j||S^{n-j}|+|l_0|\}M.$$

Using (37), we get

$$2|((P\circ f)*h)(z)|\leqslant\{\sum_{j=0}^{n}|l_j||R^j||S^{n-j}|+|l_0|\}M.$$

In particular,

$$\max_{|z|=1}|((P\circ f)*h)(z)|\leqslant\frac{1}{2}\{\sum_{j=0}^{n}|l_j||R^j||S^{n-j}|+|l_0|\}\max_{|z|=1}|P(z)|.$$

This completes the proof of the theorem. □

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(Received March 18, 2024)

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