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INEQUALITIES ON THE ESSENTIAL JOINT AND ESSENTIAL GENERALIZED SPECTRAL RADIUS

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Abstract. We prove new inequalities for the essential generalized and the essential joint spectral radius of Hadamard (Schur) weighted geometric means of bounded sets of infinite nonnegative matrices that define operators on suitable Banach sequence spaces and of bounded sets of positive kernel operators on L^2. To our knowledge the obtained inequalities are new even in the case of singelton sets.

1. Introduction

In [45], X. Zhan conjectured that, for non-negative $N \times N$ matrices A and B, the spectral radius $\rho(A \circ B)$ of the Hadamard product satisfies

$$\rho(A \circ B) \leqslant \rho(AB), \tag{1}$$

where AB denotes the usual matrix product of A and B. This conjecture was confirmed by K.M.R. Audenaert in [3] by proving

$$\rho(A \circ B) \leqslant \rho((A \circ A)(B \circ B))^{\frac{1}{2}} \leqslant \rho(AB). \tag{2}$$

These inequalities were established via a trace description of the spectral radius. Soon after, inequality (1) was reproved, generalized and refined in different ways by several authors ([5–8, 14, 19, 20, 31–33, 36, 37]). Using the fact that the Hadamard product is a principal submatrix of the Kronecker product, R. A. Horn and F. Zhang proved in [19] the inequalities

$$\rho(A \circ B) \leqslant \rho(AB \circ BA)^{\frac{1}{2}} \leqslant \rho(AB). \tag{3}$$

Applying the techniques of [19], Z. Huang proved that

$$\rho(A_1 \circ A_2 \circ \cdots \circ A_m) \leqslant \rho(A_1 A_2 \cdots A_m) \tag{4}$$

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for $n \times n$ non-negative matrices $A_1, A_2, \cdots, A_m$ (see [20]). A. R. Schep was the first one to observe that the results from [12] and [29] are applicable in this context (see [36] and [37]). He extended inequalities (2) and (3) to non-negative matrices that define bounded operators on sequence spaces (in particular on l^p spaces, $1 \leqslant p < \infty$) and proved in [36, Theorem 2.7] that

$$\rho(A \circ B) \leqslant \rho((A \circ A)(B \circ B))^{\frac{1}{2}} \leqslant \rho(AB \circ AB)^{\frac{1}{2}} \leqslant \rho(AB) \tag{5}$$

(note that there was an error in the statement of [36, Theorem 2.7], which was corrected in [37] and [31]). In [31], the second author of the current paper extended the inequality (4) to non-negative matrices that define bounded operators on Banach sequence spaces (see below for the exact definitions) and proved that the inequalities

$$\rho(A \circ B) \leqslant \rho((A \circ A)(B \circ B))^{\frac{1}{2}} \leqslant \rho(AB \circ AB)^{\frac{\beta}{2}} \rho(BA \circ BA)^{\frac{1-\beta}{2}} \leqslant \rho(AB) \tag{6}$$

and

$$\rho(A \circ B) \leqslant \rho(AB \circ BA)^{\frac{1}{2}} \leqslant \rho(AB \circ AB)^{\frac{1}{4}} \rho(BA \circ BA)^{\frac{1}{4}} \leqslant \rho(AB) \tag{7}$$

hold, where $\beta \in [0,1]$. Moreover, he generalized these inequalities to the setting of the generalized and the joint spectral radius of bounded sets of such non-negative matrices.

In [36, Theorem 2.8], A. R. Schep proved that the inequality

$$\rho\left(A^{\left(\frac{1}{2}\right)} \circ B^{\left(\frac{1}{2}\right)}\right) \leqslant \rho(AB)^{\frac{1}{2}} \tag{8}$$

holds for positive kernel operators on L^p spaces. Here $A^{\left(\frac{1}{2}\right)} \circ B^{\left(\frac{1}{2}\right)}$ denotes the Hadamard geometric mean of operators A and B. R. Drnovšek and the second author (see [14, 33]), generalized this inequality and proved that the inequalities

$$\rho\left(A_1^{\left(\frac{1}{m}\right)} \circ A_2^{\left(\frac{1}{m}\right)} \circ \cdots \circ A_m^{\left(\frac{1}{m}\right)}\right)$$

$$\leqslant \rho\left(P_1^{\left(\frac{1}{m}\right)} \circ P_2^{\left(\frac{1}{m}\right)} \circ \cdots \circ P_m^{\left(\frac{1}{m}\right)}\right)^{\frac{1}{m}} \leqslant \rho(A_1 A_2 \cdots A_m)^{\frac{1}{m}} \tag{9}$$

hold for positive kernel operators $A_1, \ldots, A_m$ on an arbitrary Banach function space L, where $P_j = A_j \ldots A_m A_1 \ldots A_{j-1}$ for $j = 1, \ldots, m$. Formally, here and throughout the article $A_{j-1} = I$ for $j = 1$ (eventhough I might not be a well defined kernel operator). The second author proved further in [33, Theorem 4.4, (4.8)]) that in the L^2 case it holds

$$\|A^{(\frac{1}{2})} \circ B^{(\frac{1}{2})}\| \leqslant \rho\left((A^*B)^{(\frac{1}{2})} \circ (B^*A)^{(\frac{1}{2})}\right)^{\frac{1}{2}} \leqslant \rho(A^*B)^{\frac{1}{2}} = \rho(AB^*)^{\frac{1}{2}}, \tag{10}$$

where $\|\cdot\|$ denotes the operator norm. In [34, Theorem 3.2], the second author showed that (9) (and thus also (8)) holds also for the essential radius ρ_{ess} under the additional condition that L and its Banach dual L^* have order continuous norms. Several additional closely related results, generalizations and refinements of the above results were obtained in [5–7, 34, 35, 46]. However, it remained unclear whether the analogues of inequalities (1)–(7) are valid for the essential spectral radius of infinite nonnegative matrices that define operators on e.g. l^2 and whether an analogue of (10) is valid for a suitable measure of non-compactness and for the essential spectral radius of positive kernel operators on L^2. In this paper (as a very special case of our results) we positively answer these questions (see Corollary 2, Theorem 13 and Corollary 5 below).

The rest of the article is organized in the following way. In Section 2 we recall definitions and results that we will use in our proofs. In Section 3 we prove the key results (Theorems 5, 6 and 7) on the Haussdorf measure of noncompactness and the essential spectral radius of ordinary products of Hadamard powers and ordinary products of Hadamard weighted geometric means of infinite nonnegative matrices that define operators on suitable Banach sequence spaces. These results are essential analogues of the known results for the operator norm and the spectral radius. By combining ideas of proofs from previously known results we prove in Theorem 8 an extension of these results to the essential joint and essential generalized spectral radius of bounded sets of infinite nonnegative matrices. In Section 4 we apply these results to obtain several essential analogues of known results on sums of Hadamard weighted geometric means, weighted geometric symmetrizations and Hadamard products of bounded sets of infinite nonnegative matrices (Theorems 9, 11, 13 and Corollary 1). In Corollary 2 we obtain the essential versions of (6) and (7), while the essential version of (4) is a very special case of Theorem 13. In Section 5 we prove new essential results for operators on Hilbert spaces. In Corollary 5 we prove the essential version of (10). We conclude the article by obtaining essential versions of several recent results from [5].

2. Preliminaries

Let μ be a σ-finite positive measure on a σ-algebra $\mathscr{M}$ of subsets of a non-void set X. Let $M(X,\mu)$ be the vector space of all equivalence classes of (almost everywhere equal) complex measurable functions on X. A Banach space $L \subseteq M(X,\mu)$ is called a *Banach function space* if $f \in L$, $g \in M(X,\mu)$, and $|g| \leqslant |f|$ imply that $g \in L$ and $\|g\| \leqslant \|f\|$. Throughout the article, it is assumed that X is the carrier of L, that is, there is no subset Y of X of strictly positive measure with the property that $f = 0$ a.e. on Y for all $f \in L$ (see [44]).

Let R denote the set $\{1,\dots,N\}$ for some $N \in \mathbb{N}$ or the set $\mathbb{N}$ of all natural numbers. Let $S(R)$ be the vector lattice of all complex sequences $(x_n)_{n\in R}$. A Banach space $L \subseteq S(R)$ is called a *Banach sequence space* if $x \in S(R)$, $y \in L$ and $|x| \leqslant |y|$ imply that $x \in L$ and $\|x\|_L \leqslant \|y\|_L$. Observe that a Banach sequence space is a Banach function space over a measure space (R,μ), where μ denotes the counting measure on R. Denote by $\mathscr{L}$ the collection of all Banach sequence spaces L satisfying the property that $e_n = \chi_{\{n\}} \in L$ and $\|e_n\|_L = 1$ for all $n \in R$. For $L \in \mathscr{L}$ the set R is the carrier of L.

Standard examples of Banach sequence spaces are Euclidean spaces, l^p spaces for $1 \leqslant p \leqslant \infty$, the space $c_0 \in \mathscr{L}$ of all null convergent sequences (equipped with the usual norms and the counting measure), while standard examples of Banach function spaces are the well-known spaces $L^p(X,\mu)$ $(1 \leqslant p \leqslant \infty)$ and other less known examples such as Orlicz, Lorentz, Marcinkiewicz and more general rearrangement-invariant spaces (see e.g. [4, 9, 22] and the references cited there), which are important e.g. in interpolation theory and in the theory of partial differential equations. Recall that the cartesian product $L = E \times F$ of Banach function spaces is again a Banach function space, equipped with the norm $\|(f,g)\|_L = \max\{\|f\|_E, \|g\|_F\}$.

If $\{f_n\}_{n\in\mathbb{N}} \subset M(X,\mu)$ is a decreasing real valued sequence and $f = \inf\{f_n \in M(X,\mu) : n \in \mathbb{N}\}$, then we write $f_n \downarrow f$. A Banach function space L has an *order continuous norm*, if $0 \leqslant f_n \downarrow 0$ implies $\|f_n\|_L \to 0$ as $n \to \infty$. It is well known that spaces $L^p(X,\mu)$, $1 \leqslant p < \infty$, have order continuous norm. Moreover, the norm of any reflexive Banach function space is order continuous. In particular, we will be interested in Banach function spaces L such that L and its Banach dual space L^* have order continuous norms. Examples of such spaces are $L^p(X,\mu)$, $1 < p < \infty$, while the space $L = c_0$ is an example of a non-reflexive Banach sequence space, such that L and $L^* = l^1$ have order continuous norms.

By an *operator* on a Banach function space L we always mean a linear operator on L. An operator A on L is said to be *positive* if it maps nonnegative functions to nonnegative ones, i.e., $AL_+ \subset L_+$, where L_+ denotes the positive cone $L_+ = \{f \in L : f \geqslant 0 \text{ a.e.}\}$. Given operators A and B on L, we write $A \geqslant B$ if the operator $A - B$ is positive.

Recall that a positive operator A is always bounded, i.e., its operator norm

$$\|A\| = \sup\{\|Ax\|_L : x \in L, \|x\|_L \leqslant 1\} = \sup\{\|Ax\|_L : x \in L_+, \|x\|_L \leqslant 1\} \tag{11}$$

is finite. Also, its spectral radius $\rho(A)$ is always contained in the spectrum.

An operator A on a Banach function space L is called a *kernel operator* if there exists a $\mu \times \mu$-measurable function $a(x,y)$ on $X \times X$ such that, for all $f \in L$ and for almost all $x \in X$,

$$\int_X |a(x,y)f(y)|\, d\mu(y) < \infty \quad \text{and} \quad (Af)(x) = \int_X a(x,y)f(y)\, d\mu(y).$$

One can check that a kernel operator A is positive iff its kernel a is non-negative almost everywhere.

Given a complex Banach space L let $B(L)$ denote the Banach algebra of bounded linear operators on L and let π be the canonical projection of $B(L)$ onto the Calkin algebra $B(L)/K(L)$, where $K(L)$ is the set of compact operators in $B(L)$. The essential spectral radius $\rho_{ess}(A)$ of $A \in B(L)$ is by definition $\rho_{ess}(A) = \rho(\pi(A))$.

Let L be a Banach function space such that L and L^* have order continuous norms and let A and B be positive kernel operators on L. By $\gamma(A)$ we denote the Hausdorff measure of non-compactness of A, i.e.,

$$\gamma(A) = \inf\{\delta > 0 : \text{ there is a finite } M \subset L \text{ such that } A(D_L) \subset M + \delta D_L\},$$

where $D_L = \{f \in L : \|f\|_L \leqslant 1\}$. Then $\gamma(A) \leqslant \|A\|$, $\gamma(A+B) \leqslant \gamma(A) + \gamma(B)$, $\gamma(AB) \leqslant \gamma(A)\gamma(B)$ and $\gamma(\alpha A) = \alpha\gamma(A)$ for $\alpha \geqslant 0$. Also $0 \leqslant A \leqslant B$ implies $\gamma(A) \leqslant \gamma(B)$ (see e.g. [23, Corollary 4.3.7 and Corollary 3.7.3]). Moreover,

$$\rho_{ess}(A) = \lim_{j\to\infty} \gamma(A^j)^{1/j} = \inf_{j\in\mathbb{N}} \gamma(A^j)^{1/j} \tag{12}$$

and $\rho_{ess}(A) \leqslant \gamma(A)$. Recall that if $L = L^2(X,\mu)$, then $\gamma(A^*) = \gamma(A)$ and $\rho_{ess}(A^*) = \rho_{ess}(A)$, where A^* denotes the adjoint of A (see e.g. [23, Proposition 4.3.3, Theorems 4.3.6 and 4.3.13 and Corollary 3.7.3], [27, Theorem 1]). Note that equalities (12) and $\rho_{ess}(A^*) = \rho_{ess}(A)$ are valid for any bounded operator A on a given complex Banach space L (see e.g. [23, Theorem 4.3.13 and Proposition 4.3.11], [27, Theorem 1]).

