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GENERALIZATIONS OF NUMERICAL RADIUS INEQUALITIES FOR HILBERT SPACE OPERATORS

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Abstract. In this work, some generalizations of numerical radius inequalities for sums and products of Hilbert space operators are presented. These new inequalities generalize some existing inequalities given in [3] and [8].

1. Introduction

Let $(\mathcal{H},\langle\cdot,\cdot\rangle)$ be a complex Hilbert space equipped with the norm $\|\cdot\|$, and let $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. A bounded linear operator T defined on $\mathcal{H}$ is self-adjoint if and only if $\langle Tx,x\rangle\in\mathbb{R}$ for all $x\in\mathcal{H}$. Recall that T is called positive if $\langle Tx,x\rangle\geqslant 0$ for all $x\in\mathcal{H}$. For a positive operator T, we write $T\geqslant 0$. We write $T>0$ to mean that T is a strictly positive operator ($T\geqslant 0$ and T is invertible). For $T\in B(\mathcal{H})$, let T^* be the adjoint of T. Also, $|T|$ and $|T^*|$ denote the positive operators $(T^*T)^{\frac{1}{2}}$ and $(TT^*)^{\frac{1}{2}}$, respectively.

The numerical range of T, denoted by $W(T)$, is defined as

$$W(T)=\{\langle Tx,x\rangle : x\in\mathcal{H},\|x\|=1\}.$$

Also, the numerical radius is defined to be

$$\omega(T)=\sup\{|\lambda| : \lambda\in W(T)\}=\sup_{\|x\|=1}|\langle Tx,x\rangle|.$$

We recall that the usual operator norm of an operator T is defined to be

$$\|T\|=\sup\{\|Tx\| : x\in\mathcal{H},\|x\|=1\}.$$

For $T\in B(\mathcal{H})$, the numerical radius satisfies the following well-known classical inequalities

$$\frac{1}{2}\|T\|\leqslant\omega(T)\leqslant\|T\|.$$

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The first inequality becomes an equality if $T^2 = 0$, and the second inequality becomes an equality if T is normal.

The study of the numerical range and numerical radius has a long and distinguished history, and has attracted the attention of many authors; for more details and recent results about the numerical radius, we refer the readers to [2–5, 7–9, 11, 23]. The subject is related and has applications to many different areas in pure and applied mathematics such as operator theory, functional analysis, Banach algebras, numerical analysis, perturbation theory, etc. Some interesting numerical radius inequalities improving and generalizing the classical inequalities have been obtained by several mathematicians (see, e.g., [6, 8, 10, 14, 20–22]).

The spectrum of an operator T is the set of all $\lambda \in \mathbb{C}$ for which the operator $\lambda I - T$ does not have a bounded liner operator inverse, and is denoted by $\sigma(T)$. The spectral radius of an operator T is defined to be

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

It is well-known that closure of the numerical range contains the spectrum, so $r(T) \leqslant \omega(T)$. Over the years, various numerical radius inequalities have been proved to improve on the classical inequalities. The classical inequalities comparing $\omega(T)$ and $\|T\|$ have been improved by Kittaneh in [16] and [19]. It has been shown that if $T \in B(\mathcal{H})$, then

$$\omega(T) \leqslant \frac{1}{2}\||T| + |T^*|\| \leqslant \frac{1}{2}\left(\|T\| + \|T^2\|^{\frac{1}{2}}\right) \leqslant \|T\|$$

and

$$\frac{1}{4}\left\||T|^2 + |T^*|^2\right\| \leqslant \omega^2(T) \leqslant \frac{1}{2}\left\||T|^2 + |T^*|^2\right\|.$$

In [13], the authors obtained a generalization of the above second inequality so that

$$\omega^p(T) \leqslant \frac{1}{2}\left\||T|^{2p\lambda} + |T^*|^{2p(1-\lambda)}\right\|$$

for $p \geqslant 1$ and $0 \leqslant \lambda \leqslant 1$.

