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INEQUALITIES FOR THE DERIVATIVE AND POLAR DERIVATIVE OF LACUNARY POLYNOMIALS

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Abstract. In this paper, we study lacunary polynomials of degree n having s-fold zeros at the origin and the remaining zeros lying on or outside the boundary of a prescribed disk. This study in turns gives generalizations and improvements of some well-known results. Besides, we also generalize as well as improves upon a result due to Aziz and Shah by extending it to the polar derivative.

1. Introduction and statement of results

If $p(z)$ is a polynomial of degree n and $p'(z)$ is its derivative, then

$$\max_{|z|=1} |p'(z)| \leqslant n \max_{|z|=1} |p(z)|, \tag{1}$$

and

$$\max_{|z|=R>1} |p(z)| \leqslant R^n \max_{|z|=1} |p(z)|. \tag{2}$$

Inequality (1) is a well-known result of S. Bernstein [6], whereas inequality (2) is a simple deduction from maximum modulus principle [14]. In both (1) and (2), equality holds only when $p(z)$ is a constant multiple of z^n.

If we restrict ourselves to the class of polynomials of degree n having no zeros in $|z| < 1$, then

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{2} \max_{|z|=1} |p(z)|, \tag{3}$$

and

$$\max_{|z|=R>1} |p(z)| \leqslant \frac{R^n+1}{2} \max_{|z|=1} |p(z)|. \tag{4}$$

Inequality (3) was conjectured by Erdős and latter verified by Lax [11], whereas Ankeny and Rivlin [1] used (3) to prove (4).

As an extension of (3), Malik [12] verified that if $p(z)$ does not vanish in $|z| < k$, $k \geqslant 1$, then

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{5}$$

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Inequality (5) was further improved by Govil [9] under the same hypothesis to

$$\max_{|z|=1}|p'(z)| \leqslant \frac{n}{1+k}\left[\max_{|z|=1}|p(z)| - \min_{|z|=k}|p(z)|\right]. \tag{6}$$

Inequalities (5) and (6) are sharp for a polynomial $p(z)=(z+k)^n$.

Chan and Malik [8] considered the lacunary type of polynomials and proved that, if $p(z)=a_0+\sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1\leqslant \mu \leqslant n$, $a_\mu \neq 0$, is a polynomial of degree n which does not vanish in $|z|<k$, $k\geqslant 1$, then

$$\max_{|z|=1}|p'(z)| \leqslant \frac{n}{1+k^\mu}\max_{|z|=1}|p(z)|. \tag{7}$$

Inequality (7) was improved by Pukhta [13] under the same hypothesis to

$$\max_{|z|=1}|p'(z)| \leqslant \frac{n}{1+k^\mu}\left[\max_{|z|=1}|p(z)| - \min_{|z|=k}|p(z)|\right]. \tag{8}$$

Inequalities (7) and (8) are sharp for a polynomial $p(z)=(z^\mu+k^\mu)^{n/\mu}$, where n is a multiple of μ.

As a generalization of inequality (5), Bidkham and Dewan [7] proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z|<k$, $k\geqslant 1$, then for $0<r\leqslant R\leqslant k$,

$$\max_{|z|=R}|p'(z)| \leqslant \frac{n(R+k)^{n-1}}{(r+k)^n}\max_{|z|=r}|p(z)|. \tag{9}$$

As a generalization of inequalities (8) and (9), Aziz and Shah [5] proved that, if $p(z)=a_0+\sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1\leqslant \mu \leqslant n$, $a_\mu \neq 0$, is a polynomial of degree n having no zeros in $|z|<k$, $k>0$, then for $0<r\leqslant R\leqslant k$,

$$\max_{|z|=R}|p'(z)| \leqslant \frac{nR^{\mu-1}(R^\mu+k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu+k^\mu)^{\frac{n}{\mu}}}\left[\max_{|z|=r}|p(z)| - \min_{|z|=k}|p(z)|\right]. \tag{10}$$

The result is best possible for a polynomial $p(z)=(z^\mu+k^\mu)^{n/\mu}$, where n is a multiple of μ.