It is well-known that kernel operators play a very important, often even central, role in a variety of applications from differential and integro-differential equations, problems from physics (in particular from thermodynamics), engineering, statistical and economic models, etc (see e.g. [21, 34] and the references cited there). For the theory of Banach function spaces and more general Banach lattices we refer the reader to the books [1, 2, 4, 23, 44].

Let A and B be positive kernel operators on a Banach function space L with kernels a and b respectively, and $\alpha \geqslant 0$. The *Hadamard (or Schur) product* $A \circ B$ of A and B is the kernel operator with kernel equal to $a(x,y)b(x,y)$ at point $(x,y) \in X \times X$ which can be defined (in general) only on some order ideal of L. Similarly, the *Hadamard (or Schur) power* $A^{(\alpha)}$ of A is the kernel operator with kernel equal to $(a(x,y))^{\alpha}$ at point $(x,y) \in X \times X$ which can be defined only on some order ideal of L.

Let $A_1, \ldots, A_m$ be positive kernel operators on a Banach function space L, and $\alpha_1, \ldots, \alpha_m$ positive numbers such that $\sum_{j=1}^{m} \alpha_j = 1$. Then the *Hadamard weighted geometric mean* $A = A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \cdots \circ A_m^{(\alpha_m)}$ of the operators $A_1, \ldots, A_m$ is a positive kernel operator defined on the whole space L, since $A \leqslant \alpha_1 A_1 + \alpha_2 A_2 + \ldots + \alpha_m A_m$ by the inequality between the weighted arithmetic and geometric means.

A matrix $A = [a_{ij}]_{i,j \in R}$ is called *nonnegative* if $a_{ij} \geqslant 0$ for all $i, j \in R$. For notational convenience, we sometimes write $a(i,j)$ instead of a_{ij}.

We say that a nonnegative matrix A defines an operator on L if $Ax \in L$ for all $x \in L$, where $(Ax)_i = \sum_{j \in R} a_{ij} x_j$. Then $Ax \in L_+$ for all $x \in L_+$ and so A defines a positive kernel operator on L.

Let us recall the following result, which was proved in [12, Theorem 2.2] and [29, Theorem 5.1 and Example 3.7] (see also e.g. [32, Theorem 2.1]).

THEOREM 1. *Let $\{A_{ij}\}_{i=1,j=1}^{k,m}$ be positive kernel operators on a Banach function space L and α_1, $\alpha_2, \ldots,$ α_m positive numbers.*

(i) If $\sum_{j=1}^{m} \alpha_j = 1$, then the positive kernel operator

$$A := \left(A_{11}^{(\alpha_1)} \circ \cdots \circ A_{1m}^{(\alpha_m)}\right) \cdots \left(A_{k1}^{(\alpha_1)} \circ \cdots \circ A_{km}^{(\alpha_m)}\right) \tag{13}$$

satisfies the following inequalities

$$A \leqslant (A_{11} \cdots A_{k1})^{(\alpha_1)} \circ \cdots \circ (A_{1m} \cdots A_{km})^{(\alpha_m)}, \tag{14}$$

$$\begin{aligned} \|A\| &\leqslant \left\| (A_{11} \cdots A_{k1})^{(\alpha_1)} \circ \cdots \circ (A_{1m} \cdots A_{km})^{(\alpha_m)} \right\| \\ &\leqslant \|A_{11} \cdots A_{k1}\|^{\alpha_1} \cdots \|A_{1m} \cdots A_{km}\|^{\alpha_m}, \end{aligned} \tag{15}$$

$$\begin{aligned}\rho(A) &\leqslant \rho\left((A_{11}\cdots A_{k1})^{(\alpha_1)}\circ\cdots\circ(A_{1m}\cdots A_{km})^{(\alpha_m)}\right)\\ &\leqslant \rho(A_{11}\cdots A_{k1})^{\alpha_1}\cdots\rho(A_{1m}\cdots A_{km})^{\alpha_m}. \end{aligned}\tag{16}$$

If, in addition, L and L^* have order continuous norms, then

$$\begin{aligned}\gamma(A) &\leqslant \gamma\left((A_{11}\cdots A_{k1})^{(\alpha_1)}\circ\cdots\circ(A_{1m}\cdots A_{km})^{(\alpha_m)}\right)\\ &\leqslant \gamma(A_{11}\cdots A_{k1})^{\alpha_1}\cdots\gamma(A_{1m}\cdots A_{km})^{\alpha_m}, \end{aligned}\tag{17}$$

$$\begin{aligned}\rho_{ess}(A) &\leqslant \rho_{ess}\left((A_{11}\cdots A_{k1})^{(\alpha_1)}\circ\cdots\circ(A_{1m}\cdots A_{km})^{(\alpha_m)}\right)\\ &\leqslant \rho_{ess}(A_{11}\cdots A_{k1})^{\alpha_1}\cdots\rho_{ess}(A_{1m}\cdots A_{km})^{\alpha_m}. \end{aligned}\tag{18}$$

(ii) If $L\in\mathscr{L}$, $\sum_{j=1}^m \alpha_j \geqslant 1$ and $\{A_{ij}\}_{i=1,j=1}^{k,m}$ are nonnegative matrices that define positive operators on L, then A from (13) defines a positive operator on L and the inequalities (14), (15) and (16) hold.

The following result is a special case of Theorem 1.

THEOREM 2. *Let $A_1,\ldots,A_m$ be positive kernel operators on a Banach function space L and $\alpha_1,\ldots,\alpha_m$ positive numbers.*

(i) If $\sum_{j=1}^m \alpha_j = 1$, then

$$\|A_1^{(\alpha_1)}\circ A_2^{(\alpha_2)}\circ\cdots\circ A_m^{(\alpha_m)}\| \leqslant \|A_1\|^{\alpha_1}\|A_2\|^{\alpha_2}\cdots\|A_m\|^{\alpha_m}\tag{19}$$

and

$$\rho(A_1^{(\alpha_1)}\circ A_2^{(\alpha_2)}\circ\cdots\circ A_m^{(\alpha_m)}) \leqslant \rho(A_1)^{\alpha_1}\rho(A_2)^{\alpha_2}\cdots\rho(A_m)^{\alpha_m}.\tag{20}$$

If, in addition, L and L^* have order continuous norms, then

$$\gamma(A_1^{(\alpha_1)}\circ A_2^{(\alpha_2)}\circ\cdots\circ A_m^{(\alpha_m)}) \leqslant \gamma(A_1)^{\alpha_1}\gamma(A_2)^{\alpha_2}\cdots\gamma(A_m)^{\alpha_m}\tag{21}$$

and

$$\rho_{ess}(A_1^{(\alpha_1)}\circ A_2^{(\alpha_2)}\circ\cdots\circ A_m^{(\alpha_m)}) \leqslant \rho_{ess}(A_1)^{\alpha_1}\rho_{ess}(A_2)^{\alpha_2}\cdots\rho_{ess}(A_m)^{\alpha_m}.\tag{22}$$

(ii) If $L\in\mathscr{L}$, $\sum_{j=1}^m \alpha_j \geqslant 1$ and if $A_1,\ldots,A_m$ are nonnegative matrices that define positive operators on L, then $A_1^{(\alpha_1)}\circ A_2^{(\alpha_2)}\circ\cdots\circ A_m^{(\alpha_m)}$ defines a positive operator on L and (19) and (20) hold.

(iii) If $L\in\mathscr{L}$, $t\geqslant 1$ and if $A,A_1,\ldots,A_m$ are nonnegative matrices that define operators on L, then $A^{(t)}$ defines an operator on L and the following inequalities hold

$$A_1^{(t)}\cdots A_m^{(t)} \leqslant (A_1\cdots A_m)^{(t)},\tag{23}$$

$$\rho(A_1^{(t)}\cdots A_m^{(t)}) \leqslant \rho(A_1\cdots A_m)^t,\tag{24}$$

$$\|A_1^{(t)}\cdots A_m^{(t)}\| \leqslant \|A_1\cdots A_m\|^t.\tag{25}$$

The following result was proved in [35, Corollary 2.10].

THEOREM 3. *Given $L \in \mathscr{L}$, let A be a nonnegative matrix that defines an operator on L and let $t \geqslant 1$. Then*

$$A^{(t)} \leqslant \|A\|_\infty^{t-1} A, \tag{26}$$

$$\|A^{(t)}\| \leqslant \|A\|_\infty^{t-1} \|A\|, \tag{27}$$

$$\rho(A^{(t)}) \leqslant \|A\|_\infty^{t-1} \rho(A). \tag{28}$$

If, in addition, L and L^* have order continuous norms, then

$$\gamma(A^{(t)}) \leqslant \|A\|_\infty^{t-1} \gamma(A), \tag{29}$$

$$\rho_{ess}(A^{(t)}) \leqslant \|A\|_\infty^{t-1} \rho_{ess}(A). \tag{30}$$

Let Σ be a bounded set of bounded operators on a complex Banach space L. For $m \geqslant 1$, let

$$\Sigma^m = \{A_1 A_2 \cdots A_m : A_i \in \Sigma\}.$$

The generalized spectral radius of Σ is defined by

$$\rho(\Sigma) = \limsup_{m\to\infty} [\sup_{A\in\Sigma^m} \rho(A)]^{1/m} \tag{31}$$

and is equal to

$$\rho(\Sigma) = \sup_{m\in\mathbb{N}} [\sup_{A\in\Sigma^m} \rho(A)]^{1/m}.$$

The joint spectral radius of Σ is defined by

$$\hat{\rho}(\Sigma) = \lim_{m\to\infty} [\sup_{A\in\Sigma^m} \|A\|]^{1/m}. \tag{32}$$

Similarly, the generalized essential spectral radius of Σ is defined by

$$\rho_{ess}(\Sigma) = \limsup_{m\to\infty} [\sup_{A\in\Sigma^m} \rho_{ess}(A)]^{1/m} \tag{33}$$

and is equal to

$$\rho_{ess}(\Sigma) = \sup_{m\in\mathbb{N}} [\sup_{A\in\Sigma^m} \rho_{ess}(A)]^{1/m}.$$

The joint essential spectral radius of Σ is defined by

$$\hat{\rho}_{ess}(\Sigma) = \lim_{m\to\infty} [\sup_{A\in\Sigma^m} \gamma(A)]^{1/m}. \tag{34}$$

It is well known that $\rho(\Sigma) = \hat{\rho}(\Sigma)$ for a precompact nonempty set Σ of compact operators on L (see e.g. [24, 40, 41]), in particular for a bounded set of complex $n \times n$ matrices (see e.g. [10, 26, 39] and the references cited there). This equality is called the Berger-Wang formula or also the generalized spectral radius theorem. It is known that also the generalized Berger-Wang formula holds, i.e, that for any precompact nonempty set Σ of bounded operators on L we have

$$\hat{\rho}(\Sigma) = \max\{\rho(\Sigma), \hat{\rho}_{ess}(\Sigma)\}$$

(see e.g. [24, 40, 41]). Observe also that it was proved in [24] that in the definition of $\hat{\rho}_{ess}(\Sigma)$ one may replace the Haussdorf measure of noncompactness by several other seminorms, for instance it may be replaced by the essential norm.

In general $\rho(\Sigma)$ and $\hat{\rho}(\Sigma)$ may differ even in the case of a bounded set Σ of compact positive operators on L (see [39] or also [32]). Also, in [18] the reader can find an example of two positive non-compact weighted shifts A and B on $L = l^2$ such that $\rho(\{A,B\}) = 0 < \hat{\rho}(\{A,B\})$. As already noted in [40] also $\rho_{ess}(\Sigma)$ and $\hat{\rho}_{ess}(\Sigma)$ may in general be different.

The theory of the generalized and the joint spectral radius has many important applications for instance to discrete and differential inclusions, wavelets, invariant subspace theory (see e.g. [10, 40, 41, 43] and the references cited there). In particular, $\hat{\rho}(\Sigma)$ plays a central role in determining stability in convergence properties of discrete and differential inclusions. In this theory the quantity $\log \hat{\rho}(\Sigma)$ is known as the maximal Lyapunov exponent (see e.g. [43]).

