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ON JENSEN–TYPE INEQUALITIES FOR HARMONIC CONVEX FUNCTIONS

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Abstract. Inequalities play a main role in pure and applied mathematics. In particular, Jensen inequality plays an important role in many fields of Mathematics. In this paper we prove two new Jensen-type inequalities for harmonic convex functions via fractional calculus, and we apply them to generalized Caputo-type fractional integrals.

1. Introduction

Integral inequalities are used in countless mathematical problems such as approximation theory and spectral analysis, statistical analysis and the theory of distributions. Studies involving integral inequalities play an important role in several areas of science and engineering.

In recent years there has been a growing interest in the study of many classical inequalities applied to integral operators associated with different types of fractional derivatives, since integral inequalities and their applications play a vital role in the theory of differential equations and applied mathematics. Some of the inequalities studied are Gronwall, Chebyshev, Hermite-Hadamard-type, Ostrowski-type, Opial-type, Grüss-type, Hardy-type, Petrović-type, Milne-type, Gagliardo-Nirenberg-type, reverse Minkowski and reverse Hölder inequalities (see, e.g., [6, 7, 11, 13, 15, 16, 18–24]).

In particular, there are many generalizations of Jensen inequality. In this work we prove two new Jensen-type inequalities for harmonic convex functions, and we apply them to the generalized Caputo-type fractional integrals defined in [5], which include most of known Caputo-type fractional integrals.

2. Generalized Caputo-type fractional derivatives

Michele Caputo proposes a new fractional derivative in [8]. This definition has an important property associated with the resolution of Differential Equations, since it is not necessary to define the initial conditions of fractional order. Multiple applications of the so-called Caputo differential operator can be found in [9].

In [5], the authors present a generalized version of the Caputo fractional derivative.

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DEFINITION 1. We say that F is an *admissible kernel* for the interval $[a,b]$ if $F:[0,b-a]\times(0,1)\to[0,\infty)$ is a non-negative continuous function such that

$$\mathbb{F}(\alpha)=\int_0^{b-a}\frac{ds}{F(s,\alpha)}<\infty$$

for each $\alpha\in(0,1)$. F is an *admissible kernel* for $[a,\infty)$ if it is admissible for $[a,b]$ for every $b>a$.

DEFINITION 2. Let $n\in\mathbb{Z}^+$, $\alpha\in(n-1,n)$, $t\in[a,b]$ and F be an admissible kernel for $[a,b]$. For a n times differentiable function $f:[a,b]\to\mathbb{R}$, the *generalized Caputo derivative* of f of order α at t is

$$^{C}D^{\alpha}_{F,a}f(t)=\int_a^t\frac{f^{(n)}(s)}{F(t-s,\alpha+1-n)}\,ds. \tag{1}$$

The next interesting compositional property follows from Definition 2. This proposition was proved in [5].

PROPOSITION 3. *Let $\alpha\in(0,1)$, $n\in\mathbb{Z}^+$, and F be an admissible kernel for $[a,b]$. If f is $(n+1)$-differentiable function on $[a,b]$, then*

$$^{C}D^{\alpha+n}_{F,a}f(t)={}^{C}D^{\alpha}_{F,a}f^{(n)}(t),$$

for every $t\in[a,b]$.

Note that the equality in Proposition 3 is interesting, since we write $^{C}D^{\alpha+n}_{F,a}$ as a composition of a local operator and a non-local operator.

The following integral operator is associated to the generalized Caputo derivative in a natural way.

DEFINITION 4. Let $\alpha\in(0,1)$, F be an admissible kernel for $[a,b]$, $f:[a,b]\to\mathbb{R}$ be a differentiable function and $t\in[a,b]$. The *generalized Caputo integral operator* of order α of the function f at the point t is

$$^{C}J^{\alpha}_{F,a}f(t)=\int_a^t\frac{f(s)}{F(t-s,\alpha)}\,ds.$$

From this definition, we have

i. $^{C}D^{\alpha}_{F,a}f(t)={}^{C}J^{\alpha}_{F,a}f'(t)$.

ii. The functional defined by $^{C}J^{\alpha}_{F}(f)={}^{C}J^{\alpha}_{F,a}f(b)=\int_a^b\frac{f(s)}{F(b-s,\alpha)}\,ds$.