The polar derivative of a polynomial $p(z)$ of degree n with respect to a complex number α, denoted by $D_\alpha p(z)$, is defined by

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z).$$

Note that $D_\alpha p(z)$ generalizes the derivative of a polynomial in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha p(z)}{\alpha} = p'(z).$$

The bounds of $|D_\alpha p(z)|$ have been studied by many researchers. For example, Aziz and Shah [3, 4] studied upper bounds of $\max_{|z|=1}|D_\alpha p(z)|$ where $p(z)$ is a polynomial of degree n having no zeros in $|z|\leqslant k$, $k\geqslant 1$ and $\alpha\in\mathbb{C}$ with $|\alpha|\geqslant 1$. Arunrat and Nakprasit [2] studied an upper bound of $\max_{|z|=1}|D_\alpha p(z)|$ where $p(z)$ is a polynomial of degree n which has some zeros in $|z|\leqslant 1$ and the remaining zeros are outside $|z|\leqslant k$, $k\geqslant 1$, and $\alpha\in\mathbb{C}$ with $|\alpha|\geqslant 1$.

In this paper, first we extend inequality (10) to the class of polynomials of degree n of type $p(z)=z^s\left(a_0+\sum_{\nu=\mu}^{n-s}a_\nu z^\nu\right)$, $1\leqslant\mu\leqslant n-s$, $0\leqslant s\leqslant n-1$, and obtain the following theorem.

THEOREM 1. (Main) *If* $p(z)=z^s\left(a_0+\sum_{\nu=\mu}^{n-s}a_\nu z^\nu\right)$, $1\leqslant\mu\leqslant n-s$, $0\leqslant s\leqslant n-1$, *is a polynomial of degree* n *having* s*-fold zeros at the origin and the remaining* $n-s$ *zeros in* $|z|\geqslant k$ *where* $k>0$, *then for* $0<r\leqslant R\leqslant k$,

$$\max_{|z|=R}|p'(z)|\leqslant\frac{A}{r^s}\max_{|z|=r}|p(z)|-\frac{A-sR^{s-1}}{k^s}\min_{|z|=k}|p(z)|, \tag{11}$$

where

$$A=\frac{R^{s-1}(nR^\mu+sk^\mu)(R^\mu+k^\mu)^{\frac{n-s}{\mu}-1}}{(r^\mu+k^\mu)^{\frac{n-s}{\mu}}}.$$

The result is best possible and equality holds for a polynomial $p(z)=z^s(z+k)^{n-s}$.

If we take $\mu=1$ in Theorem 1, we get the following result which is an improvement of a result of Bidkham and Dewan [7].

COROLLARY 1. *If* $p(z)=z^s\left(a_0+\sum_{\nu=1}^{n-s}a_\nu z^\nu\right)$, $0\leqslant s\leqslant n-1$, *is a polynomial of degree* n *having* s*-fold zeros at the origin and the remaining* $n-s$ *zeros in* $|z|\geqslant k$ *where* $k>0$, *then for* $0<r\leqslant R\leqslant k$,

$$\begin{aligned}\max_{|z|=R}|p'(z)|\leqslant{}&\frac{R^{s-1}(nR+sk)(R+k)^{n-s-1}}{r^s(r+k)^{n-s}}\max_{|z|=r}|p(z)|\\&-\frac{R^{s-1}(nR+sk)(R+k)^{n-s-1}-sR^{s-1}(r+k)^{n-s}}{k^s(r+k)^{n-s}}\min_{|z|=k}|p(z)|.\end{aligned}$$

The result is best possible and equality holds for a polynomial $p(z)=z^s(z+k)^{n-s}$.

REMARK 1. (i) If we put $s=0$ in Theorem 1, inequality (11) reduces to inequality (10).

(ii) If we put $r=R=1$ in Theorem 1, then we get a result of Kumar and Lal (see Theorem 2 in [10]).

Next, we present theorem which is a generalization as well as an extension of Theorem 1 to the polar derivative.

