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SHARP WEIGHTED POWER MEAN BOUNDS FOR TWO LEMNISCATE TYPE MEANS

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Abstract. In this paper, we present sharp weighted power mean bounds for two lemniscate type means, which were introduced by Neuman (Math Pannon **18** (1): 77–94, 2007). As a corollary, we get two new sharp power mean bounds for two lemniscate type means.

1. Introduction

The Gauss's arc lemniscate sine and the hyperbolic arc lemniscate sine functions are defined by

$$\operatorname{arcsl} x = \int_0^x \frac{dt}{\sqrt{1-t^4}}, \ |x| \leqslant 1 \tag{1.1}$$

and

$$\operatorname{arcslh} x = \int_0^x \frac{dt}{\sqrt{1+t^4}}, \ x \in \mathbb{R} \tag{1.2}$$

(cf. [3, p. 259], [5, (2.5)–(2.6)]). As is well known, the arc length s measured from the origin to a point with polar coordinates on the Bernoulli lemniscate $r^2 = \cos(2\theta)$ is $s = \operatorname{arcsl} r$. It is apparent from (1.1) and (1.2) that $x \mapsto \operatorname{arcsl} x$ is an odd function in $[-1,1]$, and strictly increasing from $[0,1]$ onto $[0,\omega]$, and $x \mapsto \operatorname{arcslh} x$ is also an odd function in $\mathbb{R}$ and increasing from $[0,+\infty)$ onto $[0,\sqrt{2}\omega]$. Here and in what follows ω is denoted by the first lemniscate constant (cf. [9, (19.20.2)]) as follows

$$\omega = \operatorname{arcsl}(1) = \frac{1}{\sqrt{2}}\mathscr{K}(1/\sqrt{2}) = \frac{[\Gamma(1/4)]^2}{4\sqrt{2}\pi} = 1.31103\cdots, \tag{1.3}$$

where

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$$

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is the classical Euler gamma function and

$$\mathcal{K}(r)=\int_0^{\pi/2}\frac{dt}{\sqrt{1-r^2\sin^2 t}}=\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;r^2\right),\ r\in(0,1)$$

is the complete elliptic integral of the first kind. And the function $F(a,b;c;x)$ is the Gaussian hypergeometric function for $a,b,c\in\mathbb{R}$ with $c\neq 0,-1,-2,\cdots$, defined by

$$F(a,b;c;x):={}_2F_1(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)}\frac{x^n}{n!},\ |x|<1,$$

where $(a,0)=1$ and $(a,n)=a(a+1)(a+2)\cdots(a+n-1)=\Gamma(a+n)/\Gamma(a)$ is the shifted factorial function. For basic properties of the above functions, the reader can refer to the literature [1, 8, 11]. Indeed, the Gauss's arc lemniscate sine and the hyperbolic arc lemniscate sine functions can be also expressed in terms of $F(a,b;c;x)$ (cf. [2]):

$$\operatorname{arcsl}x=xF\left(\frac{1}{2},\frac{1}{4};\frac{5}{4};x^4\right),\quad \operatorname{arcslh}x=xF\left(\frac{1}{2},\frac{1}{4};\frac{5}{4};-x^4\right).$$

An alternative pair of the arc lemniscate functions, Gauss' arc lemniscate tangent and the hyperbolic arc lemniscate tangent functions are defined by

$$\operatorname{arctl}x=\operatorname{arcsl}\left(\frac{x}{\sqrt[4]{1+x^4}}\right),\ x\in\mathbb{R} \tag{1.4}$$

and

$$\operatorname{arctlh}x=\operatorname{arcslh}\left(\frac{x}{\sqrt[4]{1-x^4}}\right),\ x\in(-1,1), \tag{1.5}$$

respectively (cf. [6, Proposition 3.1]). Throughout this paper, we denote by $\kappa=\operatorname{arctlh}(1)=\operatorname{arcslh}(+\infty)=\sqrt{2}\omega=1.85407\cdots$.

Let $a,b>0$ and $\varpi\in(0,1)$. Then the lemniscate mean $LM(a,b)$ and the weighted power mean $M_p(a,b;\varpi)$ are defined by

$$LM(a,b)=\begin{cases}\dfrac{\sqrt{a^2-b^2}}{\left(\operatorname{arcsl}\sqrt[4]{1-b^2/a^2}\right)^2}, & a>b,\\ \dfrac{\sqrt{b^2-a^2}}{\left(\operatorname{arcslh}\sqrt[4]{b^2/a^2-1}\right)^2}, & a<b,\\ a, & a=b\end{cases} \tag{1.6}$$

and

$$M_p(a,b;\varpi)=\begin{cases}\left[\varpi a^p+(1-\varpi)b^p\right]^{1/p}, & p\neq 0,\\ a^{\varpi}b^{1-\varpi}, & p=0,\end{cases} \tag{1.7}$$

respectively (cf. [4, 6, 7, 14]). It is known that both $LM(a,b)$ and $M_p(a,b;\varpi)$ are non-symmetric with respect to their variables a and b except for $\varpi=1/2$, and the function $p\mapsto M_p(a,b;\varpi)$ is strictly increasing on $(-\infty,+\infty)$ for any fixed $\varpi\in(0,1)$ and $a,b>$

0 with $a \neq b$. In particular, when $\varpi = 1/2$, $M_p(a,b;\varpi)$ reduces to the power mean $M_p(a,b)$. Obviously, $M_0(a,b) = G(a,b)$, $M_1(a,b) = A(a,b)$ and $M_2(a,b) = Q(a,b)$ are the classical geometric, arithmetic, and the quadratic means of a and b, respectively.