Next, some properties of the generalized Caputo derivative and its associated integral operator were presented (see [5]):

PROPOSITION 5. *Let* $\alpha \in (0,1)$, $x \in [a,b]$ *and* F *be an admissible kernel for* $[a,b]$. *If* f *is a differentiable function on* $[a,x]$ *and* $\mathbb{F}_\alpha = \min_{y\in[0,x-a]} F(y,\alpha) > 0$, *then*

$$\left\| {}^C J^\alpha_{F,a} f \right\|_{L^\infty[a,x]} \leqslant \frac{1}{\mathbb{F}_\alpha} \|f\|_{L^1[a,x]},$$
$$\left\| {}^C D^\alpha_{F,a} f \right\|_{L^\infty[a,x]} \leqslant \frac{1}{\mathbb{F}_\alpha} \|f'\|_{L^1[a,x]}.$$

By applying Proposition 5 to the function $f-g$, we obtain the following result (see [5]).

PROPOSITION 6. *Let* $\alpha \in (0,1)$, $x \in [a,b]$ *and* F *be an admissible kernel for* $[a,b]$. *If* f,g *are differentiable functions on* $[a,x]$ *and*

$$\mathbb{F}_\alpha = \min_{y\in[0,x-a]} F(y,\alpha) > 0,$$

then

$$\left\| {}^C J^\alpha_{F,a} f - {}^C J^\alpha_{F,a} g \right\|_{L^\infty[a,x]} \leqslant \frac{1}{\mathbb{F}_\alpha} \|f-g\|_{L^1[a,x]},$$
$$\left\| {}^C D^\alpha_{F,a} f - {}^C D^\alpha_{F,a} g \right\|_{L^\infty[a,x]} \leqslant \frac{1}{\mathbb{F}_\alpha} \|f'-g'\|_{L^1[a,x]}.$$

3. Jensen-type inequalities

Jensen inequality relates the value of a convex function of an integral to the integral of the convex function. It was proved in 1906 [14], and it can be stated as follows:

Let μ be a probability measure on the space X. If $f : X \to (a,b)$ is μ-integrable and φ is a convex function on (a,b), then

$$\varphi\left(\int_X f\,d\mu\right) \leqslant \int_X \varphi \circ f\,d\mu.$$

In [17] appears the following inequality.

THEOREM 7. *Let* $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$ *and let* w_k $(1 \leqslant k \leqslant n)$ *be positive weights whose sum is* 1. *If* φ *is a convex function on an interval containing* $[x_1,x_n]$, *then*

$$\varphi\left(x_1 + x_n - \sum_{k=1}^n w_k x_k\right) \leqslant \varphi(x_1) + \varphi(x_n) - \sum_{k=1}^n w_k \varphi(x_k).$$

In [4] appears the following continuous version of the above discrete inequality.

THEOREM 8. *Let* μ *be a probability measure on the space* X *and* $m \leqslant M$ *real constants. If* $f : X \to [m,M]$ *is a measurable function and* φ *is a convex function on* $[m,M]$, *then* f *and* $\varphi \circ f$ *are* μ*-integrable functions and*

$$\varphi\left(m + M - \int_X f\,d\mu\right) \leqslant \varphi(m) + \varphi(M) - \int_X \varphi \circ f\,d\mu.$$

If I is an interval in $\mathbb{R}\setminus\{0\}$, a function $\varphi : I \to \mathbb{R}$ is said to be *harmonic convex* on I if

$$\varphi\Big(\frac{xy}{tx+(1-t)y}\Big) \leqslant (1-t)\varphi(x)+t\varphi(y)$$

holds for every $x,y\in I$ and $t\in[0,1]$.

Very recently, [12] established a Jensen-type inequality for harmonic convex functions as follows.

THEOREM 9. *Let I be an interval in $(0,\infty)$ and φ be a harmonic convex function on I. If $w_1,\dots,w_n \geqslant 0$ with $\sum_{k=1}^n w_k = 1$ and $x_1,\dots,x_n\in I$, then*

$$\varphi\left(\frac{1}{\sum_{k=1}^n \frac{w_k}{x_k}}\right) \leqslant \sum_{k=1}^n \varphi(x_k)w_k.$$

In [3] appears the following inequality.

THEOREM 10. *Let I be an interval in $(0,\infty)$ and φ be a harmonic convex function on I. If $w_1,\dots,w_n \geqslant 0$ with $\sum_{k=1}^n w_k = 1$ and $x_1,\dots,x_n\in I$ is an increasing sequence, then*

$$\varphi\left(\frac{1}{\frac{1}{x_1}+\frac{1}{x_n}-\sum_{k=1}^n \frac{w_k}{x_k}}\right) \leqslant \varphi(x_1)+\varphi(x_n)-\sum_{k=1}^n \varphi(x_k)w_k.$$

In this paper we prove the following generalizations of theorems 9 and 10.