THEOREM 2. (Main) *If* $p(z) = z^s\left(a_0 + \sum_{\nu=\mu}^{n-s} a_\nu z^\nu\right)$, $1 \leqslant \mu \leqslant n-s$, $0 \leqslant s \leqslant n-1$, *is a polynomial of degree* n *having* s*-fold zeros at the origin and the remaining* $n-s$ *zeros in* $|z| \geqslant k$ *where* $k>0$, *then for every complex number* α *with* $|\alpha| \geqslant R$ *and* $0 < r \leqslant R \leqslant k$,

$$\max_{|z|=R} |D_\alpha p(z)| \leqslant \left[\frac{|\alpha|A}{r^s} + \frac{(n-s)R^s k^\mu (R^\mu + k^\mu)^{\frac{n-s}{\mu}-1}}{r^s (r^\mu + k^\mu)^{\frac{n-s}{\mu}}}\right] \max_{|z|=r} |p(z)| - \left[\frac{|\alpha|A - (|\alpha|s + (n-s)R)R^{s-1}}{k^s} + \frac{(n-s)R^s k^\mu (R^\mu + k^\mu)^{\frac{n-s}{\mu}-1}}{k^s (r^\mu + k^\mu)^{\frac{n-s}{\mu}}}\right] \min_{|z|=k} |p(z)|, \tag{12}$$

where

$$A = \frac{R^{s-1}(nR^\mu + sk^\mu)(R^\mu + k^\mu)^{\frac{n-s}{\mu}-1}}{(r^\mu + k^\mu)^{\frac{n-s}{\mu}}}.$$

The result is best possible and equality holds for a polynomial $p(z) = z^s(z+k)^{n-s}$ *where* α *is a real number with* $\alpha \geqslant 1$.

REMARK 2. Dividing both sides of inequality (12) by $|\alpha|$ and letting $|\alpha| \to \infty$, we get inequality (11) in Theorem 1.

If we put $s = 0$ in Theorem 2, we get the following result which extends inequality (10) of Aziz and Shah [5] to the polar derivative.

COROLLARY 2. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leqslant \mu \leqslant n$, *is a polynomial of degree* n *having no zeros in* $|z| < k$, $k > 0$, *then for every complex number* α *with* $|\alpha| \geqslant R$ *and* $0 < r \leqslant R \leqslant k$,

$$\max_{|z|=R} |D_\alpha p(z)| \leqslant \frac{(n|\alpha|R^{\mu-1} + nk^\mu)(R^\mu + k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu + k^\mu)^{\frac{n}{\mu}}} \max_{|z|=r} |p(z)| - \left[\frac{(n|\alpha|R^{\mu-1} + nk^\mu)(R^\mu + k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu + k^\mu)^{\frac{n}{\mu}}} - n\right] \min_{|z|=k} |p(z)|. \tag{13}$$

REMARK 3. Dividing both sides of inequality (13) in Corollary 2 by $|\alpha|$ and letting $|\alpha| \to \infty$, we get inequality (10).

2. Lemmas

In this section, we present Lemmas which are use in the proofs of our theorems. The first lemma is due to Kumar and Lal [10].

LEMMA 1. [10] *If* $p(z) = z^s\left(a_0 + \sum_{\nu=\mu}^{n-s} a_\nu z^\nu\right)$, $1 \leqslant \mu \leqslant n-s$, $0 \leqslant s \leqslant n-1$, *is a polynomial of degree* n *having* s*-fold zeros at the origin and the remaining* $n-s$ *zeros in* $|z| \geqslant k$ *where* $k \geqslant 1$, *then*

$$\max_{|z|=1}|p'(z)| \leqslant \frac{n+sk^\mu}{1+k^\mu}\max_{|z|=1}|p(z)| - \frac{(n-s)}{k^s(1+k^\mu)}\min_{|z|=k}|p(z)|.$$

Next, we apply Lemma 1 to prove the following lemma.

LEMMA 2. *If* $p(z) = z^s\left(a_0 + \sum_{\nu=\mu}^{n-s} a_\nu z^\nu\right)$, $1 \leqslant \mu \leqslant n-s$, $0 \leqslant s \leqslant n-1$, *is a polynomial of degree* n *having* s*-fold zeros at the origin and the remaining* $n-s$ *zeros in* $|z| \geqslant k$ *where* $k > 0$, *then for* $0 < r \leqslant R \leqslant k$,

$$\max_{|z|=r}|p(z)| \geqslant \left(\frac{r}{R}\right)^s \left(\frac{r^\mu + k^\mu}{R^\mu + k^\mu}\right)^{\frac{n-s}{\mu}} \max_{|z|=R}|p(z)| + \left(\frac{r}{k}\right)^s \left[1 - \left(\frac{r^\mu + k^\mu}{R^\mu + k^\mu}\right)^{\frac{n-s}{\mu}}\right] \min_{|z|=k}|p(z)|. \tag{14}$$

The result is best possible and equality holds for a polynomial $p(z) = z^s(z+k)^{n-s}$.