In 2007, Neuman [6] further discussed four symmetric and homogenous means of two variables, which were derived from the lemniscate mean by replacing (a,b) with (G,A), (A,G), (A,Q) and (Q,A). Precisely, he introduced

$$LM_{G,A}(a,b) = LM(G(a,b),A(a,b)),$$

$$LM_{A,G}(a,b) = LM(A(a,b),G(a,b)),$$

$$LM_{A,Q}(a,b) = LM(A(a,b),Q(a,b)),$$

$$LM_{Q,A}(a,b) = LM(Q(a,b),A(a,b))$$

and obtained their explicit formulas as follows

$$\begin{aligned} LM_{G,A}(a,b) &= \frac{|a-b|}{2\left(\operatorname{arctlh}\sqrt{\frac{|a-b|}{a+b}}\right)^2}, \\ LM_{A,G}(a,b) &= \frac{|a-b|}{2\left(\operatorname{arcsl}\sqrt{\frac{|a-b|}{a+b}}\right)^2}, \end{aligned} \tag{1.8}$$

$$\begin{aligned} LM_{A,Q}(a,b) &= \frac{|a-b|}{2\left(\operatorname{arcslh}\sqrt{\frac{|a-b|}{a+b}}\right)^2}, \\ LM_{Q,A}(a,b) &= \frac{|a-b|}{2\left(\operatorname{arctl}\sqrt{\frac{|a-b|}{a+b}}\right)^2}. \end{aligned} \tag{1.9}$$

It was also proved in [6, (6.10)] that the inequalities

$$\begin{aligned} &G(a,b) < L(a,b) < LM_{G,A}(a,b) < LM_{A,G}(a,b) < P(a,b) < A(a,b) \\ &< M(a,b) < LM_{A,Q}(a,b) < LM_{Q,A}(a,b) < T(a,b) < Q(a,b) \end{aligned} \tag{1.10}$$

take place for all a and b with $a \neq b$. Here $L(a,b) = (a-b)/(\log a - \log b)$, $M(a,b) = (a-b)/[2\operatorname{arcsinh}((a-b)/(a+b))]$, $P(a,b) = (a-b)/[2\arcsin((a-b)/(a+b))]$, $T(a,b) = (a-b)/[2\arctan((a-b)/(a+b))]$ represent for logarithmic, Neuman-Sándor, the first and second Seiffert's means of two distinct positive numbers a and b.

Recently, motivated by (1.10), Zhao, Shen and Chu [18] established the sharp power mean bounds for the above four lemniscate type means, that is, they showed that, for α_1, α_2, α_3, α_4, β_1, β_2, β_3, $\beta_4 \in \mathbb{R}$, the double inequalities

$$\begin{aligned} M_{\alpha_1}(a,b) < LM_{G,A}(a,b) < M_{\beta_1}(a,b), \\ M_{\alpha_2}(a,b) < LM_{A,G}(a,b) < M_{\beta_2}(a,b), \end{aligned} \tag{1.11}$$

$$M_{\alpha_3}(a,b) < LM_{A,Q}(a,b) < M_{\beta_3}(a,b),$$
$$M_{\alpha_4}(a,b) < LM_{Q,A}(a,b) < M_{\beta_4}(a,b) \tag{1.12}$$

hold for all $a,b>0$ with $a \neq b$ with the best possible parameters

$$\alpha_1 = \frac{\log 2}{\log(2\kappa^2)} = 0.359\cdots, \quad \beta_1 = \frac{2}{5},$$

$$\alpha_2 = \frac{\log 2}{\log(2\omega^2)} = 0.561\cdots, \quad \beta_2 = \frac{3}{5},$$

$$\alpha_3 = \frac{\log 2}{\log(2\sigma^2)} = 1.279\cdots, \quad \beta_3 = \frac{7}{5},$$

$$\alpha_4 = \frac{\log 2}{\log(2\tau^2)} = 1.466\cdots, \quad \beta_4 = \frac{8}{5}.$$

For more inequalities for $LM_{G,A}$, $LM_{A,G}$, $LM_{A,Q}$ and $LM_{Q,A}$, see [12, 15, 16, 17, 19].

It was worthy noting that the questions of (1.11)–(1.12) are to find the best possible exponential parameters of the weighted power mean with weight $1/2$. In this paper, instead of searching the optimal exponential parameter p with a fixed weight ω, we shall determine the optimal weights $\alpha(p)$ and $\beta(p)$, $\lambda(p)$ and $\mu(p)$ depending on p, for any given exponential order $p \in \mathbb{R}$, such that

$$M_p(a,b;\alpha(p)) \leqslant LM_{A,G}(a,b) \leqslant M_p(a,b;\beta(p)), \tag{1.13}$$
$$M_p(a,b;\lambda(p)) \leqslant LM_{G,A}(a,b) \leqslant M_p(a,b;\mu(p)) \tag{1.14}$$

hold for all $a,b>0$. With the optimal weights obtained in Theorems 2.6 and 2.7, we derive the following two other sharp power mean bounds for $LM_{A,G}(a,b)$ and $LM_{G,A}(a,b)$ as the main results of this paper.

THEOREM 1.1. *Let* $p,q>0$, $\alpha_p = [1/(2\omega^2)]^p$. *If* $a>b>0$, *then the double inequality*

$$M_p(a,b;\alpha_p) < LM_{A,G}(a,b) < M_q(a,b;\alpha_q) \tag{1.15}$$

holds if and only if $p \geqslant p_0 = (\log 2)/\log(2\omega^2) = 0.561\cdots$ *and* $q \leqslant 1/2$.

