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A TREATMENT METHOD OF A CLASS OF HALF–DISCRETE HILBERT–TYPE INEQUALITIES ON SYMMETRIC SETS

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Abstract. In this work, we first construct a special set of real numbers, and then we define a new half-discrete kernel function on symmetric sets with the parameters limited to the newly constructed set. By virtue of some techniques of real analysis, we transform the weight function to the first quadrant to estimate its upper bound, then a half-discrete Hilbert-type inequality on symmetric sets is proved, and its constant factor is proved to be optimal. Furthermore, the equivalent Hardy-type inequalities of the newly obtained Hilbert-type inequality are also considered. Lastly, assigning special values to the parameters in the kernel function, some new special half-discrete Hilbert-type inequalities are provided at the end of the paper.

1. Introduction

Throughout this work, it is assumed that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\Omega^+ := \left\{ x : x = \frac{2m+1}{2n+1}, m, n \in \mathbb{N}^+ \cup \{0\} \right\},$$

$\Omega^- = \{x : -x \in \Omega^+\}$, and $\Omega = \Omega^+ \cup \Omega^-$. Additionally, for an arbitrary integer m ($m \in \mathbb{Z}^+$), assume that

$$\mathbb{Z}_m^+ := \{x : x \geqslant m, x \in \mathbb{Z}^+\},$$

$\mathbb{Z}_m^- = \{x : -x \in \mathbb{Z}_m^+\}$, and $\mathbb{Z}_m = \mathbb{Z}_m^+ \cup \mathbb{Z}_m^-$.

Suppose that Π is a measurable set, and $f(x), \mu(x)$ are two non-negative measurable functions defined on Π. Define a function space $L_{p,\mu}(\Pi)$ as follows:

$$L_{p,\mu}(\Pi) := \left\{ f : \|f\|_{p,\mu} := \left(\int_\Pi f^p(x)\mu(x)\mathrm{d}x \right)^{1/p} < \infty \right\}.$$

Specially, we have the abbreviations: $\|f\|_p := \|f\|_{p,\mu}$ and $L_p(\Pi) := L_{p,\mu}(\Pi)$ if $\mu(x) \equiv 1$.

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Suppose that $a_n, \nu_n > 0$, $n \in \Theta \subseteq \mathbb{Z}$, $\boldsymbol{a} = \{a_n\}_{n\in\Theta}$. Define a sequence space $l_{p,\nu}$ as follows:

$$l_{p,\nu} := \left\{ \boldsymbol{a} : \|\boldsymbol{a}\|_{p,\nu} := \left(\sum_{n\in\Theta} a_n^p \nu_n \right)^{1/p} < \infty \right\}.$$

Specially, we have the abbreviations: $\|\boldsymbol{a}\|_p := \|\boldsymbol{a}\|_{p,\nu}$, and $l_p := l_{p,\nu}$ if $\nu_n \equiv 1$.

Consider two non negative real-valued sequences: $\boldsymbol{a} = \{a_m\}_{m=1}^{\infty} \in l_2$, and $\boldsymbol{b} = \{b_n\}_{n=1}^{\infty} \in l_2$, then

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \pi \|\boldsymbol{a}\|_2 \|\boldsymbol{b}\|_2, \tag{1.1}$$

where the constant factor π is optimal. Inequality (1.1) was first put forward by D. Hilbert in his lectures on integral equations in 1908, and Schur proved the integral form of (1.1) in 1911, that is,

$$\int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{x+y} \mathrm{d}x\mathrm{d}y < \pi \|f\|_2 \|g\|_2, \tag{1.2}$$

where $f, g \in L_2(\mathbb{R}^+)$, and the constant factor π is optimal.

Inequalities (1.1) and (1.2) are commonly referred to as Hilbert inequality [5]. In 1991, Hsu [6] proposed the weight coefficient method and established the following improved form of (1.1), that is,

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \|\boldsymbol{a}\|_{\mu,2} \|\boldsymbol{b}\|_{\nu,2}, \tag{1.3}$$

where $\mu_m = \pi - \frac{c_0}{\sqrt{m}}$, $\nu_n = \pi - \frac{c_0}{\sqrt{n}}$ $(c_0 = 1.1213\cdots)$.

After the 1990s, researchers established a large number of extensions and analogies regarding (1.1) and (1.2) based on Hsu's method, such as the following one proved by M. Krnić and J. Pečarić [10]:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^{\beta}} < B\left(\frac{\beta}{2}, \frac{\beta}{2}\right) \|\boldsymbol{a}\|_{p,\mu} \|\boldsymbol{b}\|_{q,\nu}, \tag{1.4}$$

where $0 < \beta \leqslant 4$, $\mu_m = m^{p(1-\beta/2)-1}$, $\nu_n = n^{q(1-\beta/2)-1}$, and $B(x,y)$ is the Beta function. In addition, Yang [16] established an extension of (1.2) as follows:

$$\int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{x^{\beta}+y^{\beta}} \mathrm{d}x\mathrm{d}y < \frac{\pi}{\beta \sin \lambda\pi} \|f\|_{p,\mu} \|g\|_{q,\nu}, \tag{1.5}$$

where $\beta, \gamma, \lambda > 0$, $\lambda + \gamma = 1$, $\mu(x) = x^{p(1-\lambda\beta)-1}$, and $\nu(x) = x^{q(1-\gamma\beta)-1}$.

Inequalities such as (1.4) and (1.5) are commonly known as Hilbert-type inequality. With regard to some other Hilbert-type inequalities, we refer to [1, 7, 11, 13, 17, 21, 22].

It should be pointed out that the kernel functions in inequalities (1.1) and (1.2) are generally referred to as homogeneous kernel [3, 8]. If a integral Hilbert-type inequality with a homogeneous kernel is proved to be true, then it is usually easy to establish a similar form with a non-homogeneous kernel, such as the following one derived from (1.2):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{1+xy}\mathrm{d}x\mathrm{d}y < \pi\|f\|_2\|g\|_2, \tag{1.6}$$

where the constant factor π is optimal. The corresponding inequality to (1.1) with a non-homogeneous kernel can also be established, that is,

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{1+mn} < \pi\|\boldsymbol{a}\|_2\|\boldsymbol{b}\|_2. \tag{1.7}$$

However, the constant factor can not be proved to be optimal (see [18], p. 315). Until now, it is still unknown whether the constant factor π is optimal. In 2005, Yang [19] proved the following half-discrete Hilbert inequality similar to (1.6) and (1.7), that is,

$$\int_0^\infty f(x) \sum_{n=1}^\infty \frac{a_n}{1+nx}\mathrm{d}x < \pi\|f\|_2\|\boldsymbol{a}\|_2, \tag{1.8}$$

where the constant factor π is optimal. In the past decade, a large number of half-discrete Hilbert inequalities were established, such as the following one with optimal constant factor $\frac{2\pi}{\beta}$ [23]:

$$\int_0^\infty f(x) \sum_{n=1}^\infty \log\left(1+\frac{1}{x^\beta n^\beta}\right) a_n \mathrm{d}x < \frac{2\pi}{\beta}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{1.9}$$

where $0<\beta<2$, $\mu(x)=x^{p(1-\beta/2)-1}$ and $\nu_n=n^{q(1-\beta/2)-1}$. With regard to some other half-discrete inequalities, we refer to [2, 9, 12, 14, 20, 24].

Generally, Hilbert-type inequalities are established in the first quadrant. However, it is not an easy task to extend a Hilbert-type inequality to the whole plane, as the non-negativity, monotonicity and integrability of a kernel function will be very complicated if we extend the range of variables to $\mathbb{R}^2$. In this work, the main objective is to provide a new half-discrete Hilbert-type inequality defined on symmetric sets with the kernel functions involving both the homogeneous and non-homogeneous cases. The paper is organized as follows: detailed lemmas will be presented in Section 2, and main results and some corollaries will be presented in Section 3 and Section 4, respectively.