We will use the following well known facts that hold for all $r \in \{\rho, \hat{\rho}, \rho_{ess}, \hat{\rho}_{ess}\}$:

$$r(\Sigma^m) = r(\Sigma)^m \ \text{ and } \ r(\Psi\Sigma) = r(\Sigma\Psi) \tag{35}$$

where $\Psi\Sigma = \{AB : A \in \Psi, B \in \Sigma\}$ and $m \in \mathbb{N}$.

Let $\Psi_1, \ldots, \Psi_m$ be bounded sets of positive kernel operators on a Banach function space L and let $\alpha_1, \ldots \alpha_m$ be positive numbers such that $\sum_{i=1}^m \alpha_i = 1$. Then the bounded set of positive kernel operators on L, defined by

$$\Psi_1^{(\alpha_1)} \circ \cdots \circ \Psi_m^{(\alpha_m)} = \{A_1^{(\alpha_1)} \circ \cdots \circ A_m^{(\alpha_m)} : A_1 \in \Psi_1, \ldots, A_m \in \Psi_m\},$$

is called the *weighted Hadamard (Schur) geometric mean* of sets $\Psi_1, \ldots, \Psi_m$. The set $\Psi_1^{(\frac{1}{m})} \circ \cdots \circ \Psi_m^{(\frac{1}{m})}$ is called the *Hadamard (Schur) geometric mean* of sets $\Psi_1, \ldots, \Psi_m$. If $L \in \mathscr{L}$, $\sum_{i=1}^m \alpha_i \geqslant 1$ and if $\Psi_1, \ldots, \Psi_m$ are bounded sets of nonnegative matrices that define operators on L, then the set $\Psi_1^{(\alpha_1)} \circ \cdots \circ \Psi_m^{(\alpha_m)}$ is a bounded set of nonnegative matrices that define operators on L by Theorem 2(ii). The following result that follows from Theorem 1 was established in [32, Theorem 3.3], [34, Theorems 3.1 and 3.8] and [7, Theorem 2.5].

THEOREM 4. *Let $\Psi_1, \ldots, \Psi_m$ be bounded sets of positive kernel operators on a Banach function space L, let $\alpha_1, \ldots, \alpha_m$ be positive numbers and $n \in \mathbb{N}$.*

(i) If $\sum_{i=1}^m \alpha_i = 1$ and $r \in \{\rho, \hat{\rho}\}$, then

$$r(\Psi_1^{(\alpha_1)} \circ \cdots \circ \Psi_m^{(\alpha_m)}) \leqslant r((\Psi_1^n)^{(\alpha_1)} \circ \cdots \circ (\Psi_m^n)^{(\alpha_m)})^{\frac{1}{n}} \leqslant r(\Psi_1)^{\alpha_1} \cdots r(\Psi_m)^{\alpha_m} \tag{36}$$

and

$$r\left(\Psi_1^{(\frac{1}{m})} \circ \cdots \circ \Psi_m^{(\frac{1}{m})}\right) \leqslant r(\Psi_1 \Psi_2 \cdots \Psi_m)^{\frac{1}{m}}. \tag{37}$$

If, in addition, L and L^* have order continuous norms, then (36) *and* (37) *hold also for each $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$.*

(ii) If $L \in \mathscr{L}$, $\sum_{j=1}^{m} \alpha_j \geqslant 1$, $r \in \{\rho, \hat{\rho}\}$ and if $\Psi, \Psi_1, \ldots, \Psi_m$ are bounded sets of nonnegative matrices that define operators on L, then Inequalities (36) *hold.*

In particular, if $t \geqslant 1$, then

$$r(\Psi^{(t)}) \leqslant r((\Psi^n)^{(t)})^{\frac{1}{n}} \leqslant r(\Psi)^t. \tag{38}$$

3. New inequalities for the Haussdorf measure of noncompactness and essential radius

In this section we prove that the essential versions of Theorems 1(ii), 2(ii)–(iii) and 4(ii) hold under the assumption that L and L^* have order continuous norms. We will need the following lemma.

LEMMA 1. *Let $L \in \mathscr{L}$ have order continuous norm. Then for each $x \in L$ it holds that $x(i) \to 0$ as $i \to \infty$.*

Proof. Suppose there exists $x \in L$ such that the entries $x(i)$ do not converge to zero as $i \to \infty$. Then there exists $\varepsilon > 0$ such that there are infinitely many positive entries of $|x|$ that are greater than ε. For $k \in \mathbb{N}$ let $x_k(i) = 0$ when $i \leqslant k$ and $x_k(i) = |x|(i)$ otherwise. Then $0 \leqslant x_k \downarrow 0$. However, $\|x_k\|$ does not converge to zero, since we have $\|x_k\| \geqslant \||x|(i) \cdot e_i\| = |x(i)| > \varepsilon$ for infinitely many $i > k$. $\square$

First we establish the essential version of Theorem 2(iii).

THEOREM 5. *Let $L \in \mathscr{L}$ such that L and L^* have order continuous norms. Let $t \geqslant 1$ and let $A, A_1, \ldots, A_m$ be nonnegative matrices that define operators on L. Then*

$$\gamma(A^{(t)}) \leqslant \gamma(A)^t, \tag{39}$$

$$\rho_{ess}(A^{(t)}) \leqslant \rho_{ess}(A)^t, \tag{40}$$

$$\gamma(A_1^{(t)} \cdots A_m^{(t)}) \leqslant \gamma(A_1 \cdots A_m)^t, \tag{41}$$

$$\rho_{ess}(A_1^{(t)} \cdots A_m^{(t)}) \leqslant \rho_{ess}(A_1 \cdots A_m)^t. \tag{42}$$

Proof. First we prove (39). If $\gamma(A) = 0$, then $\gamma(A^{(t)}) = 0$ by (29). We may assume that $t > 1$. We may also assume that $\gamma(A) = 1$ since $\gamma(\cdot)$ is positively homogeneous. Having $\gamma(A) = 1$ means that for any $\delta > 1$, there is a finite set $U \subset L$ such that the image $A(D_L)$ of the closed unit ball D_L is contained in the union $\bigcup_{u \in U}(u + \delta D_L)$. Since U is a finite set in L, then by Lemma 1 there are only finitely many entries i such that $\max_{u \in U} |u_i| > \delta^2 - \delta$. Let I denote this set of indices. For all other indices $i \notin I$, we must have $(Ax)_i \leqslant \max |u_i| + \delta \leqslant \delta^2$ for all $x \in D_L$, $x \geqslant 0$. In particular, $A_{ij} = (Ae_j)_i \leqslant \delta^2$ for all j and all $i \notin I$.

Then $\delta^{-2t}A^t_{ij} \leqslant A_{ij}$ for all $i \notin I$, $j \in \mathbb{N}$ and $t > 1$. This means that $\delta^{-2t}A_i^{(t)} \leqslant A_i$ for all rows A_i such that $i \notin I$. Let P_I be the orthogonal projection onto span$\{e_i : i \in I\}$. Then $P_I A^{(t)}$ is compact since it has finite dimensional range, and if $Q_I = \mathrm{id} - P_I$, then $\delta^{-2t}Q_I A^{(t)} \leqslant A$ and $\delta^{-2t}\gamma(A^{(t)}) = \delta^{-2t}\gamma(Q_I A^{(t)}) \leqslant \gamma(A) = 1$ (since $\gamma(\cdot)$ is invariant under compact perturbations and since it is monotone). Then $\gamma(A^{(t)}) \leqslant \delta^{2t}$. Since $\delta > 1$ can be chosen arbitrarily close to 1, we conclude that $\gamma(A^{(t)}) \leqslant 1$ for all $t > 1$. This proves (39).

Inequality (41) follows from (23), monotonicity of $\gamma(\cdot)$ and (39). Inequality (42) follows from (12) and (41) since

$$\rho_{ess}(A) = \lim_{j\to\infty} \gamma((A_1^{(t)} \cdots A_m^{(t)})^j)^{1/j} \leqslant \lim_{j\to\infty} \gamma((A_1 \cdots A_m)^j)^{t/j} = \rho_{ess}(A_1 \cdots A_m)^t.$$

Inequality (40) is a special case of (42). $\square$

REMARK 1. Observe that Theorem 5 is not a special case of [29, Lemma 4.2] since for each i and j we have $\gamma(E_{ij}) = 0$, where E_{ij} denotes the infinite matrix with 1 and at the ijth coordinate and with 0 elsewhere.

Applying standard techniques used also in [12] and [29] we establish the essential versions of Theorems 2(ii) and 1(ii).

THEOREM 6. *Let $L \in \mathscr{L}$ such that L and L^* have order continuous norms. Assume $A_1, \ldots, A_m$ are nonnegative matrices that define operators on L and let $\alpha_1, \ldots, \alpha_m$ be positive numbers such that $s_m = \sum_{j=1}^m \alpha_j \geqslant 1$. Then inequalities (21) and (22) hold.*

Proof. For $j = 1, \ldots, m$ define $\beta_j = \frac{\alpha_j}{s_m}$ and so $\sum_{j=1}^m \beta_j = 1$. Then by (39) and Theorem 2(i) we have

$$\gamma(A_1^{(\alpha_1)} \circ \cdots \circ A_m^{(\alpha_m)}) = \gamma\left(\left(A_1^{(\beta_1)} \circ \cdots \circ A_m^{(\beta_m)}\right)^{(s_m)}\right) \leqslant \gamma\left(A_1^{(\beta_1)} \circ \cdots \circ A_m^{(\beta_m)}\right)^{s_m}$$
$$\leqslant \left(\gamma(A_1)^{\beta_1} \cdots \gamma(A_m)^{\beta_m}\right)^{s_m} = \gamma(A_1)^{\alpha_1}\gamma(A_2)^{\alpha_2} \cdots \gamma(A_m)^{\alpha_m},$$

which proves (21) under our assumptions. Similarly, (22) follows from (40) and Theorem 2(i) . $\square$

THEOREM 7. *Let $L \in \mathscr{L}$ such that L and L^* have order continuous norms. Assume $\{A_{ij}\}_{i=1,j=1}^{k,m}$ are nonnegative matrices that define operators on L and let $\alpha_1, \ldots, \alpha_m$ be positive numbers such that $s_m = \sum_{j=1}^m \alpha_j \geqslant 1$. Then for A from (13) inequalities (17) and (18) hold.*

Proof. Inequalities (17) and (18) under our assumptions follow from (14) in Theorem 1(ii), monotonicity of $\gamma(\cdot)$ and $\rho_{ess}(\cdot)$ and from Theorem 6. $\square$

The following result on the joint and generalized essential radius of bounded sets of infinite nonnegative matrices generalizes Theorem 7 and is an essential version of [7, Theorem 3.3(ii)] (in the case $\sum_{j=1}^m \alpha_j = 1$ it is known by [7, Theorem 3.3(i)]). It is proved by combining ideas from the proofs of [29, Corollary 5.3], [33, Theorem 3.8] and [7, Theorem 3.3].

THEOREM 8. *Let $L \in \mathscr{L}$ such that L and L^* have order continuous norms. Assume $\alpha_1, \ldots, \alpha_m$ are positive numbers such that $\sum_{j=1}^m \alpha_j \geqslant 1$ and let $n \in \mathbb{N}$. Let $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$ and let $\Psi_1, \ldots, \Psi_m$ and $\{\Psi_{ij}\}_{i=1,j=1}^{k,m}$ be bounded sets of nonnegative matrices that define positive operators on L. Then*

$$r(\Psi_1^{(\alpha_1)} \circ \cdots \circ \Psi_m^{(\alpha_m)}) \leqslant r((\Psi_1^n)^{(\alpha_1)} \circ \cdots \circ (\Psi_m^n)^{(\alpha_m)})^{\frac{1}{n}} \leqslant r(\Psi_1)^{\alpha_1} \cdots r(\Psi_m)^{\alpha_m} \tag{43}$$

and

$$\begin{aligned} & r\left(\left(\Psi_{11}^{(\alpha_1)} \circ \cdots \circ \Psi_{1m}^{(\alpha_m)}\right) \cdots \left(\Psi_{k1}^{(\alpha_1)} \circ \cdots \circ \Psi_{km}^{(\alpha_m)}\right)\right) \\ & \leqslant r\left((\Psi_{11} \cdots \Psi_{k1})^{(\alpha_1)} \circ \cdots \circ (\Psi_{1m} \cdots \Psi_{km})^{(\alpha_m)}\right) \\ & \leqslant r\left(((\Psi_{11} \cdots \Psi_{k1})^n)^{(\alpha_1)} \circ \cdots \circ ((\Psi_{1m} \cdots \Psi_{km})^n)^{(\alpha_m)}\right)^{\frac{1}{n}} \\ & \leqslant r(\Psi_{11} \cdots \Psi_{k1})^{\alpha_1} \cdots r(\Psi_{1m} \cdots \Psi_{km})^{\alpha_m}. \end{aligned} \tag{44}$$

In particular, if $\Psi_1, \ldots, \Psi_k$ are bounded sets of nonnegative matrices that define positive operators on L and $t \geqslant 1$, then

$$r(\Psi_1^{(t)} \cdots \Psi_k^{(t)}) \leqslant r((\Psi_1 \cdots \Psi_k)^{(t)}) \leqslant r(((\Psi_1 \cdots \Psi_k)^n)^{(t)})^{\frac{1}{n}} \leqslant r(\Psi_1 \cdots \Psi_k)^t. \tag{45}$$

Proof. First we prove the inequality

$$r(\Psi_1^{(\alpha_1)} \circ \cdots \circ \Psi_m^{(\alpha_m)}) \leqslant r(\Psi_1)^{\alpha_1} \cdots r(\Psi_m)^{\alpha_m}. \tag{46}$$

Let $A \in (\Psi_1^{(\alpha_1)} \circ \cdots \circ \Psi_m^{(\alpha_m)})^l$, $l \in \mathbb{N}$. Then there are $A_{ik} \in \Psi_k$, $i = 1, \ldots, l$, $k = 1, \ldots, m$ such that

$$A = (A_{11}^{\alpha_1} \circ \cdots \circ A_{1m}^{\alpha_m}) \cdots (A_{l1}^{\alpha_1} \circ \cdots \circ A_{lm}^{\alpha_m}).$$

By Theorem 7 we have

$$\gamma(A) \leqslant \gamma(A_{11} \cdots A_{l1})^{\alpha_1} \cdots \gamma(A_{1m} \cdots A_{lm})^{\alpha_m},$$

$$\rho_{ess}(A) \leqslant \rho_{ess}(A_{11} \cdots A_{l1})^{\alpha_1} \cdots \rho_{ess}(A_{1m} \cdots A_{lm})^{\alpha_m}.$$

Since $A_{1k} \cdots A_{lk} \in \Psi_k^l$ for all $k = 1, \ldots, m$, (46) follows.