Proof of Lemma 2. Since $p(z)$ is a polynomial of degree n having s-fold zeros at the origin and the remaining $n-s$ zeros in $|z| \geqslant k$ where $k > 0$, for $0 < t \leqslant k$, $F(z) = p(tz)$ has s-fold zeros at the origin and the remaining $n-s$ zeros in $|z| \geqslant (k/t)$ where $(k/t) \geqslant 1$.

Applying Lemma 1 to a polynomial $F(z)$, we get

$$\max_{|z|=1}|F'(z)| \leqslant \frac{n + s\left(\frac{k}{t}\right)^\mu}{1+\left(\frac{k}{t}\right)^\mu}\max_{|z|=1}|F(z)| - \frac{(n-s)}{\left(\frac{k}{t}\right)^s\left(1+\left(\frac{k}{t}\right)^\mu\right)}\min_{|z|=k/t}|F(z)|.$$

Therefore,

$$\max_{|z|=t}|p'(z)| \leqslant \frac{1}{t}\cdot\frac{nt^\mu + sk^\mu}{t^\mu + k^\mu}\max_{|z|=t}|p(z)| - \frac{(n-s)t^{\mu+s-1}}{k^s(t^\mu+k^\mu)}\min_{|z|=k}|p(z)|. \tag{15}$$

Let $M(p,r) = \max_{|z|=r}|p(z)|$ and $m(p,k) = \min_{|z|=k}|p(z)|$. Then (15) is equivalent to

$$M(p',t) \leqslant \frac{1}{t}\cdot\frac{nt^\mu + sk^\mu}{t^\mu + k^\mu}M(p,t) - \frac{(n-s)t^{\mu+s-1}}{k^s(t^\mu+k^\mu)}m(p,k). \tag{16}$$

Now for $0 < r \leqslant R \leqslant k$ and $0 \leqslant \theta < 2\pi$, we have

$$p(Re^{i\theta}) - p(re^{i\theta}) = \int_r^R e^{i\theta} p'(te^{i\theta})\, dt.$$

Then $|p(Re^{i\theta})| \leqslant \left|p(re^{i\theta})\right| + \int_r^R |p'(te^{i\theta})|\, dt$, which implies that

$$M(p,R) \leqslant M(p,r) + \int_r^R M(p',t)\, dt.$$

Combining this inequality with (16), we obtain that

$$M(p,R) \leqslant M(p,r) + \int_r^R \left[\frac{nt^\mu + sk^\mu}{t(t^\mu + k^\mu)} M(p,t) - \frac{(n-s)t^{\mu+s-1}}{k^s(t^\mu + k^\mu)} m(p,k) \right] dt$$

or

$$M(p,R) \leqslant M(p,r) + \left[\int_r^R \frac{nt^\mu + sk^\mu}{t(t^\mu + k^\mu)} M(p,t)\, dt - \int_r^R \frac{(n-s)t^{\mu+s-1}}{k^s(t^\mu + k^\mu)} m(p,k)\, dt \right]. \quad (17)$$

Let

$$\phi(R) = M(p,r) + \left[\int_r^R \frac{nt^\mu + sk^\mu}{t(t^\mu + k^\mu)} M(p,t)\, dt - \int_r^R \frac{(n-s)t^{\mu+s-1}}{k^s(t^\mu + k^\mu)} m(p,k)\, dt \right].$$

Then

$$\phi'(R) = \frac{nR^\mu + sk^\mu}{R(R^\mu + k^\mu)} M(p,R) - \frac{(n-s)R^{\mu+s-1}}{k^s(R^\mu + k^\mu)} m(p,k).$$