THEOREM 11. *Let μ be a probability measure on the space X and I be an interval in $\mathbb{R}\setminus\{0\}$. If $f: X\to I$ is a measurable function, φ is a harmonic convex function on I and $\varphi\circ f$ is a μ-integrable function, then*

$$\varphi\left(\frac{1}{\int_X \frac{d\mu}{f}}\right) \leqslant \int_X \varphi\circ f\, d\mu,$$

where $\varphi(0)=\lim_{t\to 0, t\in I}\varphi(t)$ if $\int_X \frac{d\mu}{f} = \pm\infty$.

Proof. One can easily check that $\varphi(x)$ is harmonically convex on I if and only if $\varphi(-x)$ is harmonically convex on

$$-I = \big\{-x : x\in I\big\}.$$

If I is contained in $(-\infty,0)$, then $-f: X\to -I$ is a measurable function. Since $\varphi(-x)$ is harmonically convex on $-I$, it suffices to prove Theorem 11 if the interval I is contained in $(0,\infty)$.

Assume first that I is a compact interval $I=[a,b]$ and φ is a continuous function on I. Then $0<a\leqslant f\leqslant b$ and f and $1/f$ are bounded functions. Hence, $1/f$ is a μ-integrable function, since μ is a finite measure.

For each $n \geqslant 1$ and $0 \leqslant k < 2^n$, consider the intervals

$$G_{n,k} = \left[a + k2^{-n}(b-a), a + (k+1)2^{-n}(b-a)\right), \qquad G_{n,2^n} = \{b\}.$$

For each $n \geqslant 1$ and $0 \leqslant k \leqslant 2^n$, define the sets

$$F_{n,k} = f^{-1}\left(G_{n,k}\right)$$

and the constants

$$f_{n,k} = a + k2^{-n}(b-a).$$

Since f is a measurable function satisfying $a \leqslant f \leqslant b$, we have that $\{F_{n,k}\}_{k=0}^{2^n}$ are pairwise disjoint measurable sets and $X = \cup_{k=0}^{2^n} F_{n,k}$ for each n.

Recall that the characteristic function of a set A is defined as $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$. If we define

$$f_n = \sum_{k=0}^{2^n} f_{n,k}\, \chi_{F_{n,k}},$$

then

$$\int_X f_n\, d\mu = \sum_{k=0}^{2^n} f_{n,k}\, \mu(F_{n,k}).$$

It is clear that

$$a \leqslant f_n \leqslant f \leqslant b, \qquad 0 \leqslant f - f_n \leqslant 2^{-n}(b-a).$$

Hence, f_n uniformly converges to f and, since μ is a finite measure,

$$\lim_{n\to\infty} \int_X f_n\, d\mu = \int_X f\, d\mu.$$

Since $\{F_{n,k}\}_{k=0}^{2^n}$ are pairwise disjoint sets and $X = \cup_{k=0}^{2^n} F_{n,k}$, we have

$$\varphi \circ f_n = \sum_{k=0}^{2^n} \varphi(f_{n,k}) \chi_{F_{n,k}},$$

$$\int_X \varphi \circ f_n\, d\mu = \sum_{k=0}^{2^n} \varphi(f_{n,k}) \mu(F_{n,k}).$$

Since φ is a continuous function on $[a,b]$, there exists a constant M such that $|\varphi| \leqslant M$ on $[a,b]$ and so, $|\varphi \circ f| \leqslant M$ and $|\varphi \circ f_n| \leqslant M$ on X for every $n \geqslant 1$. Since μ is a finite measure, M is a μ-integrable function and dominated convergence theorem gives

$$\lim_{n\to\infty} \int_X \varphi \circ f_n\, d\mu = \int_X \varphi \circ f\, d\mu.$$

Theorem 9 gives

$$\varphi\left(\frac{1}{\sum_{k=0}^{2^n} \frac{\mu(F_{n,k})}{f_{n,k}}}\right) \leqslant \sum_{k=0}^{2^n} \varphi(f_{n,k}) \mu(F_{n,k}) = \int_X \varphi \circ f_n\, d\mu. \tag{2}$$

We are going to check that

$$\lim_{n\to\infty}\sum_{k=0}^{2^n}\frac{\mu(F_{n,k})}{f_{n,k}}=\int_X\frac{d\mu}{f}.$$

Since $\{F_{n,k}\}_{k=0}^{2^n}$ are pairwise disjoint sets and $X=\cup_{k=0}^{2^n}F_{n,k}$, we have

$$\frac{1}{f_n}=\frac{1}{\sum_{k=0}^{2^n}f_{n,k}\chi_{F_{n,k}}}=\sum_{k=0}^{2^n}\frac{1}{f_{n,k}}\chi_{F_{n,k}},$$
$$\int_X\frac{d\mu}{f_n}=\sum_{k=0}^{2^n}\frac{\mu(F_{n,k})}{f_{n,k}}.$$