2. Some lemmas

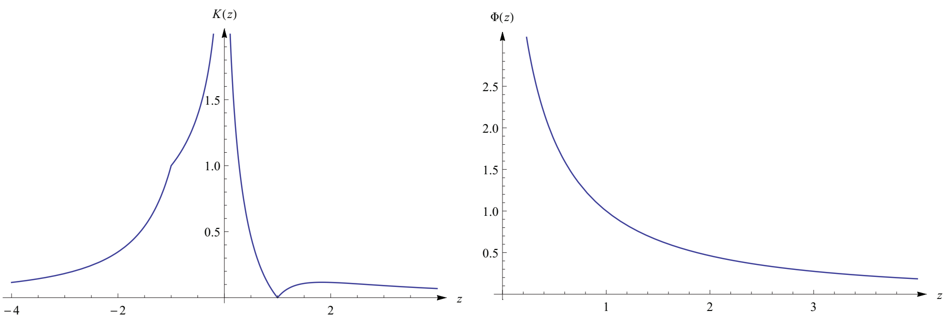
LEMMA 2.1. *Let* $\tau \in \{1,-1\}$, $\beta \in \Omega^+$, *and* $\gamma \in \mathbb{R}^+$. *Define*

$$K(z) := \frac{|\log|z||}{|\tau z^\beta + 1| \max\{1, |z|^\gamma\}}, \tag{2.1}$$

where $z \in \mathbb{R} \setminus \{-1,0\}$ *if* $\tau = 1$, *and* $z \in \mathbb{R} \setminus \{1,0\}$ *if* $\tau = -1$. *Define* $K(-1) := \frac{1}{\beta}$ *for* $\tau = 1$, *and* $K(1) := \frac{1}{\beta}$ *for* $\tau = -1$. *Let*

$$\Phi(z) := K(z) + K(-z) \ \left(z \in \mathbb{R}^+\right)$$

(*see Figure 1: The graphs of* $K(z)$ *and* $\Phi(z)$ *for* $\tau = \beta = \gamma = 1$). *Then* $\Phi(z)$ *decreases monotonically with* z.



(a) *The graph of* $K(z)$ $(z \in (-4,4))$

(b) *The graph of* $\Phi(z)$ $(z \in (0,4))$

Figure 1: *The graphs of* $K(z)$ *and* $\Phi(z)$ *for* $\tau = \beta = \gamma = 1$

Proof. It is obvious that $\Phi(1) = \frac{1}{\beta}$ whether $\tau = 1$ or $\tau = -1$. If $z \in (0,1) \cup (1,\infty)$, observing that $\beta \in \Omega^+$ and $\tau \in \{1,-1\}$, we have

$$\begin{aligned}
\Phi(z) &= K(z) + K(-z) \\
&= \frac{|\log z|}{|\tau z^\beta + 1| \max\{1, z^\gamma\}} + \frac{|\log z|}{|\tau z^\beta - 1| \max\{1, z^\gamma\}} \\
&= \frac{|\log z|}{\max\{1, z^\gamma\}} \left[\frac{1}{|\tau z^\beta + 1|} + \frac{1}{|\tau z^\beta - 1|}\right] \\
&= \frac{|\log z|}{\max\{1, z^\gamma\}} \frac{|\tau z^\beta - 1| + |\tau z^\beta + 1|}{|z^{2\beta} - 1|} \\
&= \frac{|\log z|}{\max\{1, z^\gamma\}} \frac{|z^\beta - 1| + |z^\beta + 1|}{|z^{2\beta} - 1|} \\
&= \begin{cases} \frac{2\log z}{z^{2\beta} - 1}, & z \in (0,1), \\ \frac{2z^{\beta-\gamma}\log z}{z^{2\beta} - 1}, & z \in (1,\infty). \end{cases}
\end{aligned}$$

Write

$$\psi(z) = \frac{2\log z}{z^{2\beta}-1}, \quad z \in (0,1),$$

then

$$\frac{\mathrm{d}\psi}{\mathrm{d}z} = \frac{-2z^{2\beta-1}\left(2\beta\log z + z^{-2\beta} - 1\right)}{\left(z^{2\beta}-1\right)^2} := \frac{-2z^{2\beta-1}\psi_1(z)}{\left(z^{2\beta}-1\right)^2}.$$

Since $z \in (0,1)$ and

$$\frac{\mathrm{d}\psi_1}{\mathrm{d}z} = 2\beta z^{-2\beta-1}(z^{2\beta}-1),$$

we have $\frac{\mathrm{d}\psi_1}{\mathrm{d}z} < 0$. It implies that $\psi_1(z) > \psi_1(1) = 0$ $(z \in (0,1))$, and therefore we have $\frac{\mathrm{d}\psi}{\mathrm{d}z} < 0$ and $\psi(z)$ decreases monotonically with z $(z \in (0,1))$.

Write

$$\Psi(z) = \frac{2z^{\beta-\gamma}\log z}{z^{2\beta}-1}, \quad z \in (1,\infty),$$

then

$$\frac{\mathrm{d}\Psi}{\mathrm{d}z} = \frac{-2z^{\beta-\gamma-1}\left[(\beta+\gamma)z^{2\beta}\log z + (\beta-\gamma)\log z - z^{2\beta} + 1\right]}{\left(z^{2\beta}-1\right)^2}$$

$$:= \frac{-2z^{\beta-\gamma-1}\Psi_1(z)}{\left(z^{2\beta}-1\right)^2}.$$

It can be proved that

$$\frac{\mathrm{d}\Psi_1}{\mathrm{d}z} = z^{2\beta-1}\left[2(\beta^2+\beta\gamma)\log z + (\beta-\gamma)z^{-2\beta} - (\beta-\gamma)\right] := z^{2\beta-1}\Psi_2(z).$$

Observing that $z \in (1,\infty)$, we have

$$\frac{\mathrm{d}\Psi_2}{\mathrm{d}z} = 2\beta z^{-2\beta-1}\left[\beta\left(z^{2\beta}-1\right) + \gamma\left(z^{2\beta}+1\right)\right] > 0.$$

Hence, we have $\Psi_2(z) > \Psi_2(1) = 0$ $(z \in (1,\infty))$, and it implies that $\frac{\mathrm{d}\Psi_1}{\mathrm{d}z} > 0$. Therefore, it can be obtained that $\Psi_1(z) > \Psi_1(1) = 0$ and $\frac{\mathrm{d}\Psi}{\mathrm{d}z} < 0$. It follows that $\Psi(z)$ decreases monotonically with z $(z \in (1,\infty))$.

Based on the above discussion, we have $\Phi(z)$ decreases monotonically both on $(0,1)$ and $(1,\infty)$. In view of $\Phi(1) = \frac{1}{\beta}$, it can be shown that $\Phi(z)$ is continuous on $\mathbb{R}^+$, and therefore $\Phi(z)$ decreases monotonically with z $(z \in \mathbb{R}^+)$. $\square$

LEMMA 2.2. *Let* $\tau \in \{1,-1\}$, $\alpha \in (0,1)$, $\beta \in \Omega^+$, $\gamma \in \mathbb{R}^+$, *and* $\beta+\gamma > \alpha$. *Suppose that* $K(z)$ *is defined by Lemma* 2.1, *and*

$$C(\alpha,\beta,\gamma) := \sum_{j=0}^{\infty}\left[\frac{1}{(2\beta j+\alpha)^2} + \frac{1}{(2\beta j-\alpha+\beta+\gamma)^2}\right]. \tag{2.2}$$

Then

$$\int_0^\infty [K(z)+K(-z)]z^{\alpha-1}\mathrm{d}z = C(\alpha,\beta,\gamma). \tag{2.3}$$