To prove the first inequality in (44) let $l \in \mathbb{N}$ and

$$B \in \left(\left(\Psi_{11}^{(\alpha_1)} \circ \cdots \circ \Psi_{1m}^{(\alpha_m)}\right) \cdots \left(\Psi_{k1}^{(\alpha_1)} \circ \cdots \circ \Psi_{km}^{(\alpha_m)}\right)\right)^l.$$

Then $B = A_1 \cdots A_l$, where for each $i = 1, \ldots, l$ we have

$$A_i = \left(A_{i11}^{(\alpha_1)} \circ \cdots \circ A_{i1m}^{(\alpha_m)}\right) \cdots \left(A_{ik1}^{(\alpha_1)} \circ \cdots \circ A_{ikm}^{(\alpha_m)}\right),$$

where $A_{i11} \in \Psi_{11}, \ldots, A_{i1m} \in \Psi_{1m}, \ldots, A_{ik1} \in \Psi_{k1}, \ldots, A_{ikm} \in \Psi_{km}$. Then by (14) for each $i = 1, \ldots, l$ we have

$$A_i \leqslant C_i := (A_{i11}A_{i21} \cdots A_{ik1})^{(\alpha_1)} \circ \cdots \circ (A_{i1m}A_{i2m} \cdots A_{ikm})^{(\alpha_m)},$$

where $C_i \in (\Psi_{11} \cdots \Psi_{k1})^{(\alpha_1)} \circ \cdots \circ (\Psi_{1m} \cdots \Psi_{km})^{(\alpha_m)}$. Therefore

$$B \leqslant C := C_1 \cdots C_l \in \left((\Psi_{11} \cdots \Psi_{k1})^{(\alpha_1)} \circ \cdots \circ (\Psi_{1m} \cdots \Psi_{km})^{(\alpha_m)} \right)^l,$$

$\rho_{ess}(B)^{1/l} \leqslant \rho_{ess}(C)^{1/l}$ and $\gamma(B)^{1/l} \leqslant \gamma(C)^{1/l}$, which implies the first inequality in (44).

The first inequality in (43) follows from the first inequality in (44) and (35), since

$$r(\Psi_1^{(\alpha_1)} \circ \cdots \circ \Psi_m^{(\alpha_m)}) = r\left(\left(\Psi_1^{(\alpha_1)} \circ \cdots \circ \Psi_m^{(\alpha_m)} \right)^n \right)^{\frac{1}{n}}$$

$$= r((\Psi_1^{(\alpha_1)} \circ \cdots \circ \Psi_m^{(\alpha_m)}) \cdots (\Psi_1^{(\alpha_1)} \circ \cdots \circ \Psi_m^{(\alpha_m)}))^{\frac{1}{n}}$$

$$\leqslant r((\Psi_1^n)^{(\alpha_1)} \circ \cdots \circ (\Psi_m^n)^{(\alpha_m)})^{\frac{1}{n}}.$$

The second inequality in (43) follows from (46) and (35). The second and third inequalities in (44) follow from (43). Inequalities (45) are a special case of (44). □

REMARK 2. Under the assumptions of Theorem 7 an analogue of [6, Theorem 3.4] is valid. The proof runs by following the same lines as in the proof of this result. The details are omitted.

4. Further results

By applying results of the previous section we obtain several results that are essential versions of known relatively recent results from the literature. Since the proofs are similar to the existing proofs we mostly omit them to avoid too much repetition of ideas.

Recall that for nonnegative measurable functions $\{f_{ij}\}_{i=1,j=1}^{k,m}$ and for nonnegative numbers α_j, $j = 1, \ldots, m$, such that $\sum_{j=1}^m \alpha_j \geqslant 1$ (see e.g. [25], [7]) we have

$$(f_{11}^{\alpha_1} \cdots f_{1m}^{\alpha_m}) + \cdots + (f_{k1}^{\alpha_1} \cdots f_{km}^{\alpha_m}) \leqslant (f_{11} + \cdots + f_{k1})^{\alpha_1} \cdots (f_{1m} + \cdots + f_{km})^{\alpha_m}. \quad (47)$$

The sum of bounded sets Ψ and Σ is a bounded set defined by $\Psi + \Sigma = \{A + B : A \in \Psi, B \in \Sigma\}$. The following result is an essential version of [7, Theorem 3.7(ii)] (in the case $\sum_{j=1}^m \alpha_j = 1$ it is known by [7, Theorem 3.7(i)]). It is proved in a similar way as [7, Theorem 3.7] (it follows from (47) and (43)).

THEOREM 9. *Let $L \in \mathscr{L}$ such that L and L^* have order continuous norms. Assume $\alpha_1, \ldots, \alpha_m$ are positive numbers such that $\sum_{j=1}^m \alpha_j \geqslant 1$ and let $n \in \mathbb{N}$. Let*

$r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$ and let $\{\Psi_{ij}\}_{i=1,j=1}^{k,m}$ be bounded sets of nonnegative matrices that define positive operators on L. Then

$$\begin{aligned} & r\left(\left(\Psi_{11}^{(\alpha_1)} \circ \cdots \circ \Psi_{1m}^{(\alpha_m)}\right) + \ldots + \left(\Psi_{k1}^{(\alpha_1)} \circ \cdots \circ \Psi_{km}^{(\alpha_m)}\right)\right) \\ \leqslant\ & r\left((\Psi_{11} + \cdots + \Psi_{k1})^{(\alpha_1)} \circ \cdots \circ (\Psi_{1m} + \cdots + \Psi_{km})^{(\alpha_m)}\right) \\ \leqslant\ & r\left(((\Psi_{11} + \cdots + \Psi_{k1})^n)^{(\alpha_1)} \circ \cdots \circ ((\Psi_{1m} + \cdots + \Psi_{km})^n)^{(\alpha_m)}\right)^{\frac{1}{n}} \\ \leqslant\ & r(\Psi_{11} + \cdots + \Psi_{k1})^{\alpha_1} \cdots r(\Psi_{1m} + \cdots + \Psi_{km})^{\alpha_m}. \end{aligned} \tag{48}$$

Next we turn our attention to the weighted geometic symmetrizations of sets of infinite matrices (see [7]). Let Ψ be a bounded set of nonnegative matrices that define operators on l^2 and denote $\Psi^* = \{A^* : A \in \Psi\}$. Observe that $(\Psi\Sigma)^* = \Sigma^*\Psi^*$, $(\Psi^m)^* = (\Psi^*)^m$ for all $m \in \mathbb{N}$ and $r(\Psi) = r(\Psi^*)$ for all $r \in \{\rho, \hat{\rho}, \rho_{ess}, \hat{\rho}_{ess}\}$. Let α and β be nonnegative numbers such that $\alpha + \beta \geqslant 1$. The weighted geometric symmetrization set $S_{\alpha,\beta}(\Psi) = \Psi^{(\alpha)} \circ (\Psi^*)^{(\beta)} = \{A^{(\alpha)} \circ (B^*)^{(\beta)} : A, B \in \Psi\}$ is a bounded set of nonnegative matrices that define operators on l^2 by Theorem 1(ii).

The following two results are essential versions of [7, Proposition 4.4 and Theorem 4.3]. They follow from Theorems 8 and 9 and are proved in a very similar way as [7, Proposition 4.4 and Theorem 4.3].

PROPOSITION 10. *Let Ψ, $\Psi_1, \ldots, \Psi_m$ be bounded sets of nonnegative matrices that define operators on l^2, $n \in \mathbb{N}$ and let α and β be nonnegative numbers such that $\alpha + \beta \geqslant 1$. Then we have*

$$\begin{aligned} & r(S_{\alpha,\beta}(\Psi_1) \cdots S_{\alpha,\beta}(\Psi_m)) \leqslant r\left((\Psi_1 \cdots \Psi_m)^{(\alpha)} \circ ((\Psi_m \cdots \Psi_1)^*)^{(\beta)}\right) \\ \leqslant\ & r\left(((\Psi_1 \cdots \Psi_m)^n)^{(\alpha)} \circ (((\Psi_m \cdots \Psi_1)^*)^n)^{(\beta)}\right)^{\frac{1}{n}} \\ \leqslant\ & r(\Psi_1 \cdots \Psi_m)^{\alpha}\, r(\Psi_m \cdots \Psi_1)^{\beta}, \end{aligned} \tag{49}$$

$$r(S_{\alpha,\beta}(\Psi)) \leqslant r(S_{\alpha,\beta}(\Psi^n))^{\frac{1}{n}} \leqslant r(\Psi)^{\alpha+\beta}, \tag{50}$$

$$\begin{aligned} & r(S_{\alpha,\beta}(\Psi_1) + \cdots + S_{\alpha,\beta}(\Psi_m)) \leqslant r\left(S_{\alpha,\beta}(\Psi_1 + \cdots + \Psi_m)\right) \\ \leqslant\ & r\left(S_{\alpha,\beta}((\Psi_1 + \cdots + \Psi_m)^n)\right)^{\frac{1}{n}} \leqslant r(\Psi_1 + \cdots + \Psi_m)^{\alpha+\beta}, \end{aligned} \tag{51}$$

$$\begin{aligned} & r(S_{\alpha,\beta}(\Psi_1) S_{\alpha,\beta}(\Psi_2)) \leqslant r\left((\Psi_1\Psi_2)^{(\alpha)} \circ ((\Psi_2\Psi_1)^*)^{(\beta)}\right) \\ \leqslant\ & r\left(((\Psi_1\Psi_2)^n)^{(\alpha)} \circ (((\Psi_2\Psi_1)^*)^n)^{(\beta)}\right)^{\frac{1}{n}} \leqslant r(\Psi_1\Psi_2)^{\alpha+\beta} \end{aligned} \tag{52}$$

for all $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$.

THEOREM 11. *Let Ψ be a bounded set of nonnegative matrices that define operators on l^2 and $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$. Assume α and β are nonnegative numbers such that $\alpha+\beta \geqslant 1$ and denote $r_n = r(S_{\alpha,\beta}(\Psi^{2^n}))^{2^{-n}}$ for $n \in \mathbb{N}\cup\{0\}$. Then we have*

$$r(S_{\alpha,\beta}(\Psi)) = r_0 \leqslant r_1 \leqslant \cdots \leqslant r_n \leqslant r(\Psi)^{\alpha+\beta}. \tag{53}$$

REMARK 3. Theorem 11 implies that also the essential version of [6, Theorem 2.5(ii)] is valid.

The following result is an essential version of [7, Theorem 3.1(ii)] and is proved in a similar way as this result by applying Theorem 8.

PROPOSITION 12. *Let $L \in \mathscr{L}$ such that L and L^* have order continuous norms. Assume $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$, $m,n \in \mathbb{N}$, $\alpha \geqslant 1$ and let Ψ be a bounded set of nonnegative matrices that define operators on L. Then*

$$r(\Psi^{(m)}) \leqslant r(\Psi \circ \cdots \circ \Psi) \leqslant r(\Psi^n \circ \cdots \circ \Psi^n)^{\frac{1}{n}} \leqslant r(\Psi)^m, \tag{54}$$

where in (54) the Hadamard products in $\Psi \circ \cdots \circ \Psi$ and in $\Psi^n \circ \cdots \circ \Psi^n$ are taken $m-1$ times, and

$$r(\Psi^{(\alpha)}) \leqslant r(\Psi^{(\alpha-1)} \circ \Psi) \leqslant r((\Psi^n)^{(\alpha-1)} \circ \Psi^n)^{\frac{1}{n}} \leqslant r(\Psi)^{\alpha}. \tag{55}$$

The following result is an essential version of [7, Theorem 3.6] and is proved in a similar way as this result by applying Theorems 8, 4, property (35) and [7, Theorem 3.4].