Note that

$$\left|\frac{1}{f_n}-\frac{1}{f}\right|=\frac{|f_n-f|}{f_nf}\leqslant\frac{|f_n-f|}{a^2}\leqslant\frac{2^{-n}(b-a)}{a^2}$$

and so, $1/f_n$ uniformly converges to $1/f$. Since μ is a finite measure,

$$\lim_{n\to\infty}\sum_{k=0}^{2^n}\frac{\mu(F_{n,k})}{f_{n,k}}=\lim_{n\to\infty}\int_X\frac{d\mu}{f_n}=\int_X\frac{d\mu}{f}.$$

Since $0<a\leqslant f_n\leqslant b$ and μ is a probability measure, we have

$$\frac{1}{b}\leqslant\frac{1}{f_n}\leqslant\frac{1}{a},\qquad\frac{1}{b}\leqslant\int_X\frac{d\mu}{f_n}=\sum_{k=0}^{2^n}\frac{\mu(F_{n,k})}{f_{n,k}}\leqslant\frac{1}{a},$$
$$a\leqslant\frac{1}{\int_X\frac{d\mu}{f_n}}=\frac{1}{\sum_{k=0}^{2^n}\frac{\mu(F_{n,k})}{f_{n,k}}}\leqslant b.$$

Since φ is a continuous function on $[a,b]$, the left hand side of (2) has limit

$$\varphi\left(\frac{1}{\int_X\frac{d\mu}{f}}\right).$$

This fact finishes the proof of Theorem 11 when I is a compact interval and φ is continuous on I.

We are going to remove the continuity hypothesis on φ.

Note that harmonic convex functions share with convex functions the following useful property:

$(P1)$ φ is a continuous function on (a,b), there exist $\lim_{t\to a^+}\varphi(t)$ and $\lim_{t\to b^-}\varphi(t)$; and, since $a>0$, there exists a continuous harmonic convex function φ_0 on $[a,b]$ and non-negative constants $A,B\geqslant 0$ such that

$$\varphi=\varphi_0+A\chi_{\{a\}}+B\chi_{\{b\}}.$$

Since φ_0 is continuous on $[a,b]$, we know that

$$\varphi_0\left(\frac{1}{\int_X \frac{d\mu}{f}}\right) \leqslant \int_X \varphi_0 \circ f \, d\mu.$$

Let us assume that $f = a$ μ-a.e. or $f = b$ μ-a.e. If $f = a$ μ-a.e., then

$$\varphi\left(\frac{1}{\int_X \frac{d\mu}{f}}\right) = \varphi(a) = \int_X \varphi \circ f \, d\mu \, .$$

If $f = b$ μ-a.e., then

$$\varphi\left(\frac{1}{\int_X \frac{d\mu}{f}}\right) = \varphi(b) = \int_X \varphi \circ f \, d\mu \, .$$

In other case, the sets

$$\begin{aligned}
\big\{x \in X : a < f(x) \leqslant b\big\} &= \left\{x \in X : \frac{1}{b} \leqslant \frac{1}{f(x)} < \frac{1}{a}\right\}, \\
\big\{x \in X : a \leqslant f(x) < b\big\} &= \left\{x \in X : \frac{1}{b} < \frac{1}{f(x)} \leqslant \frac{1}{a}\right\},
\end{aligned}$$

have positive measure. Thus, $1/b < \int_X d\mu/f < 1/a$ and $a < 1/(\int_X d\mu/f) < b$. Hence,

$$\varphi\left(\frac{1}{\int_X \frac{d\mu}{f}}\right) = \varphi_0\left(\frac{1}{\int_X \frac{d\mu}{f}}\right).$$

Since

$$\begin{gathered}
\varphi = \varphi_0 + A\,\chi_{\{a\}} + B\,\chi_{\{b\}}, \\
\varphi \circ f = \varphi_0 \circ f + A\,\chi_{\{f=a\}} + B\,\chi_{\{f=b\}},
\end{gathered}$$

we have

$$\int_X \varphi \circ f \, d\mu = \int_X \varphi_0 \circ f \, d\mu + A\,\mu\big(\{f=a\}\big) + B\,\mu\big(\{f=b\}\big) \geqslant \int_X \varphi_0 \circ f \, d\mu$$

and so,

$$\varphi\left(\frac{1}{\int_X \frac{d\mu}{f}}\right) = \varphi_0\left(\frac{1}{\int_X \frac{d\mu}{f}}\right) \leqslant \int_X \varphi_0 \circ f \, d\mu \leqslant \int_X \varphi \circ f \, d\mu.$$

Consider now the general case for I, with $\inf_{t\in I} = a \geqslant 0$ and $\sup_{t\in I} = b \leqslant \infty$. We can assume that $I = (a,b)$, since the cases $I = [a,b)$ and $I = (a,b]$ are similar and simpler.