Proof. Observing that $\beta\in\Omega^+$ and $\tau\in\{1,-1\}$, we have

$$\begin{aligned}
&\int_0^\infty [K(z)+K(-z)]z^{\alpha-1}\mathrm{d}z\\
&=\int_0^1\left(\frac{1}{|\tau z^\beta+1|}+\frac{1}{|\tau z^\beta-1|}\right)|\log z|z^{\alpha-1}\mathrm{d}z\\
&\quad+\int_1^\infty\left(\frac{1}{|\tau z^\beta+1|}+\frac{1}{|\tau z^\beta-1|}\right)|\log z|z^{\alpha-\gamma-1}\mathrm{d}z\\
&=\int_0^1\frac{|\tau z^\beta-1|+|\tau z^\beta+1|}{|z^{2\beta}-1|}|\log z|z^{\alpha-1}\mathrm{d}z\\
&\quad+\int_1^\infty\frac{|\tau z^\beta-1|+|\tau z^\beta+1|}{|z^{2\beta}-1|}|\log z|z^{\alpha-\gamma-1}\mathrm{d}z\\
&=2\int_0^1\frac{z^{\alpha-1}\log z}{z^{2\beta}-1}\mathrm{d}z+2\int_1^\infty\frac{z^{\alpha+\beta-\gamma-1}\log z}{z^{2\beta}-1}\mathrm{d}z\\
&=2\int_0^1\frac{z^{\alpha-1}\log z}{z^{2\beta}-1}\mathrm{d}z+2\int_0^1\frac{z^{-\alpha+\beta+\gamma-1}\log z}{z^{2\beta}-1}\mathrm{d}z.
\end{aligned}\tag{2.4}$$

Expand $\frac{1}{1-z^{2\beta}}$ $(z\in(0,1))$ into a power series, and employ Lebesgue term-by-term integration theorem, then we have

$$\int_0^1\frac{z^{\alpha-1}\log z}{z^{2\beta}-1}\mathrm{d}z=\int_0^1\sum_{j=0}^\infty z^{2\beta j+\alpha-1}\log z\mathrm{d}z=\sum_{j=0}^\infty\int_0^1 z^{2\beta j+\alpha-1}\log z\mathrm{d}z. \tag{2.5}$$

Set $\log z=\frac{-u}{2\beta j+\alpha}$ in (2.5), then we have

$$\int_0^1 z^{2\beta j+\alpha-1}\log z\mathrm{d}z=\frac{1}{(2\beta j+\alpha)^2}\int_0^\infty ue^{-u}\mathrm{d}u=\frac{1}{(2\beta j+\alpha)^2}. \tag{2.6}$$

Inserting (2.6) back into (2.5), we have

$$\int_0^1\frac{z^{\alpha-1}\log z}{z^{2\beta}-1}\mathrm{d}z=\sum_{j=0}^\infty\frac{1}{(2\beta j+\alpha)^2}. \tag{2.7}$$

Similarly, observing that $\beta+\gamma>\alpha$, it can be proved that

$$\int_0^1\frac{z^{-\alpha+\beta+\gamma-1}\log z}{z^{2\beta}-1}\mathrm{d}z=\sum_{j=0}^\infty\frac{1}{(2\beta j-\alpha+\beta+\gamma)^2}. \tag{2.8}$$

Inserting (2.7) and (2.8) back into (2.4), we arrive at (2.3). Lemma 2.2 is proved. □

LEMMA 2.3. *Let* $\gamma_0 = 0.915965\cdots$ *is the Catalan constant, then the following identity holds:*

$$\sum_{j=0}^{\infty} \frac{1}{(4j+1)^2} = \frac{\gamma_0}{2} + \frac{\pi^2}{16}. \tag{2.9}$$

Proof. In view of that [15]

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^2} = \gamma_0, \quad \sum_{j=0}^{\infty} \frac{1}{(2j+1)^2} = \frac{\pi^2}{6},$$

we have

$$\sum_{j=0}^{\infty} \frac{1}{(2j+1)^2} = \sum_{j=0}^{\infty} \left[\frac{1}{(j+1)^2} - \frac{1}{(2j+2)^2} \right] = \frac{\pi^2}{8},$$

and it follows that

$$\sum_{j=0}^{\infty} \frac{2}{(4j+1)^2} = \sum_{j=0}^{\infty} \left[\frac{(-1)^j}{(2j+1)^2} + \frac{1}{(2j+1)^2} \right] = \gamma_0 + \frac{\pi^2}{8}. \tag{2.10}$$

Identity (2.9) follows from (2.10) naturally. □

LEMMA 2.4. *Let* $k_1, k_2, k > 0,\ k_1 + k_2 = k$. *Then*

$$\sum_{j=0}^{\infty} \left[\frac{1}{(kj+k_1)^2} + \frac{1}{(kj+k_2)^2} \right] = \frac{\pi^2}{k^2} \csc^2 \left(\frac{k_1 \pi}{k} \right). \tag{2.11}$$

Proof. Observing that $\tan u$ can be expanded as follows:

$$\tan u = \sum_{j=0}^{\infty} \left[\frac{2}{(2j+1)\pi - 2u} - \frac{2}{(2j+1)\pi + 2u} \right],$$

we have

$$\sec^2 u = \sum_{j=0}^{\infty} \left[\frac{4}{((2j+1)\pi - 2u)^2} + \frac{4}{((2j+1)\pi + 2u)^2} \right]. \tag{2.12}$$

Set $u = \frac{\pi}{2} - \frac{k_1 \pi}{k}$ in (2.12), and use $k_1 + k_2 = k$, then we arrive at (2.11). □

LEMMA 2.5. *Let* $k_1, k_2, k > 0,\ k_1 + k_2 = 2k$. *Then*

$$\sum_{j=0}^{\infty} \left[\frac{1}{(kj+k_1)^2} + \frac{1}{(kj+k_2)^2} \right] = \begin{cases} \frac{\pi^2}{k^2} \csc^2 \left(\frac{k_1 \pi}{k} \right) - \frac{1}{(k-k_1)^2}, & k_1 \neq k_2, \\ \frac{\pi^2}{3k^2}, & k_1 = k_2 = k. \end{cases} \tag{2.13}$$

Proof. If $0<k_1<k$, then $k<k_2<2k$. By Lemma 2.4, we have

$$\begin{aligned}&\sum_{j=0}^{\infty}\left[\frac{1}{(kj+k_1)^2}+\frac{1}{(kj+k_2)^2}\right]\\&=\sum_{j=0}^{\infty}\left[\frac{1}{(kj+k_1)^2}+\frac{1}{(kj+k_2-k)^2}\right]-\frac{1}{(k_2-k)^2}\\&=\frac{\pi^2}{k^2}\csc^2\left(\frac{k_1\pi}{k}\right)-\frac{1}{(k-k_1)^2}.\end{aligned}$$

If $k_1=k_2=k$, then

$$\sum_{j=0}^{\infty}\left[\frac{1}{(kj+k_1)^2}+\frac{1}{(kj+k_2)^2}\right]=\frac{\pi^2}{3k^2}.$$

If $k<k_1<2k$, then $0<k_2<k$. By Lemma 2.4, we have

$$\begin{aligned}&\sum_{j=0}^{\infty}\left[\frac{1}{(kj+k_1)^2}+\frac{1}{(kj+k_2)^2}\right]\\&=\sum_{j=0}^{\infty}\left[\frac{1}{(kj+k_1-k)^2}+\frac{1}{(kj+k_2)^2}\right]-\frac{1}{(k_1-k)^2}\\&=\frac{\pi^2}{k^2}\csc^2\left(\frac{k_1\pi}{k}\right)-\frac{1}{(k-k_1)^2}.\end{aligned}$$