Recently, Bhunia and Paul in [9] obtained some inequalities for the numerical radii of bounded linear operators. In particular,

$$\omega^{2p}(T) \leqslant \left\|\frac{\alpha}{2}(|T|^{4p\lambda} + |T^*|^{4p(1-\lambda)}) + (1-\alpha)|T^*|^{2p}\right\|$$

and

$$\omega^{2p}(T) \leqslant \left\|\frac{\alpha}{2}(|T|^{4p\lambda} + |T^*|^{4p(1-\lambda)}) + (1-\alpha)|T|^{2p}\right\|$$

for $p \geqslant 1$ and $0 \leqslant \alpha, \lambda \leqslant 1$.

Also, Bhunia and Paul in [8] obtained some numerical radius inequalities for sums and products of Hilbert space operators. Let $T, S \in B(\mathcal{H})$ be such that $|T|S = S^*|T|$, and let f,g be non-negative continuous functions on $[0,\infty)$ satisfying $f(t)g(t)=t$ for all $t\in[0,\infty)$. Then

$$\omega(TS)\leqslant\frac{r(S)}{2}\left(\max\left\{\|f(|T|)\|^2,\|g|T^*|\|^2\right\}+\|f(|T|)g(|T^*|)\|\right).$$

In this paper, we present some inequalities for the numerical radii of bounded linear operators. These inequalities generalize some existing inequalities given in [3,8].

2. Numerical radius inequalities for operators

We need the following lemmas to prove our main results.

LEMMA 2.1. [18] *Let $T,S\in B(\mathcal{H})$ be such that $|T|S=S^*|T|$, and let f,g be non-negative continuous functions on $[0,\infty)$ satisfying $f(t)g(t)=t$ for all $t\in[0,\infty)$. Then $|\langle TSx,y\rangle|\leqslant r(S)\|f(|T|)x\|\|g(|T^*|)y\|$ for all $x,y\in\mathcal{H}$.*

LEMMA 2.2. [5, p. 9] *Let T be a positive operator in $B(\mathcal{H})$. Then, for any unit vector $x\in\mathcal{H}$, the following inequalities hold:*
(i) $\langle T^px,x\rangle\leqslant\langle Tx,x\rangle^p$ for $0<p\leqslant1$.
(ii) $\langle Tx,x\rangle^p\leqslant\langle T^px,x\rangle$ for $p\geqslant1$.

The following lemma follows from the convexity of the function $f(t)=t^p$ on $[0,\infty)$ for $p\geqslant1$.

LEMMA 2.3. *Let a_i be a positive real number $(i=1,2,\ldots,n)$. Then*

$$\left(\sum_{i=1}^n a_i\right)^p\leqslant n^{p-1}\sum_{i=1}^n a_i^p \text{ for } p\geqslant1.$$

LEMMA 2.4. [1] *Let $T,S\in B(\mathcal{H})$ be positive operators. Then*

$$r(TS)=\left\|T^{\frac{1}{2}}S^{\frac{1}{2}}\right\|^2.$$

LEMMA 2.5. [17] *Let $T,S\in B(\mathcal{H})$ be positive operators. Then*

$$\|T+S\|\leqslant\max\{\|T\|,\|S\|\}+\|T^{\frac{1}{2}}S^{\frac{1}{2}}\|.$$

In the following theorem, we obtain a numerical radius inequality involving n-tuples of operators.

THEOREM 2.6. *Let $T_i,S_i\in B(\mathcal{H})$ be such that $|T_i|S_i=S_i^*|T_i|$ $(i=1,2,\ldots,n)$, and let f,g be non-negative continuous function on $[0,\infty)$ which are continuous and satisfy the relation $f(t)g(t)=t$ for all $t\in[0,\infty)$. Then*

$$\omega^p(\sum_{i=1}^n T_iS_i)\leqslant\frac{n^{p-1}}{2}\left\|\sum_{i=1}^n r^p(S_i)(f^{2p}(|T_i|)+g^{2p}(|T_i^*|))\right\|$$

for $p \geqslant 1$.