THEOREM 13. *Let $L \in \mathscr{L}$ such that L and L^* have order continuous norms. Let $\Psi_1, \ldots, \Psi_m$ be bounded sets of nonnegative matrices that define operators on L and $\Phi_j = \Psi_j \ldots \Psi_m \Psi_1 \ldots \Psi_{j-1}$ for $j = 1, \ldots, m$. Assume that $\alpha \geqslant \frac{1}{m}$, $\alpha_j \geqslant 0$, $j = 1, \ldots, m$, $\sum_{j=1}^m \alpha_j \geqslant 1$ and $n \in \mathbb{N}$. If $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$ and $\Sigma_j = \Psi_j^{(\alpha m)} \ldots \Psi_m^{(\alpha m)} \Psi_1^{(\alpha m)} \ldots \Psi_{j-1}^{(\alpha m)}$ for $j = 1, \ldots, m$, then we have*

$$\begin{aligned} r\left(\Psi_1^{(\alpha)} \circ \cdots \circ \Psi_m^{(\alpha)}\right) &\leqslant r\left(\Phi_1^{(\alpha)} \circ \cdots \circ \Phi_m^{(\alpha)}\right)^{\frac{1}{m}} \\ &\leqslant r\left((\Phi_1^n)^{(\alpha)} \circ \cdots \circ (\Phi_m^n)^{(\alpha)}\right)^{\frac{1}{mn}} \leqslant r(\Psi_1 \cdots \Psi_m)^{\alpha}, \end{aligned} \tag{56}$$

$$\begin{aligned} r\left(\Psi_1^{(\alpha)} \circ \cdots \circ \Psi_m^{(\alpha)}\right) &\leqslant r\left(\Psi_1^{(\alpha m)} \cdots \Psi_m^{(\alpha m)}\right)^{\frac{1}{m}} \\ &\leqslant r\left((\Psi_1 \cdots \Psi_m)^{(\alpha m)}\right)^{\frac{1}{m}} \leqslant r\left(((\Psi_1 \cdots \Psi_m)^n)^{(\alpha m)}\right)^{\frac{1}{nm}} \leqslant r(\Psi_1 \cdots \Psi_m)^{\alpha}. \end{aligned} \tag{57}$$

If, in addition, $\alpha \geqslant 1$ then

$$\begin{aligned} r\left(\Psi_1^{(\alpha)} \circ \cdots \circ \Psi_m^{(\alpha)}\right) &\leqslant r\left(\Phi_1^{(\alpha)} \circ \cdots \circ \Phi_m^{(\alpha)}\right)^{\frac{1}{m}} \leqslant r\left((\Phi_1^n)^{(\alpha)} \circ \cdots \circ (\Phi_m^n)^{(\alpha)}\right)^{\frac{1}{mn}} \\ &\leqslant \left(r\left((\Phi_1^n)^{(m)}\right) \cdots r\left((\Phi_m^n)^{(m)}\right)\right)^{\frac{\alpha}{m^2 n}} \leqslant r(\Psi_1 \cdots \Psi_m)^{\alpha}, \end{aligned} \tag{58}$$

$$r\left(\Psi_1^{(\alpha)}\circ\cdots\circ\Psi_m^{(\alpha)}\right)\leqslant r\left(\Sigma_1^{(\frac{1}{m})}\circ\cdots\circ\Sigma_m^{(\frac{1}{m})}\right)^{\frac{1}{m}}$$
$$\leqslant r\left((\Sigma_1^n)^{(\frac{1}{m})}\circ\cdots\circ(\Sigma_m^n)^{(\frac{1}{m})}\right)^{\frac{1}{mn}}\leqslant r\left(\Psi_1^{(\alpha m)}\cdots\Psi_m^{(\alpha m)}\right)^{\frac{1}{m}}$$
$$\leqslant r\left((\Psi_1\cdots\Psi_m)^{(\alpha m)}\right)^{\frac{1}{m}}\leqslant r\left(((\Psi_1\cdots\Psi_m)^n)^{(\alpha m)}\right)^{\frac{1}{nm}}\leqslant r(\Psi_1\cdots\Psi_m)^{\alpha}. \tag{59}$$

The following consequence provides the essential version of some of the main results of [31] ([31, Theorems 3.5 and 3.7]).

COROLLARY 1. *Let $L\in\mathscr{L}$ such that L and L^* have order continuous norms. Let Ψ_1 and Ψ_2 be bounded sets of nonnegative matrices that define operators on L, let $r\in\{\rho_{ess},\hat{\rho}_{ess}\}$ and $\beta\in[0,1]$. Then*

$$r(\Psi_1\circ\Psi_2)\leqslant r(\Psi_1^{(2)}\Psi_2^{(2)})^{\frac{1}{2}}\leqslant r((\Psi_1\circ\Psi_1)(\Psi_2\circ\Psi_2))^{\frac{1}{2}}$$
$$\leqslant r(\Psi_1\Psi_2\circ\Psi_1\Psi_2)^{\frac{\beta}{2}}r(\Psi_2\Psi_1\circ\Psi_2\Psi_1)^{\frac{1-\beta}{2}}\leqslant r(\Psi_1\Psi_2) \tag{60}$$

and

$$r(\Psi_1\circ\Psi_2)\leqslant r(\Psi_1\Psi_2\circ\Psi_2\Psi_1)^{\frac{1}{2}}\leqslant r((\Psi_1\Psi_2)^{(2)})^{\frac{1}{4}}r((\Psi_2\Psi_1)^{(2)})^{\frac{1}{4}}$$
$$\leqslant r(\Psi_1\Psi_2\circ\Psi_1\Psi_2)^{\frac{1}{4}}r(\Psi_2\Psi_1\circ\Psi_2\Psi_1)^{\frac{1}{4}}\leqslant r(\Psi_1\Psi_2). \tag{61}$$

Proof. The first inequality in (60) is a special case of the first inequality in (57). The second inequality (57) is trivial, since $\Psi_i^{(2)}\subset\Psi_i\circ\Psi_i$ for $i=1,2$. The third inequality in (60) follows from the first inequality in (44) and from (35), while the fourth inequality in (60) follows from (43) and (35).

The first inequality in (61) is a special case of the first inequality in (56). To prove the second and third inequality in (61) observe that

$$\Psi_1\Psi_2\circ\Psi_2\Psi_1=((\Psi_1\Psi_2)^{(2)})^{(\frac{1}{2})}\circ((\Psi_2\Psi_1)^{(2)})^{(\frac{1}{2})}$$

It folllows from (43) and (54) that

$$r(\Psi_1\Psi_2\circ\Psi_2\Psi_1)\leqslant r((\Psi_1\Psi_2)^{(2)})^{\frac{1}{2}}r((\Psi_2\Psi_1)^{(2)})^{\frac{1}{2}}\leqslant r(\Psi_1\Psi_2\circ\Psi_1\Psi_2)^{\frac{1}{2}}r(\Psi_2\Psi_1\circ\Psi_2\Psi_1)^{\frac{1}{2}},$$

which establishes the second and third inequality in (61). The fourth inequality in (61) follows from (43) and (35), which completes the proof. □

REMARK 4. Under the assumptions of Corollary 1 one can similarly as (61) prove its variant:

$$r(\Psi_1\circ\Psi_2)\leqslant r(\Psi_1\Psi_2\circ\Psi_2\Psi_1)^{\frac{1}{2}}\leqslant r((\Psi_1\Psi_2)^{(\frac{1}{\beta})})^{\frac{\beta}{2}}r((\Psi_2\Psi_1)^{(\frac{1}{1-\beta})})^{\frac{1-\beta}{2}}\leqslant r(\Psi_1\Psi_2). \tag{62}$$

Similarly (62) is proved if $L\in\mathscr{L}$ and $r\in\{\rho,\hat{\rho}\}$.

In a special case of singelton sets $\Psi_1 = \{A\}$ and $\Psi_1 = \{B\}$ we obtain the essential versions of (6) and (7) (infact a slight generalization).

COROLLARY 2. *Let $L \in \mathscr{L}$ such that L and L^* have order continuous norms. Let A and B be nonnegative matrices that define operators on L and let $\beta \in [0,1]$. Then*

$$\rho_{ess}(A \circ B) \leqslant \rho_{ess}((A \circ A)(B \circ B))^{\frac{1}{2}} \leqslant \rho_{ess}(AB \circ AB)^{\frac{\beta}{2}} \rho_{ess}(BA \circ BA)^{\frac{1-\beta}{2}} \leqslant \rho_{ess}(AB) \tag{63}$$

and

$$\rho_{ess}(A \circ B) \leqslant \rho_{ess}(AB \circ BA)^{\frac{1}{2}} \leqslant \rho_{ess}((AB)^{\left(\frac{1}{\beta}\right)}))^{\frac{\beta}{2}} \rho_{ess}((BA)^{\left(\frac{1}{1-\beta}\right)})^{\frac{1-\beta}{2}} \leqslant \rho_{ess}(AB). \tag{64}$$

The following results is an essential version of [5, Lemma 3.16] and is proved in a similar way as this result by applying Theorem 8.

PROPOSITION 14. *Let $\alpha \geqslant \frac{1}{2}$, $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$ and let Ψ be bounded set of nonnegative matrices that define operators on l^2. Then*

$$r(\Psi^{(\alpha)} \circ (\Psi^*)^{(\alpha)}) \leqslant r(\Psi^{(\alpha)} \circ \Psi^{(\alpha)}) \leqslant r(\Psi)^{2\alpha}. \tag{65}$$

The following special case is an essential version of [6, Lemma 3.13].

COROLLARY 3. *Let $\alpha \geqslant \frac{1}{2}$ and let A be a nonnegative matrix that defines an operator on l^2. Then*

$$\rho_{ess}(A^{(\alpha)} \circ (A^*)^{(\alpha)}) \leqslant \rho_{ess}(A^{(\alpha)} \circ A^{(\alpha)}) \leqslant \rho_{ess}(A)^{2\alpha}. \tag{66}$$

5. Further results on $L^2(X,\mu)$

In this section we will apply a fact that for a bounded linear operator T defined on a Hilbert space we have

$$\rho_{ess}(T^*T) = \rho_{ess}(TT^*) = \gamma(T^*T) = \gamma(TT^*) = \gamma(T)^2. \tag{67}$$

Since we do not know if this result has previously been known or not, we prove it below in Lemma 2.

Recall that a bounded linear operator on a Hilbert space $\mathscr{H}$ is *hyponormal* if $\|Tx\| \geqslant \|T^*x\|$ for all $x \in \mathscr{H}$, or equivalently if $T^*T - TT^*$ is positive semidefinite. In particular, any normal operator is hyponormal. Since the set $K(\mathscr{H})$ of compact operators in $B(\mathscr{H})$ is a closed two-sided ideal in $B(\mathscr{H})$, the Calkin algebra is a C^*-algebra and the canonical projection π is a $*$-isomorphism. The essential norm of $T \in B(\mathscr{H})$ is by definition $\|T\|_{ess} = \|\pi(T)\|$ and recall that $\rho_{ess}(T) = \rho(\pi(T))$.

The following proposition is probably known as it combines well-known results of Nussbaum and Stampfli [27, 42], but we are unaware of a direct reference.

PROPOSITION 15. *Let $\mathscr{H}$ be a Hilbert space. If $T \in B(\mathscr{H})$ is hyponormal, then*

$$\rho_{ess}(T) = \gamma(T) = \|T\|_{ess}.$$

Proof. For any $T \in B(\mathscr{H})$ and $K \in K(\mathscr{H})$ it is clear that $\gamma(T) = \gamma(T+K) \leqslant \|T+K\|$. Therefore $\gamma(T) \leqslant \|T\|_{ess}$. By (12) ([27, Theorem 1]) and since $\gamma(T^n) \leqslant \gamma(T)^n$ for all n, it follows that

$$\rho_{ess}(T) \leqslant \gamma(T) \leqslant \|T\|_{ess}.$$

It remains to show that $\rho_{ess}(T) = \|T\|_{ess}$ when T is hyponormal. Since the spectrum of $\pi(T^*T - TT^*)$ is a subset of the spectrum of $T^*T - TT^*$ (see e.g., [17, Theorem 2.3]), it follows that $\pi(T^*T - TT^*)$ is positive and therefore $\pi(T)$ is hyponormal whenever T is hyponormal. In that case, [42, Theorem 1] says that $\rho(\pi(T)) = \|\pi(T)\|$ and therefore $\rho_{ess}(T) = \|T\|_{ess}$. □

LEMMA 2. *Let $\mathscr{H}$ be a Hilbert space and $T \in B(\mathscr{H})$. Then $\rho_{ess}(T^*T) = \gamma(T^*T) = \gamma(T)^2$. Consequently, equalities (67) and $\gamma(T) = \gamma(T^*)$ hold.*

Proof. By the polar decomposition theorem for bounded operators on a Hilbert space, $T = UN$ where U is a partial isometry and $N = \sqrt{T^*T}$. It follows immediately that $\rho_{ess}(T^*T) = \rho_{ess}(N^2) = \rho_{ess}(N)^2$.