Therefore, inequality (17) is equivalent to

$$\phi'(R) - \frac{(n-s)R^{\mu-1}}{R^\mu + k^\mu} \left[\left(1 + \frac{s(R^\mu + k^\mu)}{(n-s)R^\mu} \right) \phi(R) - \frac{R^s}{k^s} m(p,k) \right] \leqslant 0. \quad (18)$$

Multiplying both sides of (18) by $R^{-s}(R^\mu + k^\mu)^{\frac{-(n-s)}{\mu}}$, we get

$$\frac{d}{dR} \left[\left(\frac{1}{R^s} \phi(R) - \frac{1}{k^s} m(p,k) \right) (R^\mu + k^\mu)^{\frac{-(n-s)}{\mu}} \right] \leqslant 0. \quad (19)$$

From (19), we conclude that

$$g(R) := \left(\frac{1}{R^s} \phi(R) - \frac{1}{k^s} m(p,k) \right) (R^\mu + k^\mu)^{\frac{-(n-s)}{\mu}}$$

is a non-increasing function of R in $(0,k)$. Hence for $0 < r \leqslant R \leqslant k$,

$$g(r) \geqslant g(R).$$

That is,

$$\left(\frac{1}{r^s}\phi(r)-\frac{1}{k^s}m(p,k)\right)(r^\mu+k^\mu)^{\frac{-(n-s)}{\mu}} \geqslant \left(\frac{1}{R^s}\phi(R)-\frac{1}{k^s}m(p,k)\right)(R^\mu+k^\mu)^{\frac{-(n-s)}{\mu}}.$$

Then

$$\phi(r) \geqslant \left(\frac{r}{R}\right)^s \left(\frac{r^\mu+k^\mu}{R^\mu+k^\mu}\right)^{\frac{n-s}{\mu}} \phi(R) + \left(\frac{r}{k}\right)^s \left[1-\left(\frac{r^\mu+k^\mu}{R^\mu+k^\mu}\right)^{\frac{n-s}{\mu}}\right] m(p,k). \tag{20}$$

Since $\phi(R) \geqslant M(p,R)$ and $\phi(r) = M(p,r)$, it follows from (20) that

$$M(p,r) \geqslant \left(\frac{r}{R}\right)^s \left(\frac{r^\mu+k^\mu}{R^\mu+k^\mu}\right)^{\frac{n-s}{\mu}} M(p,R) + \left(\frac{r}{k}\right)^s \left[1-\left(\frac{r^\mu+k^\mu}{R^\mu+k^\mu}\right)^{\frac{n-s}{\mu}}\right] m(p,k).$$

Next, we show that the bound is best possible for a polynomial $p(z) = z^s(z+k)^{n-s}$. One can see that $\max\limits_{|z|=r}|p(z)| = r^s(z+k)^{n-s}$, $\min\limits_{|z|=k}|p(z)| = 0$, and $\max\limits_{|z|=R}|p(z)| = R^s(R+k)^{n-s}$. The right side of (14) becomes

$$\left(\frac{r}{R}\right)^s \left(\frac{r+k}{R+k}\right)^{n-s} (R^s(R+k)^{n-s}) + \left(\frac{r}{k}\right)^s \left[1-\left(\frac{r+k}{R+k}\right)^{n-s}\right](0) = r^s(z+k)^{n-s},$$

which equals $\max\limits_{|z|=r}|p(z)|$. $\square$

The next lemma is due to Arunrat and Nakprasit [2].

LEMMA 3. [2] *Let $p(z)$ be a polynomial of degree n in the form*

$$p(z) = z^s\left(a_0 + \sum_{\nu=\mu}^{n-s} a_\nu z^\nu\right), \quad 1 \leqslant \mu \leqslant n-s, \ 0 \leqslant s \leqslant n-1.$$

Let $k \geqslant 1$ and $\alpha \in \mathbb{C}$ with $|\alpha| \geqslant 1$. If all $n-s$ zeros (except a zero at the origin) are outside $|z| < k$, then

$$\max_{|z|=1}|D_\alpha p(z)| \leqslant \left[\frac{|\alpha|(n+sk^\mu)+(n-s)k^\mu}{1+k^\mu}\right]\max_{|z|=1}|p(z)| - \left[\frac{(|\alpha|-1)(n-s)}{k^s(1+k^\mu)}\right]\min_{|z|=k}|p(z)|.$$