Proof. For every unit vector $x \in \mathcal{H}$, we get

$$\begin{aligned}
\left|\left\langle (\sum_{i=1}^{n} T_i S_i)x, x \right\rangle\right|^p &= \left|\sum_{i=1}^{n} \langle T_i S_i x, x\rangle\right|^p \\
&\leqslant \left(\sum_{i=1}^{n} |\langle T_i S_i x, x\rangle|\right)^p \\
&\leqslant \left(\sum_{i=1}^{n} r(S_i)\|f(|T_i|)x\|\|g(|T_i^*|)x\|\right)^p \text{ (by Lemma 2.1)} \\
&= \left(\sum_{i=1}^{n} r(S_i)\left\langle f^2(|T_i|)x, x\right\rangle^{\frac{1}{2}}\left\langle g^2(|T_i^*|)x, x\right\rangle^{\frac{1}{2}}\right)^p \\
&\leqslant n^{p-1}\sum_{i=1}^{n} r^p(S_i)\left\langle f^2(|T_i|)x, x\right\rangle^{\frac{p}{2}}\left\langle g^2(|T_i^*|)x, x\right\rangle^{\frac{p}{2}} \text{ (by Lemma 2.3)} \\
&\leqslant n^{p-1}\sum_{i=1}^{n} r^p(S_i)\left\langle f^{2p}(|T_i|)x, x\right\rangle^{\frac{1}{2}}\left\langle g^{2p}(|T_i^*|)x, x\right\rangle^{\frac{1}{2}} \text{ (by Lemma 2.2)} \\
&\leqslant \frac{n^{p-1}}{2}\sum_{i=1}^{n} r^p(S_i)\left\langle (f^{2p}(|T_i|) + g^{2p}(|T_i^*|))x, x\right\rangle \\
&= \frac{n^{p-1}}{2}\left\langle \sum_{i=1}^{n} r^p(S_i)(f^{2p}(|T_i|) + g^{2p}(|T_i^*|))x, x\right\rangle \\
&\leqslant \frac{n^{p-1}}{2}\left\|\sum_{i=1}^{n} r^p(S_i)(f^{2p}(|T_i|) + g^{2p}(|T_i^*|))\right\|.
\end{aligned}$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$ in the above inequality, we get

$$\omega^p(\sum_{i=1}^{n} T_i S_i) \leqslant \frac{n^{p-1}}{2}\left\|\sum_{i=1}^{n} r^p(S_i)(f^{2p}(|T_i|) + g^{2p}(|T_i^*|))\right\|.$$

Therefore, the proof of our theorem is complete. □

REMARK 2.7. In Theorem 2.6, letting $n = 1$, together with Lemma 2.5, we see that if $T, S \in B(\mathcal{H})$ are such that $|T|S = S^*|T|$, then

$$\begin{aligned}
\omega^p(TS) &\leqslant \frac{1}{2}\left\|r^p(S)(f^{2p}(|T|) + g^{2p}(|T^*|))\right\| \\
&\leqslant \frac{r^p(S)}{2}\left(\max\left\{\|f(|T|)\|^{2p}, \|g|T^*|\|^{2p}\right\} + \|f^p(|T|)g^p(|T^*|)\|\right).
\end{aligned}$$

We can see that the inequality obtained in [8, Th. 2.12] is our case $p = 1$.

For $f(t) = t^{\lambda}$ and $g(t) = t^{1-\lambda}$, $0 \leqslant \lambda \leqslant 1$, in Theorem 2.6, we obtain the following inequalities.