Case (A)*:* Assume first that $b=\infty$.

For each positive integer n, let us consider the function $g_n : X \to [a+1/n, n]$ given by

$$g_n = \begin{cases} a+1/n, & \text{if } a < f < a+1/n, \\ f, & \text{if } a+1/n \leqslant f \leqslant n, \\ n, & \text{if } f > n. \end{cases}$$

Thus, $\{g_n\}$ is a converging sequence to f. We have proved that

$$\varphi\left(\frac{1}{\int_X \frac{d\mu}{g_n}}\right) \leqslant \int_X \varphi \circ g_n \, d\mu \tag{3}$$

since $g_n : X \to [a+1/n, n]$ and φ is harmonic convex on $[a+1/n, n] \subset I$.

Note that harmonic convex functions share with convex functions the following useful property:

$(P2)$ φ satisfies either:

- φ is a non-decreasing function on I,
- φ is a non-increasing function on I,
- there exists $t_0 \in I$ such that φ is non-increasing on $(a, t_0]$, and φ is non-decreasing on $[t_0, \infty)$.

By property $(P2)$, there exists a positive integer N_0 such that φ is a monotonous function on $(a, a+1/N_0]$ and on $[N_0, \infty)$. Thus, $|\varphi|$ also satisfies this property for some $N \geqslant N_0$. Since $|g_n| \leqslant \max\{|f|, |g_N|\}$ for every $n \geqslant N$, we have for every $n \geqslant N$

$$|\varphi \circ g_n| \leqslant \max\left\{|\varphi \circ f|, |\varphi \circ g_N|\right\} \leqslant |\varphi \circ f| + |\varphi \circ g_N|.$$

Since $g_N : X \to [a+1/N, N]$ and φ is continuous on $[a+1/N, N]$, $\varphi \circ g_N$ is a bounded function; since μ is a finite measure, $\varphi \circ g_N$ is μ-integrable. Since $\varphi \circ f$ is μ-integrable by hypothesis, and $|\varphi \circ g_n|$ is bounded by a μ-integrable function which does not depend on $n \geqslant N$, dominated convergence theorem gives

$$\lim_{n\to\infty} \int_X \varphi \circ g_n \, d\mu = \int_X \varphi \circ f \, d\mu. \tag{4}$$

Note that

$$h_n = \min\{f, n\} \leqslant f \leqslant \max\{f, a+1/n\} = H_n,$$
$$h_n = \min\{f, n\} \leqslant g_n \leqslant \max\{f, a+1/n\} = H_n.$$

Since h_n increases to f and H_n decreases to f, we have that $1/h_n$ decreases to $1/f$ and $1/H_n$ increases to $1/f$, and

$$\int_X \frac{d\mu}{H_n} \leqslant \int_X \frac{d\mu}{g_n} \leqslant \int_X \frac{d\mu}{h_n}, \qquad \int_X \frac{d\mu}{H_n} \leqslant \int_X \frac{d\mu}{f} \leqslant \int_X \frac{d\mu}{h_n}.$$

Note that

$$\int_X \frac{d\mu}{h_n} = \int_X \frac{d\mu}{\min\{f,n\}} = \int_X \max\left\{\frac{1}{f},\frac{1}{n}\right\} d\mu \leqslant \int_X \frac{d\mu}{f} + \int_X \frac{d\mu}{n} \leqslant \int_X \frac{d\mu}{f} + \frac{1}{n}$$

and so,

$$\int_X \frac{d\mu}{f} \leqslant \int_X \frac{d\mu}{h_n} \leqslant \int_X \frac{d\mu}{f} + \frac{1}{n}, \qquad \lim_{n\to\infty} \int_X \frac{d\mu}{h_n} = \int_X \frac{d\mu}{f}.$$

Also, monotone convergence theorem gives

$$\lim_{n\to\infty} \int_X \frac{d\mu}{H_n} = \int_X \frac{d\mu}{f}.$$

Consequently,

$$\lim_{n\to\infty} \int_X \frac{d\mu}{g_n} = \int_X \frac{d\mu}{f}. \tag{5}$$

(A.1) If $a > 0$, then $f > a$ implies

$$\frac{1}{f} < \frac{1}{a} \quad \Rightarrow \quad \int_X \frac{d\mu}{f} < \frac{1}{a} \quad \Rightarrow \quad \frac{1}{\int_X \frac{d\mu}{f}} > a.$$

Since φ is a continuous function on (a,∞),

$$\lim_{n\to\infty} \varphi\left(\frac{1}{\int_X \frac{d\mu}{g_n}}\right) = \varphi\left(\frac{1}{\int_X \frac{d\mu}{f}}\right)$$

and this fact, (3) and (4) give the desired inequality.