3. Proofs of the main theorems

Proof of Theorem 1. Let $p(z)$ be a polynomial of degree n having s-fold zeros at the origin and the remaining $n-s$ zeros in $|z| \geqslant k$ where $k > 0$. Then the polynomial $F(z) = p(Rz)$ has s-fold zeros at the origin and the remaining $n-s$ zeros in $|z| \geqslant (k/R)$ where $(k/R) \geqslant 1$. Applying Lemma 1 to a polynomial $F(z)$, we get

$$\max_{|z|=1}|F'(z)| \leqslant \frac{n+s\left(\frac{k}{R}\right)^\mu}{1+\left(\frac{k}{R}\right)^\mu}\max_{|z|=1}|F(z)| - \frac{(n-s)}{\left(\frac{k}{R}\right)^s\left(1+\left(\frac{k}{R}\right)^\mu\right)}\min_{|z|=k/R}|F(z)|.$$

Hence,

$$\max_{|z|=R}|p'(z)| \leqslant \frac{nR^\mu + sk^\mu}{R(R^\mu + k^\mu)} \max_{|z|=R}|p(z)| - \frac{(n-s)R^{\mu+s-1}}{k^s(R^\mu + k^\mu)} \min_{|z|=k}|p(z)|. \tag{21}$$

For $0 < r \leqslant R \leqslant k$, Lemma 2 implies that

$$\max_{|z|=R}|p(z)| \leqslant \left(\frac{R}{r}\right)^s \left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^{\frac{n-s}{\mu}} \max_{|z|=r}|p(z)| - \left(\frac{R}{k}\right)^s \left[\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^{\frac{n-s}{\mu}} - 1\right] \min_{|z|=k}|p(z)|. \tag{22}$$

Substituting (22) into (21), we obtain that

$$\begin{aligned}
\max_{|z|=R}|p'(z)| &\leqslant \frac{nR^\mu + sk^\mu}{R(R^\mu + k^\mu)} \left[\left(\frac{R}{r}\right)^s \left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^{\frac{n-s}{\mu}} \max_{|z|=r}|p(z)| \right.\\
&\qquad \left. - \left(\frac{R}{k}\right)^s \left[\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^{\frac{n-s}{\mu}} - 1\right] \min_{|z|=k}|p(z)|\right]\\
&\qquad - \frac{(n-s)R^{\mu+s-1}}{k^s(R^\mu + k^\mu)} \min_{|z|=k}|p(z)|\\
&= \frac{R^{s-1}(nR^\mu + sk^\mu)(R^\mu + k^\mu)^{\frac{n-s}{\mu}-1}}{r^s(r^\mu + k^\mu)^{\frac{n-s}{\mu}}} \max_{|z|=r}|p(z)|\\
&\qquad - \frac{R^{s-1}}{k^s}\left[\frac{(nR^\mu + sk^\mu)(R^\mu + k^\mu)^{\frac{n-s}{\mu}-1}}{(r^\mu + k^\mu)^{\frac{n-s}{\mu}}} - s\right] \min_{|z|=k}|p(z)|.
\end{aligned}$$

Therefore,

$$\max_{|z|=R}|p'(z)| \leqslant \frac{A}{r^s} \max_{|z|=r}|p(z)| - \frac{A - sR^{s-1}}{k^s} \min_{|z|=k}|p(z)|,$$

where $A = \dfrac{R^{s-1}(nR^\mu + sk^\mu)(R^\mu + k^\mu)^{\frac{n-s}{\mu}-1}}{(r^\mu + k^\mu)^{\frac{n-s}{\mu}}}$.

Next, we show that the upper bound is best possible for a polynomial $p(z) = z^s(z+k)^{n-s}$. One can see that $\max_{|z|=R}|nz^s + skz^{s-1}|$ and $\max_{|z|=R}|(z+k)^{n-s-1}|$ are attained at $z = R$.