COROLLARY 2.8. *Let $T_i, S_i \in B(\mathcal{H})$ be such that $|T_i|S_i = S_i^*|T_i|$ $(i = 1, 2, \ldots, n)$, $p \geqslant 1$, and $0 \leqslant \lambda \leqslant 1$. Then*

$$\omega^p(\sum_{i=1}^n T_iS_i) \leqslant \frac{n^{p-1}}{2} \left\| \sum_{i=1}^n r^p(S_i)(|T_i|^{2p\lambda} + |T_i^*|^{2p(1-\lambda)}) \right\|.$$

In particular,

$$\omega^p(\sum_{i=1}^n T_iS_i) \leqslant \frac{n^{p-1}}{2} \left\| \sum_{i=1}^n r^p(S_i)(|T_i|^p + |T_i^*|^p) \right\|.$$

REMARK 2.9. In [3, Cor. 4] Alomari proved that if $T, S \in B(\mathcal{H})$ are such that $|T|S = S^*|T|$, then

$$\omega(TS) \leqslant r(S)\|T\|.$$

The case $n = 1, \lambda = \frac{1}{2}$ in Corollary 2, gives

$$\omega^p(TS) \leqslant \frac{1}{2} \|r^p(S)(|T|^p + |T^*|^p)\| \leqslant r^p(S)\|T\|^p.$$

COROLLARY 2.10. *Let $T, S \in B(\mathcal{H})$ be such that $|T|S = S^*|T|$. Then*

$$\omega^p(TS) \leqslant \frac{1}{4} \left(\|S\|^p + \sqrt{r(|S|^p|S^*|^p)} \right) \left(\|T\|^p + \sqrt{r(|T|^p|T^*|^p)} \right)$$

for $p \geqslant 1$.

Proof. In Corollary 2.8, setting $n = 1$, we have

$$\begin{aligned} \omega^p(TS) &\leqslant \frac{1}{2} \|r^p(S)(|T|^p + |T^*|^p)\| \\ &= \frac{r^p(S)}{2} \||T|^p + |T^*|^p\| \\ &\leqslant \frac{r^p(S)}{2} \left(\|T\|^p + \||T|^{\frac{p}{2}}|T^*|^{\frac{p}{2}}\| \right) \text{ (by Lemma 2.5)} \\ &= \frac{r^p(S)}{2} \left(\|T\|^p + \sqrt{r(|T|^p|T^*|^p)} \right) \text{ (by Lemma 2.4).} \end{aligned}$$

By Remark 2.9, $r^p(S) \leqslant \omega^p(S) \leqslant \frac{1}{2} \||S|^p + |S^*|^p\|$, and so

$$\begin{aligned} \omega^p(TS) &\leqslant \frac{r^p(S)}{2} (\|T\|^p + \sqrt{r(|T|^p|T^*|^p)}) \\ &\leqslant \frac{1}{4} \||S|^p + |S^*|^p\| (\|T\|^p + \sqrt{r(|T|^p|T^*|^p)}) \\ &\leqslant \frac{1}{4} \left(\|S\|^p + \sqrt{r(|S|^p|S^*|^p)} \right) \left(\|T\|^p + \sqrt{r(|T|^p|T^*|^p)} \right). \quad \square \end{aligned}$$

REMARK 2.11. In [8, Cor. 2.13], Bhunia and Paul proved that if $T,S \in B(\mathcal{H})$ are such that $|T|S = S^*|T|$, then

$$\omega(TS) \leqslant \frac{1}{4}\left(\|S\| + \sqrt{r(|S||S^*|)}\right)\left(\|T\| + \sqrt{r(|T||T^*|)}\right).$$

We can see that our inequality in Corollary 2.10 generalizes the inequality obtained in [8, Cor. 2.13].

Now, letting $S_i = I$ in Theorem 2.6, we obtain the following inequalities.

COROLLARY 2.12. *Let $T_i \in B(\mathcal{H})$, and let f,g be as in Theorem 2.6, $p \geqslant 1$. Then*

$$\omega^p(\sum_{i=1}^n T_i) \leqslant \frac{n^{p-1}}{2}\left\|\sum_{i=1}^n (f^{2p}(|T_i|) + g^{2p}(|T_i^*|))\right\|.$$

In particular,

$$\omega^p(\sum_{i=1}^n T_i) \leqslant \frac{n^{p-1}}{2}\left\|\sum_{i=1}^n (|T_i|^p + |T_i^*|^p)\right\|.$$

REMARK 2.13. Notice that Corollary 2.12 has been proved in [2, Cor. 2.7]. However, our approach here is different from theirs.