By Proposition 15, $\rho_{ess}(N) = \gamma(N)$. Since U is a partial isometry, $\gamma(U) \leqslant \|U\| \leqslant 1$. So we have:

$$\gamma(T)^2 = \gamma(UN)^2 \leqslant \gamma(N)^2 = \rho_{ess}(N)^2 = \rho_{ess}(T^*T). \tag{68}$$

It remains to prove the reverse inequality. Since $\gamma(U^*) \leqslant \|U^*\| = \|U\| \leqslant 1$, we have

$$\gamma(T^*T) = \gamma(NU^*T) \leqslant \gamma(N)\gamma(T).$$

Since $\rho_{ess}(T^*T) = \gamma(T^*T) = \gamma(N)^2$, we conclude that $\rho_{ess}(T^*T) \leqslant \gamma(T)^2$, which together with (68) establishes $\rho_{ess}(T^*T) = \gamma(T^*T) = \gamma(T)^2$. By (35) and Proposition 15 also the remaining equalities in (67) follow. The equality $\gamma(T) = \gamma(T^*)$ follows from (67). □

Let Σ be a bounded set of bounded operators on a Hilbert space $\mathscr{H}$ and let us denote

$$\gamma(\Sigma) = \sup_{T \in \Sigma} \gamma(T) \text{ and } \|\Sigma\| = \sup_{T \in \Sigma} \|T\|.$$

By Σ^* we denote a bounded set of bounded operators on $\mathscr{H}$ defined by $\Sigma^* = \{T^* : T \in \Sigma\}$. The following lemma is an essential version of [5, Lemma 3.1.] and it also slightly generalizes it (with a similar proof).

LEMMA 3. *Let $\mathscr{H}$ be a Hilbert space and $\Sigma \subset B(\mathscr{H})$ be a bounded set. Then*

$$\gamma(\Sigma) = \rho_{ess}(\Sigma^*\Sigma)^{1/2} = \rho_{ess}(\Sigma\Sigma^*)^{1/2} = \hat{\rho}_{ess}(\Sigma^*\Sigma)^{1/2} = \hat{\rho}_{ess}(\Sigma\Sigma^*)^{1/2}, \tag{69}$$

$\gamma(\Sigma^*) = \gamma(\Sigma)$ and

$$\|\Sigma\| = \rho(\Sigma^*\Sigma)^{1/2} = \rho(\Sigma\Sigma^*)^{1/2} = \hat{\rho}(\Sigma^*\Sigma)^{1/2} = \hat{\rho}(\Sigma\Sigma^*)^{1/2}. \tag{70}$$

Proof. First we prove (69). By Lemma 2 we have

$$\begin{aligned}\gamma(\Sigma) &= \sup_{T\in\Sigma}\gamma(T) = \sup_{T\in\Sigma}\rho_{ess}(T^*T)^{\frac{1}{2}} = \big(\sup_{T\in\Sigma}\rho_{ess}((T^*T)^m)^{\frac{1}{m}}\big)^{\frac{1}{2}} \\ &\leqslant \left(\sup_{m\in\mathbb{N}}\sup_{S\in(\Sigma^*\Sigma)^m}\rho_{ess}(S)^{\frac{1}{m}}\right)^{\frac{1}{2}} = \rho_{ess}(\Sigma^*\Sigma)^{\frac{1}{2}} \leqslant \hat{\rho}_{ess}(\Sigma^*\Sigma)^{\frac{1}{2}} \leqslant \gamma(\Sigma^*\Sigma)^{\frac{1}{2}} \\ &\leqslant (\gamma(\Sigma^*)\gamma(\Sigma))^{\frac{1}{2}} = \gamma(\Sigma),\end{aligned}$$

which proves $\gamma(\Sigma) = \rho_{ess}(\Sigma^*\Sigma)^{1/2} = \hat{\rho}_{ess}(\Sigma^*\Sigma)^{1/2}$. Other equalities in (69) follow again by Lemma 2 (or also from (35)). Equality $\gamma(\Sigma^*) = \gamma(\Sigma)$ follows from (69) (or also from Lemma 2).

Equalities (70) are proved similarly. □

By applying (69) we obtain the following result, which is an essential version of [5, Theorem 3.2] and is proved in a similar way. For the sake of clarity we include the proof.

THEOREM 16. *Let $\Psi_1,\ldots,\Psi_m$ be bounded sets of positive kernel operators on $L^2(X,\mu)$ and let $r\in\{\rho_{ess},\hat{\rho}_{ess}\}$.*

If m is even, then

$$\begin{aligned}\gamma(\Psi_1^{(\frac{1}{m})}\circ\cdots\circ\Psi_m^{(\frac{1}{m})}) &\leqslant \big(r(\Psi_1^*\Psi_2\Psi_3^*\Psi_4\cdots\Psi_{m-1}^*\Psi_m)r(\Psi_1\Psi_2^*\Psi_3\Psi_4^*\cdots\Psi_{m-1}\Psi_m^*)\big)^{\frac{1}{2m}} \\ &= \big(r(\Psi_1^*\Psi_2\Psi_3^*\Psi_4\cdots\Psi_{m-1}^*\Psi_m)r(\Psi_m\Psi_{m-1}^*\cdots\Psi_4\Psi_3^*\Psi_2\Psi_1^*)\big)^{\frac{1}{2m}}.\end{aligned} \tag{71}$$

If m is odd, then

$$\begin{aligned}\gamma\ &(\Psi_1^{(\frac{1}{m})}\circ\cdots\circ\Psi_m^{(\frac{1}{m})}) \\ &\leqslant r^{\frac{1}{2m}}(\Psi_1\Psi_2^*\Psi_3\Psi_4^*\cdots\Psi_{m-2}\Psi_{m-1}^*\Psi_m\Psi_1^*\Psi_2\Psi_3^*\Psi_4\cdots\Psi_{m-2}^*\Psi_{m-1}\Psi_m^*).\end{aligned} \tag{72}$$

Proof. If m is even, then we have

$$\left(\left(\Psi_1^{(\frac{1}{m})}\circ\Psi_2^{(\frac{1}{m})}\circ\cdots\circ\Psi_m^{(\frac{1}{m})}\right)^*\left(\Psi_1^{(\frac{1}{m})}\circ\Psi_2^{(\frac{1}{m})}\circ\cdots\circ\Psi_m^{(\frac{1}{m})}\right)\right)^{\frac{m}{2}}$$

$$= \left((\Psi_1^*)^{(\frac{1}{m})}\circ(\Psi_2^*)^{(\frac{1}{m})}\circ\cdots\circ(\Psi_m^*)^{(\frac{1}{m})}\right)\left(\Psi_2^{(\frac{1}{m})}\circ\Psi_3^{(\frac{1}{m})}\circ\cdots\circ\Psi_1^{(\frac{1}{m})}\right)$$

$$\left((\Psi_3^*)^{(\frac{1}{m})}\circ(\Psi_4^*)^{(\frac{1}{m})}\circ\cdots\circ(\Psi_2^*)^{(\frac{1}{m})}\right)\left(\Psi_4^{(\frac{1}{m})}\circ\Psi_5^{(\frac{1}{m})}\circ\cdots\circ\Psi_3^{(\frac{1}{m})}\right)\cdots$$

$$\left((\Psi_{m-1}^*)^{(\frac{1}{m})}\circ(\Psi_m^*)^{(\frac{1}{m})}\circ\cdots\circ(\Psi_{m-2}^*)^{(\frac{1}{m})}\right)\left(\Psi_m^{(\frac{1}{m})}\circ\Psi_1^{(\frac{1}{m})}\circ\cdots\circ\Psi_{m-1}^{(\frac{1}{m})}\right).$$

It follows from (69), (35) and [7, Theorem 3.2(i)] that

$$\begin{aligned}
&\gamma(\Psi_1^{(\frac{1}{m})}\circ\Psi_2^{(\frac{1}{m})}\circ\cdots\circ\Psi_m^{(\frac{1}{m})})^m\\
&=r\left(\left(\Psi_1^{(\frac{1}{m})}\circ\Psi_2^{(\frac{1}{m})}\circ\cdots\circ\Psi_m^{(\frac{1}{m})}\right)^*\left(\Psi_1^{(\frac{1}{m})}\circ\Psi_2^{(\frac{1}{m})}\circ\cdots\circ\Psi_m^{(\frac{1}{m})}\right)\right)^{\frac{m}{2}}\\
&\leqslant r(\Sigma)\leqslant r(\Psi_1^*\Psi_2\Psi_3^*\Psi_4\cdots\Psi_{m-1}^*\Psi_m)^{\frac{1}{m}}r(\Psi_2^*\Psi_3\Psi_4^*\Psi_5\cdots\Psi_m^*\Psi_1)^{\frac{1}{m}}\cdots\\
&\quad r(\Psi_{m-1}^*\Psi_m\Psi_1^*\Psi_2\cdots\Psi_{m-3}^*\Psi_{m-2})^{\frac{1}{m}}r(\Psi_m^*\Psi_1\Psi_2^*\Psi_3\cdots\Psi_{m-2}^*\Psi_{m-1})^{\frac{1}{m}}\\
&=r^{\frac{1}{2}}(\Psi_1^*\Psi_2\Psi_3^*\Psi_4\cdots\Psi_{m-1}^*\Psi_m)r^{\frac{1}{2}}(\Psi_1\Psi_2^*\Psi_3\Psi_4^*\cdots\Psi_{m-1}\Psi_m^*)\\
&\quad(r(\Psi_1^*\Psi_2\Psi_3^*\Psi_4\cdots\Psi_{m-1}^*\Psi_m)r(\Psi_m\Psi_{m-1}^*\cdots\Psi_4\Psi_3^*\Psi_2\Psi_1^*))^{\frac{1}{2}},
\end{aligned}\tag{73}$$

where

$$\begin{aligned}
\Sigma:=\;&(\Psi_1^*\Psi_2\Psi_3^*\Psi_4\cdots\Psi_{m-1}^*\Psi_m)^{(\frac{1}{m})}\circ(\Psi_2^*\Psi_3\Psi_4^*\Psi_5\cdots\Psi_m^*\Psi_1)^{(\frac{1}{m})}\circ\cdots\circ\\
&(\Psi_{m-1}^*\Psi_m\Psi_1^*\Psi_2\cdots\Psi_{m-3}^*\Psi_{m-2})^{(\frac{1}{m})}\circ(\Psi_m^*\Psi_1\Psi_2^*\Psi_3\cdots\Psi_{m-2}^*\Psi_{m-1})^{(\frac{1}{m})},
\end{aligned}$$

which completes the proof of (71).

If m is odd, we have

$$\begin{aligned}
&\left(\left(\Psi_1^{(\frac{1}{m})}\circ\Psi_2^{(\frac{1}{m})}\circ\cdots\circ\Psi_m^{(\frac{1}{m})}\right)^*\left(\Psi_1^{(\frac{1}{m})}\circ\Psi_2^{(\frac{1}{m})}\circ\cdots\circ\Psi_m^{(\frac{1}{m})}\right)\right)^m\\
&=\left((\Psi_1^*)^{(\frac{1}{m})}\circ(\Psi_2^*)^{(\frac{1}{m})}\circ\cdots\circ(\Psi_m^*)^{(\frac{1}{m})}\right)\left(\Psi_2^{(\frac{1}{m})}\circ\cdots\circ\Psi_m^{(\frac{1}{m})}\circ\Psi_1^{(\frac{1}{m})}\right)\\
&\left((\Psi_3^*)^{(\frac{1}{m})}\circ(\Psi_4^*)^{(\frac{1}{m})}\circ\cdots\circ(\Psi_2^*)^{(\frac{1}{m})}\right)\left(\Psi_4^{(\frac{1}{m})}\circ\Psi_5^{(\frac{1}{m})}\circ\cdots\circ\Psi_3^{(\frac{1}{m})}\right)\cdots\\
&\left((\Psi_{m-2}^*)^{(\frac{1}{m})}\circ(\Psi_{m-1}^*)^{(\frac{1}{m})}\circ\cdots\circ(\Psi_{m-3}^*)^{(\frac{1}{m})}\right)\left(\Psi_{m-1}^{(\frac{1}{m})}\circ\Psi_m^{(\frac{1}{m})}\circ\cdots\circ\Psi_{m-2}^{(\frac{1}{m})}\right)\\
&\left((\Psi_m^*)^{(\frac{1}{m})}\circ(\Psi_1^*)^{(\frac{1}{m})}\circ\cdots\circ(\Psi_{m-1}^*)^{(\frac{1}{m})}\right)\left(\Psi_1^{(\frac{1}{m})}\circ\Psi_2^{(\frac{1}{m})}\circ\cdots\circ\Psi_{m-1}^{(\frac{1}{m})}\circ\Psi_m^{(\frac{1}{m})}\right)\\
&\left((\Psi_2^*)^{(\frac{1}{m})}\circ(\Psi_3^*)^{(\frac{1}{m})}\circ\cdots\circ(\Psi_1^*)^{(\frac{1}{m})}\right)\left(\Psi_3^{(\frac{1}{m})}\circ\Psi_4^{(\frac{1}{m})}\circ\cdots\circ\Psi_1^{(\frac{1}{m})}\circ\Psi_2^{(\frac{1}{m})}\right)\cdots\\
&\left((\Psi_{m-1}^*)^{(\frac{1}{m})}\circ(\Psi_m^*)^{(\frac{1}{m})}\circ\cdots\circ(\Psi_{m-2}^*)^{(\frac{1}{m})}\right)\left(\Psi_m^{(\frac{1}{m})}\circ\Psi_1^{(\frac{1}{m})}\circ\cdots\circ\Psi_{m-2}^{(\frac{1}{m})}\circ\Psi_{m-1}^{(\frac{1}{m})}\right).
\end{aligned}\tag{74}$$