Then $\max_{|z|=R}|p'(z)| = \max_{|z|=R}|(nz^s + skz^{s-1})(z+k)^{n-s-1}| = R^{s-1}(nR+sk)(R+k)^{n-s-1}$

and $\min\limits_{|z|=k}|p(z)|=0$. The right side of (11) becomes

$$\begin{aligned}
&\frac{R^{s-1}(nR+sk)(R+k)^{n-s-1}}{r^s(r+k)^{n-s}}\max_{|z|=r}|p(z)| \\
&\qquad -\frac{R^{s-1}}{k^s}\left[\frac{(nR+sk)(R+k)^{n-s-1}}{(r+k)^{n-s}}-s\right]\min_{|z|=k}|p(z)| \\
&=\frac{R^{s-1}(nR+sk)(R+k)^{n-s-1}}{r^s(r+k)^{n-s}}\left(r^s(r+k)^{n-s}\right) \\
&=R^{s-1}(nR+sk)(R+k)^{n-s-1}
\end{aligned}$$

which equals $\max\limits_{|z|=R}|p'(z)|$. □

Proof of Theorem 2. Let $p(z)$ be a polynomial of degree n having s-fold zeros at the origin and the remaining $n-s$ zeros in $|z|\geqslant k$ where $k>0$. Then the polynomial $F(z)=p(Rz)$ has s-fold zeros at the origin and the remaining $n-s$ zeros in $|z|\geqslant (k/R)$ where $(k/R)\geqslant 1$. Applying Lemma 3 to a polynomial $F(z)$ with $|\alpha|/R\geqslant 1$, we get

$$\begin{aligned}
\max_{|z|=1}|D_{\frac{\alpha}{R}}F(z)| &\leqslant \left[\frac{\frac{|\alpha|}{R}\left(n+s\left(\frac{k}{R}\right)^{\mu}\right)+(n-s)\left(\frac{k}{R}\right)^{\mu}}{1+\left(\frac{k}{R}\right)^{\mu}}\right]\max_{|z|=1}|F(z)| \\
&\qquad -\left[\frac{\left(\frac{|\alpha|}{R}-1\right)(n-s)}{\left(\frac{k}{R}\right)^{s}\left(1+\left(\frac{k}{R}\right)^{\mu}\right)}\right]\min_{|z|=k/R}|F(z)|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\max_{|z|=R}|D_{\alpha}p(z)| &\leqslant \left[\frac{|\alpha|(nR^{\mu}+sk^{\mu})+(n-s)Rk^{\mu}}{R(R^{\mu}+k^{\mu})}\right]\max_{|z|=R}|p(z)| \\
&\qquad -\left[\frac{(|\alpha|-R)(n-s)R^{\mu+s-1}}{k^s(R^{\mu}+k^{\mu})}\right]\min_{|z|=k}|p(z)|. \qquad (23)
\end{aligned}$$

For $0<r\leqslant R\leqslant k$, Lemma 2 implies that

$$\begin{aligned}
\max_{|z|=R}|p(z)| &\leqslant \left(\frac{R}{r}\right)^{s}\left(\frac{R^{\mu}+k^{\mu}}{r^{\mu}+k^{\mu}}\right)^{\frac{n-s}{\mu}}\max_{|z|=r}|p(z)| \\
&\qquad -\left(\frac{R}{k}\right)^{s}\left[\left(\frac{R^{\mu}+k^{\mu}}{r^{\mu}+k^{\mu}}\right)^{\frac{n-s}{\mu}}-1\right]\min_{|z|=k}|p(z)|. \qquad (24)
\end{aligned}$$