Based on Lemma 2.1, we have the following lemma.

LEMMA 2.14. [18] *Let $T \in B(\mathcal{H})$, and let f,g be non-negative continuous functions on $[0,\infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0,\infty)$. Then*

$$|\langle Tx,y\rangle| \leqslant \|f(|T|)x\|\|g(|T^*|)y\|$$

for all $x,y \in \mathcal{H}$.

The following improvement of the Cauchy-Schwarz inequality is known as Buzano's inequality.

LEMMA 2.15. [12] *Let $x,y,e \in \mathcal{H}$ with $\|e\| = 1$. Then*

$$|\langle x,e\rangle\langle e,y\rangle| \leqslant \frac{1}{2}(\|x\|\|y\| + |\langle x,y\rangle|).$$

Next, we obtain further related numerical radius inequalities.

THEOREM 2.16. *Let $T \in B(\mathcal{H})$, and let f and g be as in Theorem 2.6. Then*

$$\omega^{2p}(T) \leqslant \frac{\alpha}{2}\omega(g^{2p}(|T^*|)f^{2p}(|T|)) + \left\|\frac{\alpha}{4}(f^{4p}(|T|) + g^{4p}(|T^*|)) + (1-\alpha)|T^*|^{2p}\right\|$$

and

$$\omega^{2p}(T) \leqslant \frac{\alpha}{2}\omega(g^{2p}(|T^*|)f^{2p}(|T|)) + \left\|\frac{\alpha}{4}(f^{4p}(|T|) + g^{4p}(|T^*|)) + (1-\alpha)|T|^{2p}\right\|$$

for $p \geqslant 1$ *and* $0 \leqslant \alpha \leqslant 1$.

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$. Then by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |\langle Tx,x\rangle| &= \alpha|\langle Tx,x\rangle| + (1-\alpha)|\langle Tx,x\rangle| \\ &= \alpha|\langle Tx,x\rangle| + (1-\alpha)|\langle T^*x,x\rangle|. \end{aligned}$$

Now, by the convexity of the function $f(t) = t^{2p}$ on $[0,\infty)$, we have

$$\begin{aligned} |\langle Tx,x\rangle|^{2p} &\leqslant \alpha|\langle Tx,x\rangle|^{2p} + (1-\alpha)|\langle T^*x,x\rangle|^{2p} \\ &\leqslant \alpha\|f(|T|)x\|^{2p}\|g(|T^*|)x\|^{2p} + (1-\alpha)\|T^*x\|^{2p} \\ &\qquad \text{(by Lemma 2.14 and the Cauchy-Schwartz inequality)} \\ &= \alpha\left\langle f^2(|T|)x,x\right\rangle^p \left\langle x,g^2(|T^*|)x\right\rangle^p + (1-\alpha)\left\langle |T^*|^2x,x\right\rangle^p \\ &\leqslant \alpha\left\langle f^{2p}(|T|)x,x\right\rangle \left\langle x,g^{2p}(|T^*|)x\right\rangle + (1-\alpha)\left\langle |T^*|^{2p}x,x\right\rangle \text{ (by Lemma 2.2)} \\ &\leqslant \frac{\alpha}{2}\left(\|f^{2p}(|T|)x\|\|g^{2p}(|T^*|)x\| + \left\langle f^{2p}(|T|)x,g^{2p}(|T^*|)x\right\rangle\right) \\ &\quad + (1-\alpha)\left\langle |T^*|^{2p}x,x\right\rangle \text{ (by Lemma 2.15)} \\ &\leqslant \frac{\alpha}{2}\left(\frac{\left\langle f^{4p}(|T|)x,x\right\rangle + \left\langle g^{4p}(|T^*|)x,x\right\rangle}{2}\right) + \frac{\alpha}{2}\left\langle g^{2p}(|T^*|)f^{2p}(|T|)x,x\right\rangle \\ &\quad + (1-\alpha)\left\langle |T^*|^{2p}x,x\right\rangle \\ &\leqslant \left\langle \left(\frac{\alpha}{4}(f^{4p}(|T|) + g^{4p}(|T^*|)) + (1-\alpha)|T^*|^{2p}\right)x,x\right\rangle \\ &\quad + \frac{\alpha}{2}\omega(g^{2p}(|T^*|)f^{2p}(|T|)) \\ &\leqslant \frac{\alpha}{2}\omega(g^{2p}(|T^*|)f^{2p}(|T|)) + \left\|\frac{\alpha}{4}(f^{4p}(|T|) + g^{4p}(|T^*|)) + (1-\alpha)|T^*|^{2p}\right\|. \end{aligned}$$