THEOREM 1.2. *Let* $p,q>0$, $\lambda_p = [1/(2\kappa^2)]^p$. *If* $a>b>0$, *then the double inequality*

$$M_p(a,b;\lambda_p) < LM_{G,A}(a,b) < M_q(a,b;\lambda_q) \tag{1.16}$$

holds if and only if $p \geqslant p_1 = (\log 2)/\log(2\kappa^2) = 0.359\cdots$ *and* $q \leqslant 1/4$.

Besides, we also obtain the inequalities (1.11) with a new method. Similar questions of $LM_{A,Q}(a,b)$ and $LM_{Q,A}(a,b)$ shall be answered in a future paper.

2. Preliminaries

In order to facilitate computation, we now recall the derivative formulas of the arc lemniscate functions, by the definitions and the chain rule, as follows:

$$\frac{d\,\mathrm{arcsl}\,x}{dx} = \frac{1}{\sqrt{1-x^4}}, \quad \frac{d\,\mathrm{arctlh}\,x}{dx} = \frac{1}{(1-x^4)^{3/4}}, \quad |x|<1.$$

LEMMA 2.1. (cf. [1, Theorem 1.25]) *Let* $-\infty<a<b<\infty$, $f,g:[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b), *and* $g'(x)\neq 0$ *on* (a,b). *If* $f'(x)/g'(x)$ *is increasing (decreasing) on* (a,b), *then so are the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \textit{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ *is strict monotone, then the monotonicity in the conclusion is also strict.*

However, f'/g' is not always monotone in the whole interval but piecewise monotone. Now we introduce a useful auxiliary function $H_{f,g}$, which first appeared in [15] and is called H-function (cf. [10]) and makes a bridge between the derivatives of the ratios f/g and f'/g'. For $-\infty\leqslant a<b\leqslant\infty$, let f and g be differentiable on (a,b) and $g'\neq 0$ on (a,b). Then the function $H_{f,g}$ is defined by

$$H_{f,g} := \frac{f'}{g'}g - f. \tag{2.1}$$

For some basic properties of $H_{f,g}$, see [13, Properties 1,2]. In particular, if f and g are twice differentiable on (a,b), then we have

$$\left(\frac{f}{g}\right)' = \frac{g'}{g^2}\left(\frac{f'}{g'}g - f\right) = \frac{g'}{g^2}H_{f,g}, \tag{2.2}$$

$$H'_{f,g} = \left(\frac{f'}{g'}\right)' g. \tag{2.3}$$

LEMMA 2.2. *Let* $x\in(0,1)$. *Then the function*

$$h(x) = \frac{\sqrt{x}\left[(3-x^2)\,\mathrm{arcsl}\,\sqrt{x} - 3\sqrt{x(1-x^2)}\right]}{4\sqrt{x}\,\mathrm{arcsl}\,\sqrt{x} - 2\sqrt{1-x^2}\,[(\mathrm{arcsl}\,\sqrt{x})^2+x]}$$

is strictly increasing from $(0,1)$ *onto* $(2/5,1/2)$.

Proof. Let

$$h_1(x) = \frac{\sqrt{x}\left[(3-x^2)\,\mathrm{arcsl}\,\sqrt{x} - 3\sqrt{x(1-x^2)}\right]}{\sqrt{1-x^2}},$$

$$h_2(x) = \frac{4\sqrt{x}\,\mathrm{arcsl}\,\sqrt{x}}{\sqrt{1-x^2}} - 2[(\mathrm{arcsl}\,\sqrt{x})^2+x].$$

Then

$$h(x)=\frac{h_1(x)}{h_2(x)},\quad h_1(0)=h_2(0)=0, \tag{2.4}$$

$$h_1'(x)=\frac{(3x^4-2x^2+3)\,\mathrm{arcsl}\sqrt{x}}{2\sqrt{x}(1-x^2)^{3/2}}+\frac{5x^2-3}{2(1-x^2)},$$

$$h_2'(x)=\frac{4x^2\,\mathrm{arcsl}\sqrt{x}+2x^2\sqrt{x(1-x^2)}}{\sqrt{x}(1-x^2)^{3/2}},$$

and thereby

$$\frac{h_1'(x)}{h_2'(x)}=\frac{(3x^4-2x^2+3)\,\mathrm{arcsl}\sqrt{x}+(5x^2-3)\sqrt{x(1-x^2)}}{4x^2\left[2\,\mathrm{arcsl}\sqrt{x}+\sqrt{x(1-x^2)}\right]}:=h_3(x). \tag{2.5}$$

Let

$$h_4(x)=\frac{(3x^4-2x^2+3)\,\mathrm{arcsl}\sqrt{x}+(5x^2-3)\sqrt{x(1-x^2)}}{4x^2},$$

$$h_5(x)=2\,\mathrm{arcsl}\sqrt{x}+\sqrt{x(1-x^2)},$$

then

$$h_3(x)=\frac{h_4(x)}{h_5(x)},\quad h_4(0^+)=h_5(0)=0, \tag{2.6}$$

$$h_4'(x)=\frac{3(1+x^2)\left[(x^2-1)\,\mathrm{arcsl}\sqrt{x}+\sqrt{x(1-x^2)}\right]}{2x^3},$$

$$h_5'(x)=\frac{3\sqrt{x(1-x^2)}}{2x},$$

$$\frac{h_4'(x)}{h_5'(x)}=\frac{(1+x^2)\left[\sqrt{x}-\sqrt{1-x^2}\,\mathrm{arcsl}\sqrt{x}\right]}{x^{5/2}}:=h_6(x). \tag{2.7}$$