Substituting (24) into (23), we obtain that

$$\begin{aligned}
\max_{|z|=R}|D_\alpha p(z)| \leqslant & \left[\frac{|\alpha|(nR^\mu+sk^\mu)+(n-s)Rk^\mu}{R(R^\mu+k^\mu)}\right] \\
& \times\left[\left(\frac{R}{r}\right)^s\left(\frac{R^\mu+k^\mu}{r^\mu+k^\mu}\right)^{\frac{n-s}{\mu}}\max_{|z|=r}|p(z)|\right. \\
& \left. -\left(\frac{R}{k}\right)^s\left[\left(\frac{R^\mu+k^\mu}{r^\mu+k^\mu}\right)^{\frac{n-s}{\mu}}-1\right]\min_{|z|=k}|p(z)|\right] \\
& -\left[\frac{(|\alpha|-R)(n-s)R^{\mu+s-1}}{k^s(R^\mu+k^\mu)}\right]\min_{|z|=k}|p(z)| \\
= & \left[\frac{|\alpha|R^{s-1}(nR^\mu+sk^\mu)(R^\mu+k^\mu)^{\frac{n-s}{\mu}-1}}{r^s(r^\mu+k^\mu)^{\frac{n-s}{\mu}}}\right. \\
& \left.+\frac{(n-s)R^sk^\mu(R^\mu+k^\mu)^{\frac{n-s}{\mu}-1}}{r^s(r^\mu+k^\mu)^{\frac{n-s}{\mu}}}\right]\max_{|z|=r}|p(z)| \\
& -\left[\frac{|\alpha|R^{s-1}(nR^\mu+sk^\mu)(R^\mu+k^\mu)^{\frac{n-s}{\mu}-1}}{k^s(r^\mu+k^\mu)^{\frac{n-s}{\mu}}}\right. \\
& +\frac{(n-s)R^sk^\mu(R^\mu+k^\mu)^{\frac{n-s}{\mu}-1}}{k^s(r^\mu+k^\mu)^{\frac{n-s}{\mu}}} \\
& \left.-\frac{(s|\alpha|+(n-s)R)R^{s-1}}{k^s}\right]\min_{|z|=k}|p(z)|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\max_{|z|=R}|D_\alpha p(z)| \leqslant & \left[\frac{|\alpha|A}{r^s}+\frac{(n-s)R^sk^\mu(R^\mu+k^\mu)^{\frac{n-s}{\mu}-1}}{r^s(r^\mu+k^\mu)^{\frac{n-s}{\mu}}}\right]\max_{|z|=r}|p(z)| \\
& -\left[\frac{|\alpha|A-(|\alpha|s+(n-s)R)R^{s-1}}{k^s}\right. \\
& \left.+\frac{(n-s)R^sk^\mu(R^\mu+k^\mu)^{\frac{n-s}{\mu}-1}}{k^s(r^\mu+k^\mu)^{\frac{n-s}{\mu}}}\right]\min_{|z|=k}|p(z)|,
\end{aligned}$$

where

$$A=\frac{R^{s-1}(nR^\mu+sk^\mu)(R^\mu+k^\mu)^{\frac{n-s}{\mu}-1}}{(r^\mu+k^\mu)^{\frac{n-s}{\mu}}}.$$

Next, we show that the upper bound is best possible for a polynomial $p(z)=z^s(z+k)^{n-s}$ where α is a real number with $\alpha \geqslant 1$.

One can see that $|D_\alpha p(z)|=|(z^s((n-s)k+\alpha n)+\alpha skz^{s-1})(z+k)^{n-s-1}|$.

Note that $(n-s)k+\alpha n>0$ because $n,k,s\in\mathbb{Z}^+$ and $\alpha\in\mathbb{R}$ with $\alpha\geqslant 1$.

Then $\max_{|z|=R} |D_\alpha p(z)| = R^{s-1}\left[\alpha(nR+sk)+(n-s)Rk\right](R+k)^{n-s-1}$.

The right side of (12) becomes

$$\left[\frac{\alpha R^{s-1}(nR+sk)(R+k)^{n-s-1}+(n-s)R^s k(R+k)^{n-s-1}}{r^s(r+k)^{n-s}}\right]\left(r^s(r+k)^{n-s}\right)$$
$$= R^{s-1}\left[\alpha(nR+sk)+(n-s)Rk\right](R+k)^{n-s-1},$$

which equals $\max_{|z|=R} |D_\alpha p(z)|$. □

4. Conclusion

This paper investigates an upper bound of the maximum modulus of the derivative of $p(z) = z^s\left(a_0 + \sum_{\nu=\mu}^{n-s} a_\nu z^\nu\right)$, $1 \leqslant \mu \leqslant n-s$, $0 \leqslant s \leqslant n-1$, having s-fold zeros at the origin and the remaining zeros lie in $|z| \geqslant k$ where $k > 0$. We generalize our upper bound to the polar derivative. In particular, if $P(z)$ has all zeros in $|z| \geqslant k$, then our theorems generalize results by Aziz and Shah [5]. Furthermore, if $\mu = 1$, then we obtain a result which improves an upper bound due to Bidkham and Dewan [7].

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