It follows from (69), (35) and [7, Theorem 3.2(i)] that

$$\begin{aligned}
&\gamma(\Psi_1^{(\frac{1}{m})}\circ\Psi_2^{(\frac{1}{m})}\circ\cdots\circ\Psi_m^{(\frac{1}{m})})^{2m}\\
&=r\left(\left(\Psi_1^{(\frac{1}{m})}\circ\Psi_2^{(\frac{1}{m})}\circ\cdots\circ\Psi_m^{(\frac{1}{m})}\right)^*\left(\Psi_1^{(\frac{1}{m})}\circ\Psi_2^{(\frac{1}{m})}\circ\cdots\circ\Psi_m^{(\frac{1}{m})}\right)\right)^m\\
&\leqslant r(\Omega)\leqslant r(\Psi_1^*\Psi_2\Psi_3^*\Psi_4\cdots\Psi_{m-1}\Psi_m^*\Psi_1\Psi_2^*\Psi_3\Psi_4^*\cdots\Psi_{m-1}^*\Psi_m)\\
&=r(\Psi_1\Psi_2^*\Psi_3\cdots\Psi_{m-1}^*\Psi_m\Psi_1^*\Psi_2\cdots\Psi_m^*)
\end{aligned}$$

where

$$\Omega := (\Psi_1^*\Psi_2\Psi_3^*\Psi_4\cdots\Psi_{m-1}\Psi_m^*\Psi_1\Psi_2^*\Psi_3\cdots\Psi_{m-1}^*\Psi_m)^{(\frac{1}{m})}\circ (\Psi_2^*\Psi_3\Psi_4^*\Psi_5\cdots\Psi_m\Psi_1^*\Psi_2\Psi_3^*\cdots\Psi_m^*\Psi_1)^{(\frac{1}{m})}\circ\cdots\circ (\Psi_m^*\Psi_1\Psi_2^*\Psi_3\cdots\Psi_{m-1}^*\Psi_m\Psi_1^*\Psi_2\cdots\Psi_{m-2}^*\Psi_{m-1})^{(\frac{1}{m})}$$

which proves (72). □

The following corollary follows from (71) and (73).

COROLLARY 4. *Let Ψ and Σ be bounded sets of positive kernel operators on $L^2(X,\mu)$ and let $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$. Then*

$$\gamma(\Psi^{(\frac{1}{2})}\circ\Sigma^{(\frac{1}{2})}) \leqslant r\left((\Psi^*\Sigma)^{(\frac{1}{2})}\circ(\Sigma^*\Psi)^{(\frac{1}{2})}\right)^{\frac{1}{2}} \leqslant r(\Psi^*\Sigma)^{\frac{1}{2}} = r(\Psi\Sigma^*)^{\frac{1}{2}}.$$

To our knowledge even the following singelton set case (which as an essential version of (10) – [33, Theorem 4.4, (4.8)]) is new.

COROLLARY 5. *Let A and B be positive kernel operators on $L^2(X,\mu)$. Then*

$$\gamma(A^{(\frac{1}{2})}\circ B^{(\frac{1}{2})}) \leqslant \rho_{ess}\left((A^*B)^{(\frac{1}{2})}\circ(B^*A)^{(\frac{1}{2})}\right)^{\frac{1}{2}} \leqslant \rho_{ess}(A^*B)^{\frac{1}{2}} = \rho_{ess}(AB^*)^{\frac{1}{2}}.$$

The following result is an essential version of [5, Theorem 3.3] and is proved in a similar way as Theorem 16. It follows from (69), (35) and Theorem 8. To avoid too much repetition of ideas, the details of the proof are omitted.

THEOREM 17. *Let $\Psi_1,\ldots,\Psi_m$ be bounded sets of nonnegative matrices that define operators on l^2 and let $\alpha \geqslant \frac{1}{m}$ and $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$.*

If m is even, then

$$\gamma(\Psi_1^{(\alpha)}\circ\Psi_2^{(\alpha)}\circ\cdots\circ\Psi_m^{(\alpha)}) \leqslant r^{\frac{1}{m}}(\Sigma_\alpha) \leqslant (r(\Psi_1^*\Psi_2\Psi_3^*\Psi_4\cdots\Psi_{m-1}^*\Psi_m)\, r(\Psi_m\Psi_{m-1}^*\cdots\Psi_4\Psi_3^*\Psi_2\Psi_1^*))^{\frac{\alpha}{2}}, \tag{75}$$

where

$$\Sigma_\alpha = (\Psi_1^*\Psi_2\Psi_3^*\Psi_4\cdots\Psi_{m-1}^*\Psi_m)^{(\alpha)}\circ(\Psi_2^*\Psi_3\Psi_4^*\Psi_5\cdots\Psi_m^*\Psi_1)^{(\alpha)}\circ\cdots\circ (\Psi_{m-1}^*\Psi_m\Psi_1^*\Psi_2\cdots\Psi_{m-3}^*\Psi_{m-2})^{(\alpha)}\circ(\Psi_m^*\Psi_1\Psi_2^*\Psi_3\cdots\Psi_{m-2}^*\Psi_{m-1})^{(\alpha)}.$$

If m is odd then

$$\gamma(\Psi_1^{(\alpha)}\circ\Psi_2^{(\alpha)}\circ\cdots\circ\Psi_m^{(\alpha)}) \leqslant r^{\frac{1}{2m}}(\Omega_\alpha) \leqslant r^{\frac{\alpha}{2}}(\Psi_1\Psi_2^*\Psi_3\Psi_4^*\cdots\Psi_{m-2}\Psi_{m-1}^*\Psi_m\Psi_1^*\Psi_2\cdots\Psi_{m-1}\Psi_m^*) \tag{76}$$

where

$$\Omega_\alpha = (\Psi_1^*\Psi_2\Psi_3^*\Psi_4\cdots\Psi_{m-1}\Psi_m^*\Psi_1\Psi_2^*\Psi_3\cdots\Psi_{m-1}^*\Psi_m)^{(\alpha)}\circ (\Psi_2^*\Psi_3\Psi_4^*\Psi_5\cdots\Psi_{m-1}^*\Psi_m\Psi_1^*\Psi_2\Psi_3^*\cdots\Psi_m^*\Psi_1)^{(\alpha)}\circ\cdots\circ (\Psi_m^*\Psi_1\Psi_2^*\Psi_3\Psi_4^*\cdots\Psi_{m-1}^*\Psi_m\Psi_1^*\Psi_2\cdots\Psi_{m-2}^*\Psi_{m-1})^{(\alpha)}.$$

The following corollary follows from (75).

COROLLARY 6. *Let Ψ and Σ be bounded sets of nonnegative matrices that define operators on l^2, let $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$ and $\alpha \geqslant \frac{1}{2}$. Then*

$$\gamma(\Psi^{(\alpha)} \circ \Sigma^{(\alpha)}) \leqslant r\left((\Psi^*\Sigma)^{(\alpha)} \circ (\Sigma^*\Psi)^{(\alpha)}\right)^{\frac{1}{2}} \leqslant r(\Psi^*\Sigma)^{\alpha} = r(\Psi\Sigma^*)^{\alpha}.$$

The following result is an essential version of [5, Corollary 3.17]. It follows from Corollary 6 and Proposition 14.

COROLLARY 7. *Let $\alpha \geqslant \frac{1}{2}$ and let Ψ and Σ be bounded sets of nonnegative matrices that define operators on l^2. If $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$, then*

$$\gamma(\Psi^{(\alpha)} \circ \Sigma^{(\alpha)}) \leqslant r((\Psi^*\Sigma)^{(\alpha)} \circ (\Sigma^*\Psi)^{(\alpha)})^{\frac{1}{2}}$$
$$\leqslant r((\Psi^*\Sigma)^{(\alpha)} \circ (\Psi^*\Sigma)^{(\alpha)})^{\frac{1}{2}} \leqslant r(\Psi^*\Sigma)^{\alpha}.$$

Again, to our knowledge even the following singelton set case (which as an essential version of [33, Theorem 4.4, (4.9)]) is new.

COROLLARY 8. *Let A and B be nonnegative matrices that define operators on l^2 and $\alpha \geqslant \frac{1}{2}$. Then*

$$\gamma(A^{(\alpha)} \circ B^{(\alpha)}) \leqslant \rho_{ess}\left((A^*B)^{(\alpha)} \circ (B^*A)^{(\alpha)}\right)^{\frac{1}{2}}$$
$$\leqslant \rho_{ess}\left((A^*B)^{(\alpha)} \circ (A^*B)^{(\alpha)}\right)^{\frac{1}{2}} \leqslant \rho_{ess}(A^*B)^{\alpha} = \rho_{ess}(AB^*)^{\alpha}.$$

We conclude the article by stating additional results that are essential versions of [5, Theorems 3.5 and 3.6, Corollary 3.7, Theorems 3.8, 3.11 and 3.13, Corollary 3.15], respectively. The results follow from (69), [7, Theorem 3.2(i)] and Theorem 8 and are proved in a similar way than results in [5]. To avoid repetition of ideas we omit the details of the proof.

THEOREM 18. *Let m be odd and let $\Psi_1, \ldots, \Psi_m$ be bounded sets of positive kernel operators on $L^2(X, \mu)$. For $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$ we have*

$$\begin{aligned}
&\gamma(\Psi_1^{(\frac{1}{m})} \circ \cdots \circ \Psi_m^{(\frac{1}{m})}) \\
&\leqslant r((\Psi_1\Psi_2^*)^{(\frac{1}{m})} \circ \cdots \circ (\Psi_m\Psi_1^*)^{(\frac{1}{m})} \circ (\Psi_2\Psi_3^*)^{(\frac{1}{m})} \circ \cdots \circ (\Psi_{m-1}\Psi_m^*)^{(\frac{1}{m})})^{\frac{1}{2}} \\
&\leqslant r(\Omega_1^{(\frac{1}{m})} \circ \cdots \circ \Omega_m^{(\frac{1}{m})})^{\frac{1}{2m}} \\
&\leqslant r(\Psi_1\Psi_2^* \cdots \Psi_{m-2}\Psi_{m-1}^*\Psi_m\Psi_1^* \cdots \Psi_{m-2}^*\Psi_{m-1}\Psi_m^*)^{\frac{1}{2m}},
\end{aligned} \tag{77}$$

where

$$\Omega_j = \Psi_{2j-1}\Psi_{2j}^* \cdots \Psi_{m-2}\Psi_{m-1}^*\Psi_m\Psi_1^*\Psi_2\Psi_3^* \cdots \Psi_{m-1}\Psi_m^*\Psi_1\Psi_2^* \cdots \Psi_{2j-3}\Psi_{2j-2}^*$$

for $1 \leqslant j \leqslant \frac{m-1}{2}$, *and*

$$\Omega_{\frac{m+1}{2}} = \Psi_m \Psi_1^* \Psi_2 \Psi_3^* \cdots \Psi_{m-1} \Psi_m^* \Psi_1 \Psi_2^* \Psi_3 \Psi_4^* \cdots \Psi_{m-2} \Psi_{m-1}^*,$$
$$\Omega_j = \Psi_{2j-m-1} \Psi_{2j-m}^* \cdots \Psi_{m-1} \Psi_m^* \Psi_1 \Psi_2^* \Psi_3 \Psi_4^* \cdots \Psi_m \Psi_1^* \cdots \Psi_{2j-m-3} \Psi_{2j-m-2}^*$$

for $\frac{m+3}{2} \leqslant j \leqslant m$.