Based on the above discussion, we complete the proof of Lemma 2.5. □

3. Main results

THEOREM 3.1. *Let $\tau\in\{1,-1\}$, $\alpha\in(0,1)$, $\beta_1\in\Omega$, $\beta_2,\beta\in\Omega^+$, $\gamma\in\mathbb{R}^+$, $\beta+\gamma>\alpha$ and $\beta_2\alpha<1$. Let $0\leqslant a\leqslant 1$ and $S=(-\infty,-a)\cup(a,\infty)$ when $\beta_1\in\Omega^-$. Let $a\geqslant 1$ and $S=(-a,0)\cup(0,a)$ when $\beta_1\in\Omega^+$. Suppose that $\mu(x)=|x|^{p(1-\beta_1\alpha)-1}$ and $\nu_n=|n|^{q(1-\beta_2\alpha)-1}$, where $n\in\mathbb{Z}_m$. Let $f(x)$, $a_n\geqslant 0$ with $f(x)\in L_{p,\mu}(S)$ and $\boldsymbol{a}=\{a_n\}_{n\in\mathbb{Z}_m}\in l_{q,\nu}$. Let $K(z)$ and $C(\alpha,\beta,\gamma)$ be defined by (2.1) and (2.2), respectively. Then*

$$\begin{aligned}\sum_{n\in\mathbb{Z}_m}a_n\int_S K\left(x^{\beta_1}n^{\beta_2}\right)f(x)\mathrm{d}x&=\int_S f(x)\sum_{n\in\mathbb{Z}_m}K\left(x^{\beta_1}n^{\beta_2}\right)a_n\mathrm{d}x\\&<|\beta_1|^{-\frac{1}{q}}\beta_2^{-\frac{1}{p}}C(\alpha,\beta,\gamma)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu},\end{aligned}\tag{3.1}$$

where the constant factor $|\beta_1|^{-\frac{1}{q}}\beta_2^{-\frac{1}{p}}C(\alpha,\beta,\gamma)$ is optimal.

Proof. Set $\hat{K}\left(x^{\beta_1}y^{\beta_2}\right):=K\left(x^{\beta_1}n^{\beta_2}\right)$, $g(y):=a_n$, and $\omega(y):=n$ for $y\in[n,n+1)$, $n\in\mathbb{Z}_m^+$. Set $\hat{K}\left(x^{\beta_1}y^{\beta_2}\right):=K\left(x^{\beta_1}n^{\beta_2}\right)$, $g(y):=a_n$, and $\omega(y):=|n|$ for $y\in[n-1,n)$,

$n \in \mathbb{Z}_m^-$. Let $D = (-\infty, -m) \cup (m, \infty)$. By Hölder's inequality for double integrals, we have

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}_m} a_n \int_S K\left(x^{\beta_1} n^{\beta_2}\right) f(x)\mathrm{d}x \\
&\quad = \int_S f(x) \sum_{n\in\mathbb{Z}_m} K\left(x^{\beta_1} n^{\beta_2}\right) a_n \mathrm{d}x \\
&\quad = \int_D \int_S \hat{K}\left(x^{\beta_1} y^{\beta_2}\right) f(x) g(y)\mathrm{d}x\mathrm{d}y \\
&\quad = \int_D \int_S \left[\hat{K}\left(x^{\beta_1} y^{\beta_2}\right)\right]^{1/p} (\omega(y))^{(\beta_2\alpha-1)/p} |x|^{(1-\beta_1\alpha)/q} f(x) \\
&\qquad \times \left[\hat{K}\left(x^{\beta_1} y^{\beta_2}\right)\right]^{1/q} |x|^{(\beta_1\alpha-1)/q} (\omega(y))^{(1-\beta_2\alpha)/p} g(y)\mathrm{d}x\mathrm{d}y \\
&\quad \leqslant \left[\int_S \int_D \hat{K}\left(x^{\beta_1} y^{\beta_2}\right) (\omega(y))^{\beta_2\alpha-1} |x|^{p(1-\beta_1\alpha)/q} f^p(x)\mathrm{d}y\mathrm{d}x\right]^{1/p} \\
&\qquad \times \left[\int_D \int_S \hat{K}\left(x^{\beta_1} y^{\beta_2}\right) |x|^{\beta_1\alpha-1} (\omega(y))^{q(1-\beta_2\alpha)/p} g^q(y)\mathrm{d}x\mathrm{d}y\right]^{1/q} \\
&\quad = \left[\int_S F(x) |x|^{p(1-\beta_1\alpha)/q} f^p(x)\mathrm{d}x\right]^{1/p} \left[\sum_{n\in\mathbb{Z}_m} G(n) |n|^{q(1-\beta_2\alpha)/p} a_n^q\right]^{1/q}, \qquad (3.2)
\end{aligned}$$

where

$$F(x) = \sum_{n\in\mathbb{Z}_m} K\left(x^{\beta_1} n^{\beta_2}\right) |n|^{\beta_2\alpha-1},$$

$$G(n) = \int_S K\left(x^{\beta_1} n^{\beta_2}\right) |x|^{\beta_1\alpha-1} \mathrm{d}x.$$

It follows from $\beta_2 \in \Omega^+$ that

$$\begin{aligned}
F(x) &= \sum_{n\in\mathbb{Z}_m} K\left(x^{\beta_1} n^{\beta_2}\right) |n|^{\beta_2\alpha-1} \\
&= \sum_{n\in\mathbb{Z}_m^+} K\left(x^{\beta_1} n^{\beta_2}\right) |n|^{\beta_2\alpha-1} + \sum_{n\in\mathbb{Z}_m^-} K\left(x^{\beta_1} n^{\beta_2}\right) |n|^{\beta_2\alpha-1} \\
&= \sum_{n\in\mathbb{Z}_m^+} \left[K\left(x^{\beta_1} n^{\beta_2}\right) + K\left(-x^{\beta_1} n^{\beta_2}\right)\right] n^{\beta_2\alpha-1}.
\end{aligned}$$

Observing that $\beta_1 \in \Omega$, $\beta_2 \in \Omega^+$, and employing Lemma 2.1, it can be easy to show that

$$K\left(x^{\beta_1} n^{\beta_2}\right) + K\left(-x^{\beta_1} n^{\beta_2}\right)$$

decreases monotonically with n $(n \in \mathbb{Z}_m^+)$ for a fixed x, whether $x > 0$ or $x < 0$. Additionally, $n^{\beta_2\alpha-1}$ decreases monotonically with n $(n \in \mathbb{Z}_m^+)$ owing to $\beta_2\alpha < 1$. Therefore, setting $|x|^{\beta_1} y^{\beta_2} = z$, and using Lemma 2.2, we have

$$\begin{aligned} F(x) &< \int_0^{\infty} \left[K\left(x^{\beta_1} y^{\beta_2}\right) + K\left(-x^{\beta_1} y^{\beta_2}\right)\right] y^{\beta_2 \alpha - 1} \mathrm{d}y \\ &= \beta_2^{-1} |x|^{-\beta_1 \alpha} \int_0^{\infty} [K(z) + K(-z)] z^{\alpha - 1} \mathrm{d}z \\ &= \beta_2^{-1} |x|^{-\beta_1 \alpha} C(\alpha, \beta, \gamma). \end{aligned} \tag{3.3}$$

In what follows, we will estimate the upper bound of $G(n)$. It follows from variable substitution $z = x^{\beta_1} |n|^{\beta_2}$ that

$$\begin{aligned} G(n) &= \int_S K\left(x^{\beta_1} n^{\beta_2}\right) |x|^{\beta_1 \alpha - 1} \mathrm{d}x \\ &= \int_{S \cap \mathbb{R}^+} K\left(x^{\beta_1} n^{\beta_2}\right) |x|^{\beta_1 \alpha - 1} \mathrm{d}x \\ &\quad + \int_{S \cap \mathbb{R}^-} K\left(x^{\beta_1} n^{\beta_2}\right) |x|^{\beta_1 \alpha - 1} \mathrm{d}x \\ &= \int_{S \cap \mathbb{R}^+} \left[K\left(x^{\beta_1} n^{\beta_2}\right) + K\left(-x^{\beta_1} n^{\beta_2}\right)\right] x^{\beta_1 \alpha - 1} \mathrm{d}x \\ &\leqslant \int_0^{\infty} \left[K\left(x^{\beta_1} n^{\beta_2}\right) + K\left(-x^{\beta_1} n^{\beta_2}\right)\right] x^{\beta_1 \alpha - 1} \mathrm{d}x \\ &= |\beta_1|^{-1} |n|^{-\beta_2 \alpha} \int_0^{\infty} [K(z) + K(-z)] z^{\alpha - 1} \mathrm{d}z \\ &= |\beta_1|^{-1} |n|^{-\beta_2 \alpha} C(\alpha, \beta, \gamma). \end{aligned} \tag{3.4}$$

Plugging (3.3) and (3.4) back into (3.2), we arrive at (3.1).