(A.2) Assume that $a = 0$. If $\int_X d\mu/f < \infty$, then

$$\frac{1}{\int_X \frac{d\mu}{f}} > 0$$

and the previous argument also gives the inequality. If $\int_X d\mu/f = \infty$, then (5) gives

$$\lim_{n\to\infty} \frac{1}{\int_X \frac{d\mu}{g_n}} = \frac{1}{\int_X \frac{d\mu}{f}} = 0.$$

Since property $(P1)$ guarantees that there exists $\varphi(0) = \lim_{t\to 0^+} \varphi(t) \leqslant \infty$, we conclude that

$$\lim_{n\to\infty} \varphi\left(\frac{1}{\int_X \frac{d\mu}{g_n}}\right) = \lim_{t\to 0^+} \varphi(t) = \varphi(0)$$

and so, (3) and (4) give

$$\varphi\left(\frac{1}{\int_X \frac{d\mu}{f}}\right) = \varphi(0) \leqslant \int_X \varphi \circ f \, d\mu.$$

In this case, since $\varphi \circ f$ is a μ-integrable function, we conclude that $\varphi(0) < \infty$.

Case (B)*:* Assume now that $b<\infty$.

In this case, for each positive integer $n>\frac{2}{b-a}$, let us consider the function $g_n : X\to[a+1/n,b-1/n]$ given by

$$g_n=\begin{cases} a+1/n, & \text{if } \ a<f<a+1/n,\\ f, & \text{if } \ a+1/n\leqslant f\leqslant b-1/n,\\ b-1/n, & \text{if } \ b-1/n<f<b.\end{cases}$$

Now, the argument in the proof of the case (A) $(b=\infty)$ gives the result. □

THEOREM 12. *Let μ be a probability measure on the space X and $[a,b]$ be a compact interval in $\mathbb{R}\setminus\{0\}$. If $f:X\to[a,b]$ is a measurable function and φ is a harmonic convex function on $[a,b]$, then $1/f$ and $\varphi\circ f$ are μ-integrable functions and*

$$\varphi\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\int_X\frac{d\mu}{f}}\right)\leqslant\varphi(a)+\varphi(b)-\int_X\varphi\circ f\,d\mu.$$

First of all, we need the following version of Theorem 10, without the hypotheses on the order of $x_1,\dots,x_n$.

LEMMA 13. *Let $0<a\leqslant b$ and φ be a continuous harmonic convex function on $[a,b]$. If $w_1,\dots,w_n\geqslant 0$ with $\sum_{k=1}^n w_k=1$ and $x_1,\dots,x_n\in[a,b]$, then*

$$\varphi\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\sum_{k=1}^n\frac{w_k}{x_k}}\right)\leqslant\varphi(a)+\varphi(b)-\sum_{k=1}^n\varphi(x_k)w_k.$$

Proof. Let σ be a permutation of $x_1,\dots,x_n$ such that $x_{\sigma(1)},\dots,x_{\sigma(n)}$ is an increasing sequence. Fix $0<\varepsilon<1$ and define

$$\begin{aligned} z_0&=a, \qquad & z_1&=x_{\sigma(1)},\dots,z_n=x_{\sigma(n)}, \qquad & z_{n+1}&=b,\\ w_0^*&=\varepsilon/2, \qquad & w_1^*&=(1-\varepsilon)w_1,\dots,w_n^*=(1-\varepsilon)w_n, \qquad & w_{n+1}^*&=\varepsilon/2.\end{aligned}$$

Since $z_0,z_1,\dots,z_{n+1}\in[a,b]$ is an increasing sequence and $\sum_{k=0}^{n+1}w_k^*=1$, Theorem 10 gives

$$\begin{aligned}\varphi&\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{\varepsilon/2}{a}-\frac{\varepsilon/2}{b}-(1-\varepsilon)\sum_{k=1}^n\frac{w_k}{x_k}}\right)\\ &=\varphi\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\sum_{k=0}^{n+1}\frac{w_k^*}{z_k}}\right)\\ &\leqslant\varphi(a)+\varphi(b)-\sum_{k=0}^{n+1}\varphi(z_k)w_k^*\\ &=\varphi(a)+\varphi(b)-\frac{\varepsilon}{2}\varphi(a)-\frac{\varepsilon}{2}\varphi(b)-(1-\varepsilon)\sum_{k=1}^n\varphi(x_k)w_k.\end{aligned}$$

Since φ is a continuous function on $[a,b]$, we obtain the desired result by taking the limit as $\varepsilon \to 0^+$ in the previous inequality. □

Let us start the proof of Theorem 12.