THEOREM 19. *Let $m \in \mathbb{N}$ be odd and let $\Psi_1, \ldots, \Psi_m$ be bounded sets of nonnegative matrices that define operators on l^2. If $\alpha \geqslant \frac{1}{m}$ and if $\Omega_1, \ldots, \Omega_m$ are sets defined in Theorem 18, then for $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$*

$$\begin{aligned} &\gamma(\Psi_1^{(\alpha)} \circ \Psi_2^{(\alpha)} \circ \cdots \circ \Psi_m^{(\alpha)}) \\ &\leqslant r((\Psi_1\Psi_2^*)^{(\alpha)} \circ \cdots \circ (\Psi_{m-2}\Psi_{m-1}^*)^{(\alpha)} \circ (\Psi_m\Psi_1^*)^{(\alpha)} \circ \cdots \circ (\Psi_{m-1}\Psi_m^*)^{(\alpha)})^{\frac{1}{2}} \\ &\leqslant r(\Omega_1^{(\alpha)} \circ \cdots \circ \Omega_m^{(\alpha)})^{\frac{1}{2m}} \\ &\leqslant r(\Psi_1\Psi_2^*\Psi_3\Psi_4^* \cdots \Psi_{m-2}\Psi_{m-1}^*\Psi_m\Psi_1^*\Psi_2\Psi_3^* \cdots \Psi_{m-1}\Psi_m^*)^{\frac{\alpha}{2}}. \end{aligned} \tag{78}$$

COROLLARY 9. *(i) Let Ψ and Σ be bounded sets of positive kernel operators on $L^2(X,\mu)$ and $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$. Then*

$$\begin{aligned} &\gamma(\Psi^{(\frac{1}{3})} \circ (\Sigma^*)^{(\frac{1}{3})} \circ \Psi^{(\frac{1}{3})}) \leqslant r((\Psi^*\Sigma^*)^{(\frac{1}{3})} \circ (\Psi^*\Psi)^{(\frac{1}{3})} \circ (\Sigma\Psi)^{(\frac{1}{3})})^{\frac{1}{2}} \\ &\leqslant r((\Psi^*\Sigma^*\Psi^*\Psi\Sigma\Psi)^{(\frac{1}{3})} \circ (\Psi^*\Psi\Sigma\Psi\Psi^*\Sigma^*)^{(\frac{1}{3})} \circ (\Sigma\Psi\Psi^*\Sigma^*\Psi^*\Psi)^{(\frac{1}{3})})^{\frac{1}{6}} \leqslant \gamma(\Psi\Sigma\Psi)^{\frac{1}{3}}. \end{aligned} \tag{79}$$

(ii) If Ψ and Σ are bounded sets of nonnegative matrices that define operators on l^2 and if $\alpha \geqslant \frac{1}{3}$ then

$$\begin{aligned} &\gamma(\Psi^{(\alpha)} \circ (\Sigma^*)^{(\alpha)} \circ \Psi^{(\alpha)}) \leqslant r((\Psi^*\Sigma^*)^{(\alpha)} \circ (\Psi^*\Psi)^{(\alpha)} \circ (\Sigma\Psi)^{(\alpha)})^{\frac{1}{2}} \\ &\leqslant r((\Psi^*\Sigma^*\Psi^*\Psi\Sigma\Psi)^{(\alpha)} \circ (\Psi^*\Psi\Sigma\Psi\Psi^*\Sigma^*)^{(\alpha)} \circ (\Sigma\Psi\Psi^*\Sigma^*\Psi^*\Psi)^{(\alpha)})^{\frac{1}{6}} \leqslant \gamma(\Psi\Sigma\Psi)^{\alpha}. \end{aligned} \tag{80}$$

Let S_m denote the group of permutations of the set $\{1, \ldots, m\}$.

THEOREM 20. *Let m be even, $\tau, \nu \in S_m$, and let $\Psi_1, \ldots, \Psi_m$ be bounded sets of positive kernel operators on $L^2(X,\mu)$. Denote $\Sigma_j = \Psi_{\tau(2j-1)}^* \Psi_{\tau(2j)}$ and $\Sigma_{\frac{m}{2}+j} = \Psi_{\tau(2j)}^* \Psi_{\tau(2j-1)} = \Sigma_j^*$ for $j = 1, \ldots, \frac{m}{2}$. Let $\Omega_i = \Sigma_{\nu(i)} \cdots \Sigma_{\nu(m)} \Sigma_{\nu(1)} \cdots \Sigma_{\nu(i-1)}$ for $i = 1, \ldots, m$ and $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$.*

(i) Then

$$\begin{aligned} &\gamma(\Psi_1^{(\frac{1}{m})} \circ \cdots \circ \Psi_m^{(\frac{1}{m})}) \leqslant r(\Sigma_1^{(\frac{1}{m})} \circ \cdots \circ \Sigma_m^{(\frac{1}{m})})^{\frac{1}{2}} \\ &\leqslant r(\Omega_1^{(\frac{1}{m})} \circ \cdots \circ \Omega_m^{(\frac{1}{m})}) \leqslant r^{\frac{1}{2m}}(\Sigma_{\nu(1)} \cdots \Sigma_{\nu(m)})^{\frac{1}{2m}}. \end{aligned} \tag{81}$$

(ii) If $\Psi_1,\dots,\Psi_m$ are bounded sets of nonnegative matrices that define operators on l^2 and if $\alpha \geqslant \frac{1}{m}$, then

$$\gamma(\Psi_1^{(\alpha)}\circ\cdots\circ\Psi_m^{(\alpha)}) \leqslant r(\Sigma_1^{(\alpha)}\circ\cdots\circ\Sigma_m^{(\alpha)})^{\frac{1}{2}} \leqslant r(\Omega_1^{(\alpha)}\circ\cdots\circ\Omega_m^{(\alpha)})^{\frac{1}{2m}} \leqslant r(\Sigma_{\nu(1)}\cdots\Sigma_{\nu(m)})^{\frac{\alpha}{2}}. \tag{82}$$

THEOREM 21. *Let $m\in\mathbb{N}$ be even, $\alpha \geqslant \frac{2}{m}$, $\tau\in S_m$ and let $\Psi_1,\dots,\Psi_m$ be bounded sets of nonnegative matrices that define operators on l^2. Let Σ_j for $j=1,\dots,m$ be as in Theorem 20 and denote $\Theta_i = \Sigma_i\cdots\Sigma_{\frac{m}{2}}\Sigma_1\cdots\Sigma_{i-1}$ for $i=1,\dots,\frac{m}{2}$. If $r\in\{\rho_{ess},\hat{\rho}_{ess}\}$, then*

$$\gamma(\Psi_1^{(\alpha)}\circ\cdots\circ\Psi_m^{(\alpha)}) \leqslant r(\Sigma_1^{(\alpha)}\circ\cdots\circ\Sigma_m^{(\alpha)})^{\frac{1}{2}} \leqslant r(\Sigma_1^{(\alpha)}\circ\cdots\circ\Sigma_{\frac{m}{2}}^{(\alpha)})$$
$$= r((\Psi_{\tau(1)}^*\Psi_{\tau(2)})^{(\alpha)}\circ(\Psi_{\tau(3)}^*\Psi_{\tau(4)})^{(\alpha)}\circ\cdots\circ(\Psi_{\tau(m-1)}^*\Psi_{\tau(m)})^{(\alpha)})$$
$$\leqslant r(\Theta_1^{(\alpha)}\circ\Theta_2^{(\alpha)}\circ\cdots\circ\Theta_{\frac{m}{2}}^{(\alpha)})^{\frac{2}{m}} \leqslant r(\Psi_{\tau(1)}^*\Psi_{\tau(2)}\Psi_{\tau(3)}^*\Psi_{\tau(4)}\cdots\Psi_{\tau(m-1)}^*\Psi_{\tau(m)})^{\alpha}. \tag{83}$$

THEOREM 22. *Let $\Psi_1,\dots,\Psi_m$ be bounded sets of positive kernel operators on $L^2(X,\mu)$ and $\tau,\nu\in S_m$. Denote $\Omega_j = \Psi_{\tau(j)}^*\Psi_{\nu(j)}\cdots\Psi_{\tau(m)}^*\Psi_{\nu(m)}\cdots\Psi_{\tau(j-1)}^*\Psi_{\nu(j-1)}$ for $j=1,\dots,m$. Let $r\in\{\rho_{ess},\hat{\rho}_{ess}\}$.*

(i) Then

$$\gamma(\Psi_1^{(\frac{1}{m})}\circ\cdots\circ\Psi_m^{(\frac{1}{m})}) \leqslant r((\Psi_{\tau(1)}^*\Psi_{\nu(1)})^{(\frac{1}{m})}\circ\cdots\circ(\Psi_{\tau(m)}^*\Psi_{\nu(m)})^{(\frac{1}{m})})^{\frac{1}{2}}$$
$$\leqslant r((\Omega_1)^{(\frac{1}{m})}\circ\cdots\circ(\Omega_m)^{(\frac{1}{m})})^{\frac{1}{2m}} \leqslant r(\Psi_{\tau(1)}^*\Psi_{\nu(1)}\cdots\Psi_{\tau(m)}^*\Psi_{\nu(m)})^{\frac{1}{2m}}. \tag{84}$$

(ii) If $\Psi_1,\dots,\Psi_m$ are bounded sets of nonnegative matrices that define operators on l^2 and if $\alpha\geqslant\frac{1}{m}$, then

$$\gamma(\Psi_1^{(\alpha)}\circ\cdots\circ\Psi_m^{(\alpha)}) \leqslant r((\Psi_{\tau(1)}^*\Psi_{\nu(1)})^{(\alpha)}\circ\cdots\circ(\Psi_{\tau(m)}^*\Psi_{\nu(m)})^{(\alpha)})^{\frac{1}{2}}$$
$$\leqslant r(\Omega_1^{(\alpha)}\circ\cdots\circ\Omega_m^{(\alpha)})^{\frac{1}{2m}} \leqslant r(\Psi_{\tau(1)}^*\Psi_{\nu(1)}\cdots\Psi_{\tau(m)}^*\Psi_{\nu(m)})^{\frac{\alpha}{2}}. \tag{85}$$

COROLLARY 10. *Let m be odd and let $\Psi_1,\dots,\Psi_m$ be bounded sets of positive kernel operators on $L^2(X,\mu)$. Let Ω_j for $j=1,\dots,m$ be as in Theorem 22 and let $r\in\{\rho_{ess},\hat{\rho}_{ess}\}$.*

(i) Then

$$\gamma(\Psi_1^{(\frac{1}{m})}\circ\cdots\circ\Psi_m^{(\frac{1}{m})})$$
$$\leqslant r((\Psi_1^*\Psi_2)^{(\frac{1}{m})}\circ\cdots\circ(\Psi_{m-2}^*\Psi_{m-1})^{(\frac{1}{m})}\circ(\Psi_m^*\Psi_1)^{(\frac{1}{m})}\circ(\Psi_2^*\Psi_3)^{(\frac{1}{m})}\circ\cdots\circ$$
$$(\Psi_{m-1}^*\Psi_m)^{(\frac{1}{m})})^{\frac{1}{2}} \leqslant r(\Omega_1^{(\frac{1}{m})}\circ\cdots\circ\Omega_m^{(\frac{1}{m})})^{\frac{1}{2m}}$$
$$\leqslant r(\Psi_1^*\Psi_2\cdots\Psi_{m-2}^*\Psi_{m-1}\Psi_m^*\Psi_1\Psi_2^*\Psi_3\cdots\Psi_{m-1}^*\Psi_m)^{\frac{1}{2m}}$$
$$= r(\Psi_1\Psi_2^*\Psi_3\cdots\Psi_{m-1}^*\Psi_m\Psi_1^*\Psi_2\cdots\Psi_{m-2}^*\Psi_{m-1}\Psi_m^*)^{\frac{1}{2m}}.$$

(ii) If $\Psi_1, \ldots, \Psi_m$ are nonnegative matrices that define operators on l^2 and if $\alpha \geqslant \frac{1}{m}$, then

$$\gamma(\Psi_1^{(\alpha)} \circ \cdots \circ \Psi_m^{(\alpha)})$$
$$\leqslant r((\Psi_1^*\Psi_2)^{(\alpha)} \circ \cdots \circ (\Psi_{m-2}^*\Psi_{m-1})^{(\alpha)} \circ (\Psi_m^*\Psi_1)^{(\alpha)} \circ (\Psi_2^*\Psi_3)^{(\alpha)} \circ$$
$$\cdots \circ (\Psi_{m-1}^*\Psi_m)^{(\alpha)})^{\frac{1}{2}} \leqslant r(\Omega_1^{(\alpha)} \circ \cdots \circ \Omega_m^{(\alpha)})^{\frac{1}{2m}}$$
$$\leqslant r(\Psi_1^*\Psi_2 \cdots \Psi_{m-2}^*\Psi_{m-1}\Psi_m^*\Psi_1\Psi_2^*\Psi_3 \cdots \Psi_{m-1}^*\Psi_m)^{\frac{\alpha}{2}}$$
$$= r(\Psi_1\Psi_2^*\Psi_3 \cdots \Psi_{m-1}^*\Psi_m\Psi_1^*\Psi_2 \cdots \Psi_{m-2}^*\Psi_{m-1}\Psi_m^*)^{\frac{\alpha}{2}}.$$

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