In what follows, it will be proved the constant factor $|\beta_1|^{-\frac{1}{q}} \beta_2^{-\frac{1}{p}} C(\alpha, \beta, \gamma)$ in (3.1) is optimal. Let

$$\hat{f}(x) := \begin{cases} |x|^{\beta_1 \alpha - 1 + \frac{2\beta_1}{ps}} & x \in E \\ 0 & x \in S \setminus E \end{cases},$$

where $E := \left\{x : |x|^{\operatorname{sgn} \beta_1} < 1\right\}$. Additionally, let

$$\hat{\boldsymbol{a}} := \{\hat{a}_n\}_{n \in \mathbb{Z}_m} := \left\{|n|^{\beta_2 \alpha - 1 - \frac{2\beta_2}{qs}}\right\}_{n \in \mathbb{Z}_m},$$

where s is a sufficiently large natural number.

Write $E^+ := \{x : x \in E, x > 0\}$ and $E^- := \{x : x \in E, x < 0\}$. Observe that $\beta \in \Omega^+$, $\beta_1 \in \Omega$, then it follows that

$$\begin{aligned} \int_S \hat{f}(x) \sum_{n \in \mathbb{Z}_m} K\left(x^{\beta_1} n^{\beta_2}\right) \hat{a}_n \mathrm{d}x = &\int_{E^+} \hat{f}(x) \sum_{n \in \mathbb{Z}_m^+} K\left(x^{\beta_1} n^{\beta_2}\right) \hat{a}_n \mathrm{d}x \\ &+ \int_{E^+} \hat{f}(x) \sum_{n \in \mathbb{Z}_m^-} K\left(x^{\beta_1} n^{\beta_2}\right) \hat{a}_n \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&+\int_{E^-}\hat{f}(x)\sum_{n\in\mathbb{Z}_m^+}K\left(x^{\beta_1}n^{\beta_2}\right)\hat{a}_n\mathrm{d}x\\
&+\int_{E^-}\hat{f}(x)\sum_{n\in\mathbb{Z}_m^-}K\left(x^{\beta_1}n^{\beta_2}\right)\hat{a}_n\mathrm{d}x\\
=&\,2\int_{E^+}\hat{f}(x)\sum_{n\in\mathbb{Z}_m^+}K\left(x^{\beta_1}n^{\beta_2}\right)\hat{a}_n\mathrm{d}x\\
&+2\int_{E^+}\hat{f}(x)\sum_{n\in\mathbb{Z}_m^+}K\left(-x^{\beta_1}n^{\beta_2}\right)\hat{a}_n\mathrm{d}x\\
=&\,2\int_{E^+}\hat{f}(x)\sum_{n\in\mathbb{Z}_m^+}\left[K\left(x^{\beta_1}n^{\beta_2}\right)+K\left(-x^{\beta_1}n^{\beta_2}\right)\right]\hat{a}_n\mathrm{d}x.
\end{aligned}
$$

By Lemma 2.1, it is obvious that $K\left(x^{\beta_1}n^{\beta_2}\right)+K\left(-x^{\beta_1}n^{\beta_2}\right)$ decreases monotonically with n $(n\in\mathbb{Z}_m^+)$ for a fixed x $(x\in E^+)$. Additionally, we also have that $\hat{a}_n=n^{\beta_2\alpha-1-\frac{2\beta_2}{qs}}$ decreases monotonically with n $(n\in\mathbb{Z}_m^+)$. Therefore, setting $x^{\beta_1}y^{\beta_2}=z$, and using Lemma 2.2, we have

$$
\begin{aligned}
&\int_S\hat{f}(x)\sum_{n\in\mathbb{Z}_m}K\left(x^{\beta_1}n^{\beta_2}\right)\hat{a}_n\mathrm{d}x\\
&>2\int_{E^+}x^{\beta_1\alpha-1+\frac{2\beta_1}{ps}}\int_m^{\infty}\left[K\left(x^{\beta_1}y^{\beta_2}\right)+K\left(-x^{\beta_1}y^{\beta_2}\right)\right]y^{\beta_2\alpha-1-\frac{2\beta_2}{qs}}\mathrm{d}y\mathrm{d}x\\
&=\frac{2}{\beta_2}\int_{E^+}x^{-1+\frac{2\beta_1}{s}}\int_{x^{\beta_1}m^{\beta_2}}^{\infty}\left[K(z)+K(-z)\right]z^{\alpha-1-\frac{2}{qs}}\mathrm{d}z\mathrm{d}x\\
&=\frac{2}{\beta_2}\int_{E^+}x^{-1+\frac{2\beta_1}{s}}\int_1^{\infty}\left[K(z)+K(-z)\right]z^{\alpha-1-\frac{2}{qs}}\mathrm{d}z\mathrm{d}x\\
&\quad+\frac{2}{\beta_2}\int_{E^+}x^{-1+\frac{2\beta_1}{s}}\int_{x^{\beta_1}m^{\beta_2}}^{1}\left[K(z)+K(-z)\right]z^{\alpha-1-\frac{2}{qs}}\mathrm{d}z\mathrm{d}x.
\end{aligned}
\tag{3.5}
$$

Consider the case that $\beta_1\in\Omega^+$. Then

$$
\int_{E^+}x^{-1+\frac{2\beta_1}{s}}\mathrm{d}x=\int_0^1x^{-1+\frac{2\beta_1}{s}}\mathrm{d}x=\frac{s}{2|\beta_1|}. \tag{3.6}
$$

Furthermore, by Fubini's theorem, we have

$$
\begin{aligned}
&\int_{E^+}x^{-1+\frac{2\beta_1}{s}}\int_{x^{\beta_1}m^{\beta_2}}^{1}\left[K(z)+K(-z)\right]z^{\alpha-1-\frac{2}{qs}}\mathrm{d}z\mathrm{d}x\\
&=\int_0^1x^{-1+\frac{2\beta_1}{s}}\int_{x^{\beta_1}m^{\beta_2}}^{1}\left[K(z)+K(-z)\right]z^{\alpha-1-\frac{2}{qs}}\mathrm{d}z\mathrm{d}x\\
&=\int_0^1\left[K(z)+K(-z)\right]z^{\alpha-1-\frac{2}{qs}}\int_0^{z^{1/\beta_1}m^{-\beta_2/\beta_1}}x^{-1+\frac{2\beta_1}{s}}\mathrm{d}x\mathrm{d}z\\
&=\frac{s}{2|\beta_1|}m^{-\frac{2\beta_1}{s}}\int_0^1\left[K(z)+K(-z)\right]z^{\alpha-1+\frac{2}{ps}}\mathrm{d}z.
\end{aligned}
\tag{3.7}
$$

If $\beta_1 \in \Omega^-$, it can also be proved that (3.6) and (3.7) hold true. Therefore, inserting (3.6) and (3.7) back into (3.5), we have

$$\begin{aligned}
&\int_S \hat{f}(x) \sum_{n\in\mathbb{Z}_m} K\left(x^{\beta_1} n^{\beta_2}\right) \hat{a}_n \mathrm{d}x \\
&> \frac{s}{|\beta_1\beta_2|}\left[\int_1^{\infty} [K(z)+K(-z)]\, z^{\alpha-1-\frac{2}{qs}}\mathrm{d}z\mathrm{d}x \right.\\
&\left. + m^{-\frac{2\beta_1}{s}} \int_0^1 [K(z)+K(-z)]\, z^{\alpha-1+\frac{2}{ps}}\mathrm{d}z\mathrm{d}x\right]. \qquad (3.8)
\end{aligned}$$