Proof. The argument at the beginning of the proof of Theorem 11 gives that it suffices to prove Theorem 12 if the compact interval $[a,b]$ is contained in $(0,\infty)$.

Assume first that φ is a continuous function on $[a,b]$ and let's consider the sequence of functions $\{f_n\}$ defined in the proof of Theorem 11.

Since $f_{n,k} = a + k2^{-n}(b-a) \in [a,b]$ for each $n \geqslant 1$ and $0 \leqslant k \leqslant 2^n$, $\{F_{n,k}\}_{k=0}^{2^n}$ is a partition of X for each $n \geqslant 1$, μ is a probability measure on X, and φ is a continuous harmonic convex function on $[a,b]$, Lemma13 gives

$$\varphi\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\sum_{k=0}^{2^n}\frac{\mu(F_{n,k})}{f_{n,k}}}\right) \leqslant \varphi(a)+\varphi(b)-\sum_{k=0}^{2^n}\varphi(f_{n,k})\mu(F_{n,k}).$$

We know, by the argument in the proof of Theorem 11, that

$$\int_X f_n\,d\mu = \sum_{k=0}^{2^n} f_{n,k}\,\mu(F_{n,k}), \qquad \int_X \varphi\circ f_n\,d\mu = \sum_{k=0}^{2^n}\varphi(f_{n,k})\mu(F_{n,k}),$$
$$\int_X \frac{d\mu}{f_n} = \sum_{k=0}^{2^n}\frac{\mu(F_{n,k})}{f_{n,k}},$$

and so,

$$\varphi\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\int_X\frac{d\mu}{f_n}}\right) \leqslant \varphi(a)+\varphi(b)-\int_X \varphi\circ f_n\,d\mu$$

for every $n \geqslant 1$.

We also know, by the argument in the proof of Theorem 11, that

$$\lim_{n\to\infty}\int_X f_n\,d\mu = \int_X f\,d\mu, \qquad \lim_{n\to\infty}\int_X \varphi\circ f_n\,d\mu = \int_X \varphi\circ f\,d\mu,$$
$$\lim_{n\to\infty}\int_X\frac{d\mu}{f_n} = \int_X\frac{d\mu}{f}.$$

Since φ is a continuous function on $[a,b]$, these facts finish the proof in this case.

Finally, let φ be a (not necessarily continuous) harmonic convex function on $[a,b]$.

Property $(P1)$ in the proof of Theorem 11 gives that there exists a continuous harmonic convex function φ_0 on $[a,b]$ and non-negative constants $A,B \geqslant 0$ such that

$$\varphi = \varphi_0 + A\chi_{\{a\}} + B\chi_{\{b\}}.$$

We have proved above

$$\varphi_0\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\int_X\frac{d\mu}{f_n}}\right) \leqslant \varphi_0(a)+\varphi_0(b)-\int_X \varphi_0\circ f_n\,d\mu.$$

Assume that $f = a$ μ-a.e. or $f = b$ μ-a.e. If $f = a$ μ-a.e., then

$$\begin{aligned}\varphi\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\int_X \frac{d\mu}{f}}\right) &= \varphi\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{a}}\right) = \varphi(b) \\ &= \varphi(a)+\varphi(b)-\varphi(a) = \varphi(a)+\varphi(b)-\int_X \varphi\circ f\, d\mu\,.\end{aligned}$$

If $f = b$ μ-a.e., then

$$\begin{aligned}\varphi\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\int_X \frac{d\mu}{f}}\right) &= \varphi\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{b}}\right) = \varphi(a) \\ &= \varphi(a)+\varphi(b)-\varphi(b) = \varphi(a)+\varphi(b)-\int_X \varphi\circ f\, d\mu\,.\end{aligned}$$

In other case, $1/b < \int_X d\mu/f < 1/a$ and so,

$$\frac{1}{b} < \frac{1}{a}+\frac{1}{b}-\int_X \frac{d\mu}{f} < \frac{1}{a}$$

and

$$\varphi\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\int_X \frac{d\mu}{f}}\right) = \varphi_0\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\int_X \frac{d\mu}{f}}\right).$$