Taking the supremum over all $x \in \mathcal{H}$ with $\|x\| = 1$, we have

$$\omega^{2p}(T) \leqslant \frac{\alpha}{2}\omega(g^{2p}(|T^*|)f^{2p}(|T|)) + \left\|\frac{\alpha}{4}(f^{4p}(|T|) + g^{4p}(|T^*|)) + (1-\alpha)|T^*|^{2p}\right\|.$$

By similar arguments as above, we have

$$\omega^{2p}(T) \leqslant \frac{\alpha}{2}\omega(g^{2p}(|T^*|)f^{2p}(|T|)) + \left\|\frac{\alpha}{4}(f^{4p}(|T|) + g^{4p}(|T^*|)) + (1-\alpha)|T|^{2p}\right\|.$$

Therefore, the proof of our theorem is complete. □

For $f(t) = t^\lambda$ and $g(t) = t^{1-\lambda}$, $0 \leqslant \lambda \leqslant 1$, in Theorem 2.16, we obtain the following inequalities.

COROLLARY 2.17. *Let* $T \in B(\mathcal{H})$. *Then*

$$\omega^{2p}(T) \leqslant \frac{\alpha}{2}\omega(|T^*|^{2p(1-\lambda)}|T|^{2p\lambda}) + \left\|\frac{\alpha}{4}(|T|^{4p\lambda} + |T^*|^{4p(1-\lambda)}) + (1-\alpha)|T^*|^{2p}\right\|$$

and

$$\omega^{2p}(T) \leqslant \frac{\alpha}{2}\omega(|T^*|^{2p(1-\lambda)}|T|^{2p\lambda}) + \left\|\frac{\alpha}{4}(|T|^{4p\lambda}+|T^*|^{4p(1-\lambda)}) + (1-\alpha)|T|^{2p}\right\|$$

for $p \geqslant 1$ *and* $0 \leqslant \alpha, \lambda \leqslant 1$.

THEOREM 2.18. *Let* $T \in B(\mathcal{H})$, *and let* f *and* g *be as in Theorem 2.6. Then*

$$\omega^{2p}(T) \leqslant \left\|\alpha\Big(\frac{f^4(|T|)+g^4(|T^*|)}{4} + \frac{g^2(|T^*|)f^2(|T|)}{2}\Big)^p + (1-\alpha)|T^*|^{2p}\right\|$$

and

$$\omega^{2p}(T) \leqslant \left\|\alpha\Big(\frac{f^4(|T|)+g^4(|T^*|)}{4} + \frac{g^2(|T^*|)f^2(|T|)}{2}\Big)^p + (1-\alpha)|T|^{2p}\right\|$$

for $p \geqslant 1$ *and* $0 \leqslant \alpha \leqslant 1$.