Assume that there exists a constant C satisfying

$$0 < C \leqslant |\beta_1|^{-\frac{1}{q}} \beta_2^{-\frac{1}{p}} C(\alpha,\beta,\gamma) \qquad (3.9)$$

so that (3.1) holds true if we replace the constant factor $|\beta_1|^{-\frac{1}{q}} \beta_2^{-\frac{1}{p}} C(\alpha,\beta,\gamma)$ with C, that is,

$$\begin{aligned}
\sum_{n\in\mathbb{Z}_m} a_n \int_S K\left(x^{\beta_1} n^{\beta_2}\right) f(x)\mathrm{d}x &= \int_S f(x) \sum_{n\in\mathbb{Z}_m} K\left(x^{\beta_1} n^{\beta_2}\right) a_n \mathrm{d}x \\
&< C\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}. \qquad (3.10)
\end{aligned}$$

Let $a_n = \hat{a}_n$ and $f(x) = \hat{f}(x)$ in (3.10), then we have

$$\begin{aligned}
&\int_S \hat{f}(x) \sum_{n\in\mathbb{Z}_m} K\left(x^{\beta_1} n^{\beta_2}\right) \hat{a}_n \mathrm{d}x < C\|\hat{f}\|_{p,\mu}\|\hat{\boldsymbol{a}}\|_{q,\nu} \\
&= C\left[\int_S |x|^{\frac{2\beta_1}{s}-1}\mathrm{d}x\right]^{\frac{1}{p}} \left[\sum_{n\in\mathbb{Z}_m} |n|^{\frac{-2\beta_2}{s}-1}\right]^{\frac{1}{q}} \\
&= C\left[2\int_{S\cap\mathbb{R}^+} x^{\frac{2\beta_1}{s}-1}\mathrm{d}x\right]^{\frac{1}{p}} \left[2m^{\frac{-2\beta_2}{s}-1} + 2\sum_{n=m+1}^{\infty} n^{\frac{-2\beta_2}{s}-1}\right]^{\frac{1}{q}} \\
&< 2C\left[\int_{S\cap\mathbb{R}^+} x^{\frac{2\beta_1}{s}-1}\mathrm{d}x\right]^{\frac{1}{p}} \left[m^{\frac{-2\beta_2}{s}-1} + \int_m^{\infty} x^{\frac{-2\beta_2}{s}-1}\mathrm{d}x\right]^{\frac{1}{q}} \\
&= 2C\left[\frac{s}{2|\beta_1|}\right]^{\frac{1}{p}} \left[m^{\frac{-2\beta_2}{s}-1} + \frac{s}{2\beta_2} m^{\frac{-2\beta_2}{s}}\right]^{\frac{1}{q}}. \qquad (3.11)
\end{aligned}$$

Combine (3.8) and (3.11), then we have

$$\begin{aligned}
&\int_1^{\infty} [K(z)+K(-z)]\, z^{\alpha-1-\frac{2}{qs}}\mathrm{d}z \\
&+ m^{-\frac{2\beta_1}{s}} \int_0^1 [K(z)+K(-z)]\, z^{\alpha-1+\frac{2}{ps}}\mathrm{d}z \\
&< C|\beta_1|^{\frac{1}{q}} \beta_2^{\frac{1}{p}} \left[\frac{2\beta_2 m^{\frac{-2\beta_2}{s}-1}}{s} + m^{\frac{-2\beta_2}{s}}\right]^{\frac{1}{q}}. \qquad (3.12)
\end{aligned}$$

Applying Fatou's lemma to (3.12), and using Lemma 2.2, it follows that

$$\begin{aligned}C(\alpha,\beta,\gamma) &= \int_0^{\infty}[K(z)+K(-z)]z^{\alpha-1}\mathrm{d}z \\ &= \int_0^1 \lim_{s\to\infty} m^{-\frac{2\beta_1}{s}}\left[K(z)+K(-z)\right]z^{\alpha-1+\frac{2}{ps}}\mathrm{d}z \\ &\quad + \int_1^{\infty}\lim_{s\to\infty}\left[K(z)+K(-z)\right]z^{\alpha-1-\frac{2}{qs}}\mathrm{d}z \\ &\leqslant \varliminf_{s\to\infty}\left\{\int_0^1 m^{-\frac{2\beta_1}{s}}\left[K(z)+K(-z)\right]z^{\alpha-1+\frac{2}{ps}}\mathrm{d}z \right. \\ &\quad \left. + \int_1^{\infty}\left[K(z)+K(-z)\right]z^{\alpha-1-\frac{2}{qs}}\mathrm{d}z\right\} \\ &\leqslant \varliminf_{s\to\infty}\left[C|\beta_1|^{\frac{1}{q}}\beta_2^{\frac{1}{p}}\left(\frac{2\beta_2 m^{\frac{-2\beta_2}{s}-1}}{s}+m^{\frac{-2\beta_2}{s}}\right)^{\frac{1}{q}}\right] = C|\beta_1|^{\frac{1}{q}}\beta_2^{\frac{1}{p}}.\end{aligned}$$

It implies that

$$C \geqslant |\beta_1|^{-\frac{1}{q}}\beta_2^{-\frac{1}{p}}C(\alpha,\beta,\gamma). \tag{3.13}$$

Combining (3.9) and (3.13), we have

$$C = |\beta_1|^{-\frac{1}{q}}\beta_2^{-\frac{1}{p}}C(\alpha,\beta,\gamma).$$

It follows therefore that the constant factor $|\beta_1|^{-\frac{1}{q}}\beta_2^{-\frac{1}{p}}C(\alpha,\beta,\gamma)$ in inequality (3.1) is optimal. Theorem 3.1 is proved. □

By Theorem 3.1, we can derive the following half-discrete Hardy-type inequalities on symmetric sets.

THEOREM 3.2. *Under the conditions of Theorem 3.1, the following two Hardy-type inequalities hold:*

$$\sum_{n\in\mathbb{Z}_m}|n|^{p\alpha\beta_2-1}\left[\int_S K\left(x^{\beta_1}n^{\beta_2}\right)f(x)\mathrm{d}x\right]^p < \left[|\beta_1|^{-\frac{1}{q}}\beta_2^{-\frac{1}{p}}C(\alpha,\beta,\gamma)\right]^p\|f\|_{p,\mu}^p, \tag{3.14}$$

$$\int_S |x|^{q\alpha\beta_1-1}\left[\sum_{n\in\mathbb{Z}_m}K\left(x^{\beta_1}n^{\beta_2}\right)a_n\right]^q\mathrm{d}x < \left[|\beta_1|^{-\frac{1}{q}}\beta_2^{-\frac{1}{p}}C(\alpha,\beta,\gamma)\right]^q\|\boldsymbol{a}\|_{q,\nu}^q, \tag{3.15}$$

where the constant factors $\left[|\beta_1|^{-\frac{1}{q}}\beta_2^{-\frac{1}{p}}C(\alpha,\beta,\gamma)\right]^p$ *and* $\left[|\beta_1|^{-\frac{1}{q}}\beta_2^{-\frac{1}{p}}C(\alpha,\beta,\gamma)\right]^q$ *are optimal.*

Proof. Let $\boldsymbol{y} = \{y_n\}_{n\in\mathbb{Z}_m}$, where

$$y_n := |n|^{p\alpha\beta_2-1}\left[\int_S K\left(x^{\beta_1}n^{\beta_2}\right)f(x)\mathrm{d}x\right]^{p-1}.$$