Differentiating $h_6(x)$ yields

$$h_6'(x)=\frac{(x^4-2x^2+5)\,\mathrm{arcsl}\sqrt{x}}{2x^{7/2}\sqrt{1-x^2}}-\frac{x^2+5}{2x^3}:=\frac{(x^4-2x^2+5)}{2x^{7/2}\sqrt{1-x^2}}h_7(x), \tag{2.8}$$

where

$$h_7(x)=\mathrm{arcsl}\sqrt{x}-\frac{(x^2+5)\sqrt{x(1-x^2)}}{x^4-2x^2+5}, \tag{2.9}$$

$$h_7(0)=0, \tag{2.10}$$

$$h_7'(x)=\frac{16x^{7/2}(3-x^2)}{\sqrt{1-x^2}(x^4-2x^2+5)^2}>0 \tag{2.11}$$

for all $x\in(0,1)$. This, together with (2.8)–(2.11), shows that $h_6'(x)>0$ for all $x\in(0,1)$, so that $h_6(x)$ is strictly increasing on $(0,1)$. By (2.4)–(2.7) and application of

Lemma 2.1 twice, the monotonicity of h on $(0,1)$ follows. For the limiting values, we have $h(1^-)=1/2$, and by l'Hôpital's rule we get

$$\begin{aligned}\lim_{x\to 0^+} h(x) &= \lim_{x\to 0^+} h_6(x) = \lim_{x\to 0^+} \frac{\sqrt{x}-\sqrt{1-x^2}\,\mathrm{arcsl}\sqrt{x}}{x^{5/2}} \\ &= \lim_{x\to 0^+} \frac{\sqrt{x}(1-x^2)^{-1/2}-\mathrm{arcsl}\sqrt{x}}{x^{5/2}}x^{5/2} \\ &= \lim_{x\to 0^+} \frac{2}{5(1-x^2)^{3/2}} = \frac{2}{5}.\end{aligned}$$

This completes the proof. □

LEMMA 2.3. *Let $x\in(0,1)$. Then the function*

$$g(x)=\frac{3\sqrt{x}\left[\mathrm{arctlh}\sqrt{x}-\sqrt{x}(1-x^2)^{1/4}\right]}{4\sqrt{x}\,\mathrm{arctlh}\sqrt{x}-2(1-x^2)^{3/4}\left[(\mathrm{arctlh}\sqrt{x})^2+x(1-x^2)^{-1/2}\right]}$$

is strictly increasing from $(0,1)$ onto $(3/5,3/4)$.

Proof. Let

$$g_1(x)=\frac{3\sqrt{x}\left[\mathrm{arctlh}\sqrt{x}-\sqrt{x}(1-x^2)^{1/4}\right]}{(1-x^2)^{3/4}},$$

$$g_2(x)=\frac{4\sqrt{x}\,\mathrm{arctlh}\sqrt{x}}{(1-x^2)^{3/4}}-2\left[(\mathrm{arctlh}\sqrt{x})^2+x(1-x^2)^{-1/2}\right].$$

Then

$$g(x)=\frac{g_1(x)}{g_2(x)},\quad g_1(0)=g_2(0)=0, \tag{2.12}$$

$$g_1'(x)=\frac{3(1+2x^2)\,\mathrm{arctlh}\sqrt{x}-3\sqrt{x}(1-x^2)^{1/4}}{2\sqrt{x}(1-x^2)^{7/4}},$$

$$g_2'(x)=\frac{6x^{3/2}\,\mathrm{arctlh}\sqrt{x}}{(1-x^2)^{7/4}},$$

and thereby

$$\frac{g_1'(x)}{g_2'(x)}=\frac{(1+2x^2)\,\mathrm{arctlh}\sqrt{x}-\sqrt{x}(1-x^2)^{1/4}}{4x^2\,\mathrm{arctlh}\sqrt{x}}=g_3(x). \tag{2.13}$$

Let

$$g_4(x)=\frac{(1+2x^2)\,\mathrm{arctlh}\sqrt{x}-\sqrt{x}(1-x^2)^{1/4}}{4x^2},$$

$$g_5(x)=\mathrm{arctlh}\sqrt{x},$$

then

$$g_3(x)=\frac{g_4(x)}{g_5(x)},\quad g_4(0^+)=g_5(0)=0, \tag{2.14}$$

$$g_4'(x) = \frac{\sqrt{x} - (1-x^2)^{3/4} \operatorname{arctlh} \sqrt{x}}{2(1-x^2)^{3/4} x^3},$$

$$g_5'(x) = \frac{1}{2\sqrt{x}(1-x^2)^{3/4}},$$

and

$$\frac{g_4'(x)}{g_5'(x)} = \frac{x - \sqrt{x}(1-x^2)^{3/4} \operatorname{arctlh} \sqrt{x}}{x^3} := g_6(x). \tag{2.15}$$

Differentiating g_6 yields

$$g_6'(x) = \frac{(5-2x^2) \operatorname{arctlh} \sqrt{x} - 5\sqrt{x}(1-x^2)^{1/4}}{2x^{7/2}(1-x^2)^{1/4}} := \frac{5-2x^2}{2x^{7/2}(1-x^2)^{1/4}} g_7(x), \tag{2.16}$$

where

$$g_7(x) = \operatorname{arctlh} \sqrt{x} - \frac{5\sqrt{x}(1-x^2)^{1/4}}{5-2x^2}, \quad g_7(0) = 0, \tag{2.17}$$

$$g_7'(x) = \frac{24x^{7/2}}{2(1-x^2)^{3/4}(5-2x^2)^2} > 0 \tag{2.18}$$

for all $x \in (0,1)$. This, together with (2.16)–(2.18), shows that $g_6'(x) > 0$ for all $x \in (0,1)$, so that $g_6(x)$ is strictly increasing on $(0,1)$. By (2.12)–(2.15) and application of Lemma 2.1 twice, the monotonicity of g on $(0,1)$ follows. For the limiting values, we have $g(1^-) = 3/4$, and by l'Hôpital's rule we have