Proof. By the convexity of the function $f(t) = t^{2p}$ on $[0,\infty)$, we have

$$\begin{aligned}
|\langle Tx,x\rangle|^{2p} &\leqslant \alpha|\langle Tx,x\rangle|^{2p} + (1-\alpha)|\langle T^*x,x\rangle|^{2p} \\
&\leqslant \alpha(\|f(|T|)x\|\|g(|T^*|)x\|)^{2p} + (1-\alpha)\|T^*x\|^{2p} \text{ (by Lemma 2.14)} \\
&= \alpha\left(\left\langle f^2(|T|)x,x\right\rangle\left\langle x,g^2(|T^*|)x\right\rangle\right)^p + (1-\alpha)\left\langle |T^*|^2x,x\right\rangle^p \\
&\leqslant \alpha\left(\frac{\|f^2(|T|)x\|\|g^2(|T^*|)x\| + \left\langle f^2(|T|)x, g^2(|T^*|)x\right\rangle}{2}\right)^p \\
&\quad + (1-\alpha)\left\langle |T^*|^{2p}x,x\right\rangle \text{ (by Lemma 2.15)} \\
&\leqslant \alpha\left\langle\Big(\frac{f^4(|T|)+g^4(|T^*|)}{4} + \frac{g^2(|T^*|)f^2(|T|)}{2}\Big)x,x\right\rangle^p \\
&\quad + (1-\alpha)\left\langle |T^*|^{2p}x,x\right\rangle \\
&\leqslant \alpha\left\langle\Big(\frac{f^4(|T|)+g^4(|T^*|)}{4} + \frac{g^2(|T^*|)f^2(|T|)}{2}\Big)^p x,x\right\rangle \\
&\quad + (1-\alpha)\left\langle |T^*|^{2p}x,x\right\rangle \text{ (by Lemma 2.2)} \\
&= \left\langle\left(\alpha\Big(\frac{f^4(|T|)+g^4(|T^*|)}{4} + \frac{g^2(|T^*|)f^2(|T|)}{2}\Big)^p + (1-\alpha)|T^*|^{2p}\right)x,x\right\rangle \\
&\leqslant \left\|\alpha\Big(\frac{f^4(|T|)+g^4(|T^*|)}{4} + \frac{g^2(|T^*|)f^2(|T|)}{2}\Big)^p + (1-\alpha)|T^*|^{2p}\right\|.
\end{aligned}$$

Taking the supremum over all $x \in \mathcal{H}$ with $\|x\| = 1$, we have

$$\omega^{2p}(T) \leqslant \left\|\alpha\Big(\frac{f^4(|T|)+g^4(|T^*|)}{4} + \frac{g^2(|T^*|)f^2(|T|)}{2}\Big)^p + (1-\alpha)|T^*|^{2p}\right\|.$$

By similar arguments as above, we have

$$\omega^{2p}(T) \leqslant \left\| \alpha\Big(\frac{f^4(|T|)+g^4(|T^*|)}{4}+\frac{g^2(|T^*|)f^2(|T|)}{2}\Big)^p+(1-\alpha)|T|^{2p}\right\|.$$

Therefore, the proof of our theorem is complete. □

For $f(t)=t^{\lambda}$ and $g(t)=t^{1-\lambda}$, $0\leqslant\lambda\leqslant 1$, in Theorem 2.18, we obtain the following inequalities.

COROLLARY 2.19. *Let $T\in B(\mathcal{H})$. Then*

$$\omega^{2p}(T) \leqslant \left\| \alpha\Big(\frac{|T|^{4\lambda}+|T^*|^{4(1-\lambda)}}{4}+\frac{|T^*|^{2(1-\lambda)}|T|^{2\lambda}}{2}\Big)^p+(1-\alpha)|T^*|^{2p}\right\|$$

and

$$\omega^{2p}(T) \leqslant \left\| \alpha\Big(\frac{|T|^{4\lambda}+|T^*|^{4(1-\lambda)}}{4}+\frac{|T^*|^{2(1-\lambda)}|T|^{2\lambda}}{2}\Big)^p+(1-\alpha)|T|^{2p}\right\|$$

for $p\geqslant 1$ and $0\leqslant\alpha,\lambda\leqslant 1$.

It should be noted here that the proofs of Theorems 2.16 and 2.18 can be modified to yield the following related results.