By virtue of Theorem 3.1, we have

$$\begin{aligned}&\sum_{n\in\mathbb{Z}_m}|n|^{p\alpha\beta_2-1}\left[\int_S K\left(x^{\beta_1}n^{\beta_2}\right)f(x)\mathrm{d}x\right]^p\\&=\sum_{n\in\mathbb{Z}_m}y_n\int_S K\left(x^{\beta_1}n^{\beta_2}\right)f(x)\mathrm{d}x\\&<|\beta_1|^{-\frac{1}{q}}\beta_2^{-\frac{1}{p}}C(\alpha,\beta,\gamma)\|f\|_{p,\mu}\|\boldsymbol{y}\|_{q,\nu}.\end{aligned}\tag{3.16}$$

It can be easy to show that

$$\|\boldsymbol{y}\|_{q,\nu}^q=\sum_{n\in\mathbb{Z}_m}|n|^{p\alpha\beta_2-1}\left[\int_S K\left(x^{\beta_1}n^{\beta_2}\right)f(x)\mathrm{d}x\right]^p.\tag{3.17}$$

Combining (3.16) and (3.17), inequality (3.14) holds true obviously. Furthermore, set

$$J(x) := |x|^{q\alpha\beta_1-1}\left[\sum_{n\in\mathbb{Z}_m}K\left(x^{\beta_1}n^{\beta_2}\right)a_n\right]^{q-1},$$

and employ Theorem 3.1, then we have

$$\begin{aligned}&\int_S|x|^{q\alpha\beta_1-1}\left[\sum_{n\in\mathbb{Z}^0}K\left(x^{\beta_1}n^{\beta_2}\right)a_n\right]^q\mathrm{d}x\\&=\int_S J(x)\sum_{n\in\mathbb{Z}_m}K\left(x^{\beta_1}n^{\beta_2}\right)a_n\mathrm{d}x\\&<|\beta_1|^{-\frac{1}{q}}\beta_2^{-\frac{1}{p}}C(\alpha,\beta,\gamma)\|J\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}.\end{aligned}\tag{3.18}$$

Inserting identity

$$\|J\|_{p,\mu}=\left[\int_S|x|^{q\alpha\beta_1-1}\left(\sum_{n\in\mathbb{Z}_m}K\left(x^{\beta_1}n^{\beta_2}\right)a_n\right)^q\mathrm{d}x\right]^{1/p}$$

back into (3.18), we arrive at (3.15). Theorem 3.2 is proved. □

4. Corollaries

In this section, we will present some special cases of Theorem 3.1.

Let $\gamma = 0$, $\beta_1 = \beta_2 = 1$ in Theorem 3.1. Then $a \geqslant 1$ and $S = (-a,0) \cup (0,a)$. Write

$$C_0(\alpha,\beta) = \sum_{j=0}^{\infty}\left[\frac{1}{(2\beta j+\alpha)^2} + \frac{1}{(2\beta j-\alpha+\beta)^2}\right].$$

Then Theorem 3.1 is transformed into the following Hilbert-type inequality with a non-homogeneous kernel.

COROLLARY 4.1. *Let $\beta \in \Omega^+$, and $0 < \alpha < \min\{1,\beta\}$. Let $a \geqslant 1$ and $S = (-a,0) \cup (0,a)$. Suppose that $\mu(x) = |x|^{p(1-\alpha)-1}$ and $\nu_n = |n|^{q(1-\alpha)-1}$, where $n \in \mathbb{Z}_m$. Let $f(x)$, $a_n \geqslant 0$ with $f(x) \in L_{p,\mu}(S)$ and $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{Z}_m} \in l_{q,\nu}$. Then*

$$\int_S f(x) \sum_{n\in\mathbb{Z}_m} \frac{|\log|xn||}{|1 \pm x^\beta n^\beta|} a_n \mathrm{d}x < C_0(\alpha,\beta)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.1}$$

where the constant factor $C_0(\alpha,\beta)$ is optimal.

Set $\alpha = \frac{1}{2}\beta$ in Corollary 4.1, then $\beta < 2$ $(\beta \in \Omega^+)$. By Lemma 2.3, we have

$$\int_S f(x) \sum_{n\in\mathbb{Z}_m} \frac{|\log|xn||}{|1 + x^\beta n^\beta|} a_n \mathrm{d}x < \left(\frac{\pi^2}{2\beta^2} + \frac{4\gamma_0}{\beta^2}\right)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.2}$$

where $\mu(x) = |x|^{p(1-\beta/2)-1}$, $\nu_n = |n|^{q(1-\beta/2)-1}$, and $\gamma_0 = 0.915965\cdots$ is the Catalan constant.

Let $\gamma = 0$, $\beta_1 = -1$, $\beta_2 = 1$ in Theorem 3.1. Then $0 \leqslant a \leqslant 1$ and $S = (-\infty,-a) \cup (a,\infty)$. Additionally, replace $f(x)|x|^\beta$ with $f(x)$, then Theorem 3.1 is transformed into the following Hilbert-type inequality with a homogeneous kernel.

COROLLARY 4.2. *Let $\beta \in \Omega^+$, and $0 < \alpha < \min\{1,\beta\}$. Let $0 \leqslant a \leqslant 1$ and $S = (-\infty,-a) \cup (a,\infty)$. Suppose that $\mu(x) = |x|^{p(1+\alpha-\beta)-1}$ and $\nu_n = |n|^{q(1-\alpha)-1}$, where $n \in \mathbb{Z}_m$. Let $f(x)$, $a_n \geqslant 0$ with $f(x) \in L_{p,\mu}(S)$ and $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{Z}_m} \in l_{q,\nu}$. Then*

$$\int_S f(x) \sum_{n\in\mathbb{Z}_m} \frac{|\log|\frac{n}{x}||}{|x^\beta \pm n^\beta|} a_n \mathrm{d}x < C_0(\alpha,\beta)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.3}$$

where the constant factor $C_0(\alpha,\beta)$ is optimal.

Let $\gamma = \beta$, $\beta_1 = \beta_2 = 1$ in Theorem 3.1. Then $a \geqslant 1$ and $S = (-a,0) \cup (0,a)$. Additionally, it follows from Lemma 2.4 that

$$C(\alpha,\beta,\gamma) = \sum_{j=0}^{\infty}\left[\frac{1}{(2\beta j+\alpha)^2} + \frac{1}{(2\beta j-\alpha+2\beta)^2}\right] = \frac{\pi^2}{4\beta^2}\csc^2\left(\frac{\alpha\pi}{2\beta}\right).$$

Then Theorem 3.1 is transformed into the following Hilbert-type inequality with a non-homogeneous kernel.

COROLLARY 4.3. *Let $\beta \in \Omega^+$, and $0<\alpha<\min\{1,2\beta\}$. Let $a \geqslant 1$ and $S=(-a,0)\cup(0,a)$. Suppose that $\mu(x)=|x|^{p(1-\alpha)-1}$ and $\nu_n=|n|^{q(1-\alpha)-1}$, where $n\in\mathbb{Z}_m$. Let $f(x)$, $a_n \geqslant 0$ with $f(x)\in L_{p,\mu}(S)$ and $\boldsymbol{a}=\{a_n\}_{n\in\mathbb{Z}_m}\in l_{q,\nu}$. Then*

$$\int_S f(x)\sum_{n\in\mathbb{Z}_m}\frac{|\log|xn||}{|1\pm x^\beta n^\beta|\max\{1,|xn|^\beta\}}a_n\mathrm{d}x<\frac{\pi^2}{4\beta^2}\csc^2\left(\frac{\alpha\pi}{2\beta}\right)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \quad (4.4)$$

where the constant factor $\frac{\pi^2}{4\beta^2}\csc^2\left(\frac{\alpha\pi}{2\beta}\right)$ is optimal.