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} g_6(x) = \lim_{x \to 0^+} \frac{\sqrt{x}(1-x^2)^{-3/4} - \operatorname{arctlh} \sqrt{x}}{x^{5/2}}$$
$$= \lim_{x \to 0^+} \frac{3}{5(1-x^2)^{7/4}} = \frac{3}{5}.$$

This completes the proof. □

THEOREM 2.4. *Let $p \in \mathbb{R}$ with $p \neq 0$. Define the function F_p on $(0,1)$ by*

$$F_p(x) = \frac{(1+x)^p - \left[x/(\operatorname{arcsl} \sqrt{x})^2\right]^p}{(1+x)^p - (1-x)^p}. \tag{2.19}$$

Then the following statements hold

(1) If $p \geqslant 3/5$, then F_p is strictly increasing from $(0,1)$ onto $(1/2, 1 - \left[1/(2\omega^2)\right]^p)$.

(2) If $p \leqslant 1/2$, then F_p is strictly decreasing on $(0,1)$. Furthermore, in this case $F_p(0^+) = 1/2$, and $F_p(1^-) = 1 - \left[1/(2\omega^2)\right]^p$ if $0 < p \leqslant 1/2$ and $F_p(1^-) = 0$ if $p < 0$.

(3) If $1/2 < p < 3/5$, then there exists a unique point $x_0^* = x_0^*(p) \in (0,1)$ such that F_p is strictly decreasing on $(0,x_0^*)$, and strictly increasing on $(x_0^*,1)$. Consequently, inequalities

$$\sigma_0^* \leqslant F_p(x) < 1/2, \text{ if } p \in \left(\frac{1}{2}, p_0\right], \tag{2.20}$$

$$\sigma_0^* \leqslant F_p(x) < 1 - \left(\frac{1}{2\omega^2}\right)^p, \text{ if } p \in (p_0, 3/5) \tag{2.21}$$

hold for all $x \in (0,1)$. Here $p_0 = (\log 2)/\log(2\omega^2)$ and $\sigma_0^* = F_p(x_0^*) < \min\{1/2, 1 - [1/(2\omega^2)]^p\}$. The right-hand side of (2.20) (resp. (2.21)) can be arrived at as $r \to 0^+$ (resp. $r \to 1^-$).

Proof. Let

$$\psi_1(x) = 1 - \left[\frac{x}{(1+x)(\operatorname{arcsl}\sqrt{x})^2}\right]^p, \quad \psi_2(x) = 1 - \left(\frac{1-x}{1+x}\right)^p.$$

Then

$$F_p(x) = \frac{\psi_1(x)}{\psi_2(x)}, \quad \psi_1(0^+) = \psi_2(0) = 0, \tag{2.22}$$

$$\psi_1'(x) = p\left[\frac{x}{(1+x)(\operatorname{arcsl}\sqrt{x})^2}\right]^{p-1} \frac{\sqrt{x}(1+x) - \sqrt{1-x^2}\operatorname{arcsl}\sqrt{x}}{\sqrt{1-x^2}(1+x)^2(\operatorname{arcsl}\sqrt{x})^3},$$

$$\psi_2'(x) = 2p\left(\frac{1-x}{1+x}\right)^{p-1} \frac{1}{(1+x)^2},$$

and

$$\frac{\psi_1'(x)}{\psi_2'(x)} = \left[\frac{x}{(1-x)(\operatorname{arcsl}\sqrt{x})^2}\right]^{p-1} \frac{\sqrt{x}(1+x) - \sqrt{1-x^2}\operatorname{arcsl}\sqrt{x}}{2\sqrt{1-x^2}(\operatorname{arcsl}\sqrt{x})^3} := \psi_3(x). \tag{2.23}$$

By logarithmic differentiation, we obtain

$$\frac{\psi_3'(x)}{\psi_3(x)} = \frac{\sqrt{1+x}\operatorname{arcsl}\sqrt{x} - \sqrt{x(1-x)}}{x\sqrt{1+x}(1-x)\operatorname{arcsl}\sqrt{x}}[p - 1 + h(x)], \tag{2.24}$$

where $h(x)$ is defined in Lemma 2.2.

Next we divide the proof into three cases.

Case 1. $p \geqslant 3/5$. Since it can be easily know that $x \mapsto \sqrt{x(1-x^2)}\operatorname{arcsl}\sqrt{x} - x(1-x)$ is positive on $(0,1)$. This, together with (2.24) and Lemma 2.2, leads to the conclusion that $\psi_3'(x) > 0$ for all $x \in (0,1)$, so that $\psi_3(x)$, as well as $\psi_1'(x)/\psi_2'(x)$, is strictly increasing on $(0,1)$ due to (2.23). Therefore, the monotonicity of F_p follows from (2.22) and Lemma 2.1 immediately. Moreover, $F_p(1^-) = 1 - \left[1/(2\omega^2)\right]^p$, and

$$\lim_{x\to 0^+} F_p(x) = \lim_{x\to 0^+} \frac{\psi_1'(x)}{\psi_2'(x)} = \lim_{x\to 0^+} \frac{\sqrt{x}(1+x) - \sqrt{1-x^2}\,\mathrm{arcsl}\sqrt{x}}{2x^{3/2}}$$
$$= \lim_{x\to 0^+} \frac{\sqrt{x}(1+x)(1-x^2)^{-1/2} - \mathrm{arcsl}\sqrt{x}}{2x^{3/2}}$$
$$= \lim_{x\to 0^+} \frac{3-x}{6(1-x)\sqrt{1-x^2}} = \frac{1}{2}.$$

Case 2. $p \leqslant 1/2$. Then from (2.24) and Lemma 2.2 one has that $\psi_3'(x) < 0$ for all $x \in (0,1)$, and therefore $\psi_3(x)$ is strictly decreasing on $(0,1)$. So is $F_p(x)$ by (2.22), (2.23) and application of Lemma 2.1. Also in this case we have $F_p(0^+) = 1/2$, and while $F_p(1^-) = 1 - \left[1/(2\omega^2)\right]^p$ if $0 < p \leqslant 1/2$ and $F_p(1^-) = 0$ if $p < 0$.