The equality

$$\varphi = \varphi_0 + A\chi_{\{a\}} + B\chi_{\{b\}}$$

implies

$$\varphi\circ f = \varphi_0\circ f + A\chi_{\{f=a\}} + B\chi_{\{f=b\}}.$$

Thus,

$$\int_X \varphi\circ f\, d\mu = \int_X \varphi_0\circ f\, d\mu + A\mu\big(\{f=a\}\big) + B\mu\big(\{f=b\}\big)$$

and so,

$$\begin{aligned}&\varphi\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\int_X \frac{d\mu}{f}}\right) = \varphi_0\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\int_X \frac{d\mu}{f}}\right) \leqslant \varphi_0(a)+\varphi_0(b)-\int_X \varphi_0\circ f\, d\mu \\ &= \varphi(a)-A+\varphi(b)-B-\int_X \varphi\circ f\, d\mu + A\mu\big(\{f=a\}\big)+B\mu\big(\{f=b\}\big) \\ &= \varphi(a)+\varphi(b)-\int_X \varphi\circ f\, d\mu - A\big(1-\mu\big(\{f=a\}\big)\big) - B\big(1-\mu\big(\{f=b\}\big)\big) \\ &\leqslant \varphi(a)+\varphi(b)-\int_X \varphi\circ f\, d\mu.\end{aligned}$$

This finishes the proof of Theorem 12. □

Theorem 12 has the following consequence, improving Lemma 13.

COROLLARY 14. *Let φ be a harmonic convex function on a compact interval $[a,b] \subset \mathbb{R}\setminus\{0\}$. If $w_1,\dots,w_n \geqslant 0$ with $\sum_{k=1}^n w_k = 1$ and $x_1,\dots,x_n \in [a,b]$, then*

$$\varphi\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\sum_{k=1}^n \frac{w_k}{x_k}}\right) \leqslant \varphi(a)+\varphi(b)-\sum_{k=1}^n \varphi(x_k)w_k.$$

4. Inequalities for general fractional integrals of Caputo type

Theorems 11 and 12 have the following direct consequences for general fractional integrals of Caputo type.

PROPOSITION 15. *Let $\alpha \in (0,1)$. If $f:[a,b]\to I$ is a measurable function where I is an interval in $\mathbb{R}\setminus\{0\}$, F is an admissible kernel for the interval $[a,b]$ with*

$$\mathbb{F}(\alpha)=\int_a^b \frac{1}{F(b-s,\alpha)}\,ds=\int_0^{b-a}\frac{ds}{F(s,\alpha)}<\infty,$$

φ is a harmonic convex function on I, and $\varphi(f(s))/F(b-s,\alpha)\in L^1[a,b]$, then

$$\varphi\left(\frac{\mathbb{F}(\alpha)}{\int_a^b \frac{ds}{f(s)F(b-s,\alpha)}}\right)\leqslant \frac{1}{\mathbb{F}(\alpha)}\int_a^b \frac{\varphi(f(s))}{F(b-s,\alpha)}\,ds,$$

i.e.,

$$\varphi\left(\frac{\mathbb{F}(\alpha)}{{}^{C}J_F^{\alpha}(1/f)}\right)\leqslant \frac{1}{\mathbb{F}(\alpha)}\,{}^{C}J_F^{\alpha}(\varphi\circ f).$$

PROPOSITION 16. *Let $\alpha\in(0,1)$. If $f:[a,b]\to[A,B]$ is a measurable function where $[A,B]$ is a compact interval in $\mathbb{R}\setminus\{0\}$, F is an admissible kernel for the interval $[a,b]$ with*

$$\mathbb{F}(\alpha)=\int_a^b \frac{1}{F(b-s,\alpha)}\,ds=\int_0^{b-a}\frac{ds}{F(s,\alpha)}<\infty,$$

and φ is a harmonic convex function on $[A,B]$, then

$$\frac{1}{f(s)F(b-s,\alpha)}\in L^1[a,b], \qquad \frac{\varphi(f(s))}{F(b-s,\alpha)}\in L^1[a,b],$$

and

$$\varphi\left(\frac{1}{\frac{1}{A}+\frac{1}{B}-\frac{1}{\mathbb{F}(\alpha)}\int_a^b\frac{ds}{f(s)F(b-s,\alpha)}}\right)\leqslant \varphi(A)+\varphi(B)-\frac{1}{\mathbb{F}(\alpha)}\int_a^b\frac{\varphi(f(s))}{F(b-s,\alpha)}\,ds,$$

i.e.,

$$\varphi\left(\frac{1}{\frac{1}{A}+\frac{1}{B}-\frac{1}{\mathbb{F}(\alpha)}\,{}^{C}J_F^{\alpha}\left(\frac{1}{f}\right)}\right)\leqslant \varphi(A)+\varphi(B)-\frac{1}{\mathbb{F}(\alpha)}\,{}^{C}J_F^{\alpha}(\varphi\circ f).$$

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