Set $\alpha=\frac{1}{3}\beta$ in Corollary 4.3, then $\beta<3$ $(\beta\in\Omega^+)$. Therefore, it follows from (4.4) that

$$\int_S f(x)\sum_{n\in\mathbb{Z}_m}\frac{|\log|xn||}{|1+x^\beta n^\beta|\max\{1,|xn|^\beta\}}a_n\mathrm{d}x<\frac{\pi^2}{\beta^2}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \quad (4.5)$$

where $\mu(x)=|x|^{p(1-\beta/3)-1}$, $\nu_n=|n|^{q(1-\beta/3)-1}$.

Set $\alpha=\beta$ in Corollary 4.3, then $\beta<1$ $(\beta\in\Omega^+)$. It follows that

$$\int_S f(x)\sum_{n\in\mathbb{Z}_m}\frac{|\log|xn||}{|1+x^\beta n^\beta|\max\{1,|xn|^\beta\}}a_n\mathrm{d}x<\frac{\pi^2}{4\beta^2}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \quad (4.6)$$

where $\mu(x)=|x|^{p(1-\beta)-1}$, $\nu_n=|n|^{q(1-\beta)-1}$.

Let $\gamma=\beta$, $\beta_1=-1$, $\beta_2=1$ in Theorem 3.1. Then $0\leqslant a\leqslant 1$ and $S=(-\infty,-a)\cup(a,\infty)$. Additionally, replace $f(x)|x|^{2\beta}$ with $f(x)$, then Theorem 3.1 is transformed into the following corollary.

COROLLARY 4.4. *Let $\beta\in\Omega^+$, and $0<\alpha<\min\{1,2\beta\}$. Let $0\leqslant a\leqslant 1$ and $S=(-\infty,-a)\cup(a,\infty)$. Suppose that $\mu(x)=|x|^{p(1+\alpha-2\beta)-1}$ and $\nu_n=|n|^{q(1-\alpha)-1}$, where $n\in\mathbb{Z}_m$. Let $f(x)$, $a_n\geqslant 0$ with $f(x)\in L_{p,\mu}(S)$ and $\boldsymbol{a}=\{a_n\}_{n\in\mathbb{Z}_m}\in l_{q,\nu}$. Then*

$$\int_S f(x)\sum_{n\in\mathbb{Z}_m}\frac{|\log|\frac{n}{x}||}{|x^\beta\pm n^\beta|\max\{|x|^\beta,|n|^\beta\}}a_n\mathrm{d}x<\frac{\pi^2}{4\beta^2}\csc^2\left(\frac{\alpha\pi}{2\beta}\right)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \quad (4.7)$$

where the constant factor $\frac{\pi^2}{4\beta^2}\csc^2\left(\frac{\alpha\pi}{2\beta}\right)$ is optimal.

Let $\gamma=3\beta$, $\beta_1=\beta_2=1$ in Theorem 3.1. Then $a\geqslant 1$ and $S=(-a,0)\cup(0,a)$. Additionally, by Lemma 2.5, we have

$$\begin{aligned} C_1(\alpha,\beta) &:= \sum_{j=0}^{\infty}\left[\frac{1}{(2\beta j+\alpha)^2}+\frac{1}{(2\beta j-\alpha+4\beta)^2}\right] \\ &= \begin{cases} \frac{\pi^2}{4\beta^2}\csc^2\left(\frac{\alpha\pi}{2\beta}\right)-\frac{1}{(2\beta-\alpha)^2}, & \alpha\neq 2\beta, \\ \frac{\pi^2}{12\beta^2}, & \alpha=2\beta. \end{cases} \end{aligned}$$

Hence, Theorem 3.1 reduces to the following corollary.

COROLLARY 4.5. *Let $\beta \in \Omega^+$, and $0 < \alpha < \min\{1, 4\beta\}$. Let $a \geqslant 1$ and $S = (-a, 0) \cup (0, a)$. Suppose that $\mu(x) = |x|^{p(1-\alpha)-1}$ and $\nu_n = |n|^{q(1-\alpha)-1}$, where $n \in \mathbb{Z}_m$. Let $f(x)$, $a_n \geqslant 0$ with $f(x) \in L_{p,\mu}(S)$ and $\boldsymbol{a} = \{a_n\}_{n \in \mathbb{Z}_m} \in l_{q,\nu}$. Then*

$$\int_S f(x) \sum_{n \in \mathbb{Z}_m} \frac{|\log|xn||}{|1 \pm x^\beta n^\beta| \max\{1, |xn|^{3\beta}\}} a_n \mathrm{d}x < C_1(\alpha, \beta) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}, \tag{4.8}$$

where the constant factor $C_1(\alpha, \beta)$ is optimal.

Set $\alpha = \beta$ in Corollary 4.5, then $\beta < 1$ $(\beta \in \Omega^+)$, and it follows from (4.8) that

$$\int_S f(x) \sum_{n \in \mathbb{Z}_m} \frac{|\log|xn||}{|1 + x^\beta n^\beta| \max\{1, |xn|^{3\beta}\}} a_n \mathrm{d}x < \frac{\pi^2 - 4}{4\beta^2} \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}, \tag{4.9}$$

where $\mu(x) = |x|^{p(1-\beta)-1}$, $\nu_n = |n|^{q(1-\beta)-1}$.

Set $\alpha = 2\beta$ in Corollary 4.5, then $\beta < \frac{1}{2}$ $(\beta \in \Omega^+)$, and it follows from (4.8) that

$$\int_S f(x) \sum_{n \in \mathbb{Z}_m} \frac{|\log|xn||}{|1 + x^\beta n^\beta| \max\{1, |xn|^{3\beta}\}} a_n \mathrm{d}x < \frac{\pi^2}{12\beta^2} \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}, \tag{4.10}$$

where $\mu(x) = |x|^{p(1-2\beta)-1}$, $\nu_n = |n|^{q(1-2\beta)-1}$.

Let $\gamma = 3\beta$, $\beta_1 = -1$, $\beta_2 = 1$ in Theorem 3.1. Then $0 \leqslant a \leqslant 1$ and $S = (-\infty, -a) \cup (a, \infty)$. Replace $f(x)|x|^{4\beta}$ with $f(x)$, then Theorem 3.1 reduces to Corollary 4.6.

COROLLARY 4.6. *Let $\beta \in \Omega^+$, and $0 < \alpha < \min\{1, 4\beta\}$. Let $0 \leqslant a \leqslant 1$ and $S = (-\infty, -a) \cup (a, \infty)$. Suppose that $\mu(x) = |x|^{p(1+\alpha-4\beta)-1}$ and $\nu_n = |n|^{q(1-\alpha)-1}$, where $n \in \mathbb{Z}_m$. Let $f(x)$, $a_n \geqslant 0$ with $f(x) \in L_{p,\mu}(S)$ and $\boldsymbol{a} = \{a_n\}_{n \in \mathbb{Z}_m} \in l_{q,\nu}$. Then*

$$\int_S f(x) \sum_{n \in \mathbb{Z}_m} \frac{|\log|\frac{n}{x}||}{|x^\beta \pm n^\beta| \max\{|x|^{3\beta}, |n|^{3\beta}\}} a_n \mathrm{d}x < C_1(\alpha, \beta) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}, \tag{4.11}$$

where the constant factor $C_1(\alpha, \beta)$ is optimal.

Set $\alpha = \frac{1}{3}\beta$ in Corollary 4.6, then $\beta < 3$ $(\beta \in \Omega^+)$, and it follows from (4.11) that

$$\int_S f(x) \sum_{n \in \mathbb{Z}_m} \frac{|\log|\frac{n}{x}||}{|x^\beta + n^\beta| \max\{|x|^{3\beta}, |n|^{3\beta}\}} a_n \mathrm{d}x < \frac{25\pi^2 - 9}{25\beta^2} \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu},$$

where $\mu(x) = |x|^{p(1-11\beta/3)-1}$, $\nu_n = |n|^{q(1-\beta/3)-1}$.

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