Case 3. $1/2 < p < 3/5$. Then by (2.1), (2.2), (2.3) and simple computations we obtain

$$F_p'(x) = \left[\frac{\psi_1(x)}{\psi_2(x)}\right]' = \frac{\psi_2'(x)}{\psi_2^2(x)} H_{\psi_1,\psi_2}(x), \tag{2.25}$$

$$H_{\psi_1,\psi_2}'(x) = \left[\frac{\psi_1'(x)}{\psi_2'(x)}\right]' \psi_2(x) = \psi_3'(x)\psi_2(x), \tag{2.26}$$

Moreover, in this case, by (2.23)

$$\lim_{x\to 0^+} \frac{\psi_1'(x)}{\psi_2'(x)} = \frac{1}{2}, \quad \lim_{x\to 1^-} \frac{\psi_1'(x)}{\psi_2'(x)} = +\infty.$$

$$H_{\psi_1,\psi_2}(0^+) = 0, \quad H_{\psi_1,\psi_2}(1^-) = +\infty. \tag{2.27}$$

Equation (2.24) together with Lemma 2.2 shows that there exists $x_0 \in (0,1)$ such that $\psi_3'(x) < 0$ for $x \in (0,x_0)$ and $\psi_3'(x) > 0$ for $x \in (x_0,1)$. Since $\psi_2(x)$ is strictly increasing and positive on $(0,1)$ for $1/2 < p < 3/5$, then from (2.26) and (2.27) we can conclude that H_{ψ_1,ψ_2} decreases on $(0,x_0)$ and then increases on $(x_0,1)$, and there exists $x_0^* \in (x_0,1)$ such that $H_{\psi_1,\psi_2}(x) < 0$ for $x \in (0,x_0^*)$ and $H_{\psi_1,\psi_2}(x) > 0$ for $x \in (x_0^*,1)$, so that F_p is also first decreasing then increasing due to (2.25). Consequently, inequalities

$$\sigma_0^* := F_p(x_0^*) \leqslant F_p(x) < \max\{F_p(0^+), F_p(1^-)\} = \max\left\{\frac{1}{2}, 1 - \left(\frac{1}{2\omega^2}\right)^p\right\}$$

take place for all $x \in (0,1)$. It was observed that $p_0 = (\log 2)/\log(2\omega^2)$ is the unique root of the equation $1 - \left[1/(2\omega^2)\right]^p = 1/2$ on $(1/2, 3/5)$, and $p \mapsto 1 - \left[1/(2\omega^2)\right]^p$ is strictly increasing on $(-\infty,+\infty)$. This yields that $F_{p_0}(0^+) = F_{p_0}(1^-) = 1/2$, $F_p(0^+) < F_p(1^-)$ if $p \in (1/2, p_0)$ and $F_p(0^+) > F_p(1^-)$ if $p \in (p_0, 3/5)$, and therefore inequalities (2.20) and (2.21) follows. The remaining assertions in Theorem 2.4 are clear. □

THEOREM 2.5. *Let* $p \in \mathbb{R}$ *with* $p \neq 0$. *Define the function* G_p *on* $(0,1)$ *by*

$$G_p(x) = \frac{(1+x)^p - \left[x/(\operatorname{arctlh}\sqrt{x})^2\right]^p}{(1+x)^p - (1-x)^p}. \tag{2.28}$$

Then the following statements hold

(1) If $p \geqslant 2/5$, *then* G_p *is strictly increasing from* $(0,1)$ *onto* $\left(1/2, 1-\left(1/2\kappa^2\right)^p\right)$.

(2) If $p \leqslant 1/4$, *then* G_p *is strictly decreasing on* $(0,1)$. *Furthermore, in this case* $G_p(0^+) = 1/2$, *and* $G_p(1^-) = 1 - \left[1/(2\kappa^2)\right]^p$ *if* $0 < p \leqslant 1/4$ *and* $G_p(1^-) = 0$ *if* $p < 0$.

(3) If $1/4 < p < 2/5$, *then there exists a unique point* $x_1^* = x_1^*(p) \in (0,1)$ *such that* G_p *is strictly decreasing on* $(0, x_1^*)$, *and strictly increasing on* $(x_1^*, 1)$. *Consequently, inequalities*

$$\sigma_1^* \leqslant G_p(x) < 1/2, \ \text{ if } \ p \in \left(\frac{1}{4}, p_1\right], \tag{2.29}$$

$$\sigma_1^* \leqslant G_p(x) < 1 - \left(\frac{1}{2\kappa^2}\right)^p, \ \text{ if } \ p \in \left(p_1, \frac{2}{5}\right) \tag{2.30}$$

hold for all $x \in (0,1)$. *Here* $p_1 = (\log 2)/\log(2\kappa^2) = 0.359\cdots$ *and* $\sigma_1^* = G_p(x_1^*) < \min\{1/2, 1 - \left[1/(2\kappa^2)\right]^p\}$. *The right-hand side of* (2.29) (*resp.* (2.30)) *can be arrived at as* $r \to 0^+$ (*resp.* $r \to 1^-$).