THEOREM 2.20. *Let $T\in B(\mathcal{H})$, and let f and g be as in Theorem 2.6. Then*

$$\omega^{2p}(T) \leqslant \left\| \frac{\alpha}{2}(f^{4p}(|T|)+g^{4p}(|T^*|))+(1-\alpha)|T^*|^{2p}\right\|$$

and

$$\omega^{2p}(T) \leqslant \left\| \frac{\alpha}{2}(f^{4p}(|T|)+g^{4p}(|T^*|))+(1-\alpha)|T|^{2p}\right\|$$

for $p\geqslant 1$ and $0\leqslant\alpha\leqslant 1$.

REMARK 2.21. In [9, Th 2.5] Bhunia and Paul proved that if $T\in B(\mathcal{H})$, $p\geqslant 1$ and $0\leqslant\alpha,\lambda\leqslant 1$, then

$$\omega^{2p}(T) \leqslant \left\| \frac{\alpha}{2}(|T|^{4p\lambda}+|T^*|^{4p(1-\lambda)})+(1-\alpha)|T^*|^{2p}\right\|$$

and

$$\omega^{2p}(T) \leqslant \left\| \frac{\alpha}{2}(|T|^{4p\lambda}+|T^*|^{4p(1-\lambda)})+(1-\alpha)|T|^{2p}\right\|.$$

It should be mentioned here that the inequalities in Theorem 2.20 generalize the results in [9, Th 2.5].

THEOREM 2.22. *Let $T \in B(\mathcal{H})$, and let f and g be as in Theorem 2.6. Then*

$$\omega^{2p}(T) \leqslant \left\| \alpha \left(\frac{f^2(|T|) + g^2(|T^*|)}{2} \right)^{2p} + (1-\alpha)|T^*|^{2p} \right\|$$

and

$$\omega^{2p}(T) \leqslant \left\| \alpha \left(\frac{f^2(|T|) + g^2(|T^*|)}{2} \right)^{2p} + (1-\alpha)|T|^{2p} \right\|$$

for $p \geqslant 1$ and $0 \leqslant \alpha \leqslant 1$.

We remark here that, in view of the operator convexity of the function $f(t) = t^2$ on $[0,\infty)$, it is evident that for $p = 1$, the inequalities in Theorem 2.22 are sharper than their counterparts in Theorem 2.20.

For $f(t) = t^{\lambda}$ and $g(t) = t^{1-\lambda}$, $0 \leqslant \lambda \leqslant 1$, in Theorem 2.22, we obtain the following inequalities.

COROLLARY 2.23. *Let $T \in B(\mathcal{H})$. Then*

$$\omega^{2p}(T) \leqslant \left\| \alpha \left(\frac{|T|^{2\lambda} + |T^*|^{2(1-\lambda)}}{2} \right)^{2p} + (1-\alpha)|T^*|^{2p} \right\|$$

and

$$\omega^{2p}(T) \leqslant \left\| \alpha \left(\frac{|T|^{2\lambda} + |T^*|^{2(1-\lambda)}}{2} \right)^{2p} + (1-\alpha)|T|^{2p} \right\|$$

for $p \geqslant 1$ and $0 \leqslant \alpha, \lambda \leqslant 1$.

REMARK 2.24. In [9, Th 2.14] Bhunia and Paul proved that if $T \in B(\mathcal{H})$, $p \geqslant 1$ and $0 \leqslant \alpha \leqslant 1$, then

$$\omega^{2p}(T) \leqslant \left\| \alpha \left(\frac{|T| + |T^*|}{2} \right)^{2p} + (1-\alpha)|T^*|^{2p} \right\|$$

and

$$\omega^{2p}(T) \leqslant \left\| \alpha \left(\frac{|T| + |T^*|}{2} \right)^{2p} + (1-\alpha)|T|^{2p} \right\|.$$

It should be mentioned here that the inequalities in Corollary 2.23 generalize the results in [9, Th 2.14]. In Corollary 2.23, taking $\lambda = \frac{1}{2}$, we get the inequalities in [9, Th 2.14].

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