Proof. Let

$$\varphi_1(x) = 1 - \left[\frac{x}{(1+x)(\operatorname{arctlh}\sqrt{x})^2}\right]^p, \quad \varphi_2(x) = 1 - \left(\frac{1-x}{1+x}\right)^p.$$

Then

$$G_p(x) = \frac{\varphi_1(x)}{\varphi_2(x)}, \quad \varphi_1(0^+) = \varphi_2(0) = 0, \tag{2.31}$$

$$\varphi_1'(x) = p\left[\frac{x}{(1+x)(\operatorname{arctlh}\sqrt{x})^2}\right]^{p-1} \frac{\sqrt{x}(1+x) - (1-x^2)^{3/4}\operatorname{arctlh}\sqrt{x}}{(1-x^2)^{3/4}(1+x)^2(\operatorname{arctlh}\sqrt{x})^3},$$

$$\varphi_2'(x) = 2p\left(\frac{1-x}{1+x}\right)^{p-1} \frac{1}{(1+x)^2},$$

and

$$\frac{\varphi_1'(x)}{\varphi_2'(x)} = \left[\frac{x}{(1-x)(\operatorname{arctlh}\sqrt{x})^2}\right]^{p-1} \frac{\sqrt{x}(1+x) - (1-x^2)^{3/4}\operatorname{arctlh}\sqrt{x}}{2(1-x^2)^{3/4}(\operatorname{arctlh}\sqrt{x})^3} := \varphi_3(x). \tag{2.32}$$

By logarithmic differentiation, we obtain

$$\begin{aligned}\frac{\varphi_3'(x)}{\varphi_3(x)} &= (p-1)\frac{(1-x^2)^{3/4}\operatorname{arctlh}\sqrt{x}-\sqrt{x}(1-x)}{x(1-x)(1-x^2)^{3/4}\operatorname{arctlh}\sqrt{x}} \\ &\quad + \frac{3\operatorname{arctlh}\sqrt{x}-3\sqrt{x}(1-x^2)^{1/4}}{2(1-x)\operatorname{arctlh}\sqrt{x}\left[x(1+x)-\sqrt{x}(1-x^2)^{3/4}\operatorname{arctlh}\sqrt{x}\right]} \\ &= \frac{(1-x^2)^{3/4}\operatorname{arctlh}\sqrt{x}-\sqrt{x}(1-x)}{x(1-x)(1-x^2)^{3/4}\operatorname{arctlh}\sqrt{x}}\left[p-1+g(x)\right], \end{aligned} \tag{2.33}$$

where $g(x)$ is defined in Lemma 2.3.

Following we can also divide the proof into three cases.

Case 1. $p \geqslant 2/5$. Since it can be easily know that $x \mapsto (1-x^2)^{3/4}\operatorname{arctlh}\sqrt{x} - \sqrt{x}(1-x)$ is positive on $(0,1)$, we conclude from (2.33) and Lemma 2.3 that $\varphi_3'(x) > 0$ for all $x \in (0,1)$. Hence $\varphi_3(x)$, as well as $\varphi_1'(x)/\varphi_2'(x)$, is strictly increasing on $(0,1)$ due to (2.32). Therefore, the monotonicity of G_p follows from (2.31) and Lemma 2.1 immediately. Moreover, $G_p(1^-) = 1-\left[1/(2\kappa^2)\right]^p$, and

$$\begin{aligned}\lim_{x\to 0^+} G_p(x) &= \lim_{x\to 0^+}\frac{\varphi_1'(x)}{\varphi_2'(x)} = \lim_{x\to 0^+}\frac{\sqrt{x}(1+x)-(1-x^2)^{3/4}\operatorname{arctlh}\sqrt{x}}{2x^{3/2}} \\ &= \lim_{x\to 0^+}\frac{\sqrt{x}(1+x)(1-x^2)^{-3/4}-\operatorname{arctlh}\sqrt{x}}{2x^{3/2}} \\ &= \lim_{x\to 0^+}\frac{(1+x)(1+2x)-(1-x^2)}{6x} = \frac{1}{2}.\end{aligned}$$

Case 2. $p \leqslant 1/4$. Then from (2.33) and Lemma 2.3 one has that $\varphi_3'(x) < 0$ for all $x \in (0,1)$, and therefore $\varphi_3(x)$ is strictly decreasing on $(0,1)$. So is $G_p(x)$ by (2.31), (2.32) and application of Lemma 2.1. Also in this case we have $G_p(0^+) = 1/2$, and while $G_p(1^-) = 1-\left[1/(2\kappa^2)\right]^p$ if $0 < p \leqslant 1/4$ and $G_p(1^-) = 0$ if $p < 0$.

Case 3. $1/4 < p < 2/5$. Then by (2.1), (2.2), (2.3) and simple computations we obtain

$$G_p'(x) = \left[\frac{\varphi_1(x)}{\varphi_2(x)}\right]' = \frac{\varphi_2'(x)}{\varphi_2^2(x)}H_{\varphi_1,\varphi_2}(x), \tag{2.34}$$

$$H_{\varphi_1,\varphi_2}'(x) = \left[\frac{\varphi_1'(x)}{\varphi_2'(x)}\right]'\varphi_2(x) = \varphi_3'(x)\varphi_2(x), \tag{2.35}$$

$$H_{\varphi_1,\varphi_2}(0^+) = 0, \qquad H_{\varphi_1,\varphi_2}(1^-) = +\infty. \tag{2.36}$$

Equation (2.33) together with Lemma 2.3 shows that there exists $x_1 \in (0,1)$ such that $\varphi_3'(x) < 0$ for $x \in (0,x_1)$ and $\varphi_3'(x) > 0$ for $x \in (x_1,1)$. Since $\varphi_2(x)$ is strictly increasing and positive on $(0,1)$ for $1/4 < p < 2/5$, then from (2.35) and (2.36) we can conclude that H_{φ_1,φ_2} decreases on $(0,x_1)$ and then increases on $(x_1,1)$, and there exists

$x_1^* \in (x_1, 1)$ such that $H_{\varphi_1,\varphi_2}(x) < 0$ for $x \in (0, x_1^*)$ and $H_{\varphi_1,\varphi_2}(x) > 0$ for $x \in (x_1^*, 1)$, so that G_p is also first decreasing then increasing due to (2.34). Consequently, inequalities

$$\sigma_1^* := G_p(x_1^*) \leqslant G_p(x) < \max\{G_p(0^+), G_p(1^-)\} = \max\left\{\frac{1}{2}, 1 - \left(\frac{1}{2\kappa^2}\right)^p\right\}$$

are valid for all $x \in (0,1)$. It was observed that $p_1 = (\log 2)/\log(2\kappa^2) = 0.359\cdots$ is the unique root of the equation $1 - \left[1/(2\kappa^2)\right]^p = 1/2$ on $(1/4, 2/5)$, and $p \mapsto 1 - \left[1/(2\kappa^2)\right]^p$ is strictly increasing on $(-\infty, +\infty)$. This yields that $G_{p_1}(0^+) = G_{p_1}(1^-) = 1/2$, $G_p(0^+) \geqslant G_p(1^-)$ if $p \in (1/4, p_1]$ and $G_p(0^+) < G_p(1^-)$ if $p \in (p_1, 2/5)$, and therefore inequalities (2.29) and (2.30) follows. The remaining assertions in Theorem 2.5 are clear. □

THEOREM 2.6. *Let $\alpha, \beta \in (0,1)$, $p_0 = (\log 2)/\log(2\omega^2) = 0.561\cdots$, and σ_0^* be defined in Theorem 2.4. Then for each fixed $p \in \mathbb{R}$, inequality*

$$M_p(a,b;\alpha) < LM_{A,G}(a,b) \leqslant M_p(a,b;\beta) \tag{2.37}$$

holds for all $a > b > 0$ if and only if $\alpha \leqslant \alpha^*(p)$ and $\beta \geqslant \beta^*(p)$, where

$$\alpha^*(p) = \begin{cases} 1/2, & p \in (-\infty, p_0], \\ \left[1/(2\omega^2)\right]^p, & p \in (p_0, \infty), \end{cases}$$
$$\beta^*(p) = \begin{cases} 1, & p \in (-\infty, 0], \\ \left[1/(2\omega^2)\right]^p, & p \in (0, 1/2], \\ 1 - \sigma_0^*, & p \in (1/2, 3/5), \\ 1/2, & p \in [3/5, \infty). \end{cases} \tag{2.38}$$

In particular, the equality of (2.37) holds only for $p \in (1/2, 3/5)$, $\beta = \beta^*(p)$ and some $(a.b)$ satisfying $F_p'(\frac{a-b}{a+b}) = 0$.

Proof. Since both $LM_{A,G}(a,b)$ and $M_p(a,b;\omega)$ are homogenous of degree one means of a and b, without loss of generality, we may assume that $a = 1+x > b = 1-x$ for $x \in (0,1)$, then $LM_{A,G}(a,b) = x/\left(\operatorname{arcsl}\sqrt{x}\right)^2$. In the following we divide into two cases $p = 0$ and $p \neq 0$ to complete the proof.

Case 1. $p = 0$. Then the inequality (2.37) can be written as

$$\alpha < 1 - \frac{\log\left[\frac{x}{(1+x)(\operatorname{arcsl}\sqrt{x})^2}\right]}{\log\left(\frac{1-x}{1+x}\right)} \leqslant \beta, \quad x \in (0,1). \tag{2.39}$$

It suffices to prove that the function $x \mapsto \log\left[\frac{x}{(1+x)(\operatorname{arcsl}\sqrt{x})^2}\right]/\log\left(\frac{1-x}{1+x}\right)$ is strictly decreasing from $(0,1)$ onto $(1/2, 1)$. Indeed, for $x \in (0,1)$, if we let

$$\eta(x) = \frac{\log\left[\frac{x}{(1+x)(\operatorname{arcsl}\sqrt{x})^2}\right]}{\log\left(\frac{1-x}{1+x}\right)}, \quad \eta_1(x) = \log\left[\frac{x}{(1+x)(\operatorname{arcsl}\sqrt{x})^2}\right],$$

$$\eta_2(x) = \log\left(\frac{1-x}{1+x}\right),$$

then

$$\eta(x) = \frac{\eta_1(x)}{\eta_2(x)}, \quad \eta_1(0^+) = \eta_2(0^+) = 0, \tag{2.40}$$

$$\frac{\eta_1'(x)}{\eta_2'(x)} = \left[\frac{x}{(1-x)(\operatorname{arcsl}\sqrt{x})^2}\right]^{-1} \frac{\sqrt{x}(1+x) - \sqrt{1-x^2}\operatorname{arcsl}\sqrt{x}}{2\sqrt{1-x^2}(\operatorname{arcsl}\sqrt{x})^3}, \tag{2.41}$$

which is (2.23) in the case of $p=0$. It follows from the Case 2 of Theorem 2.4 that $\eta_1'(x)/\eta_2'(x)$ is strictly decreasing on $(0,1)$, so is $\eta(x)$ by (2.40) and Lemma 2.1. For the limiting values, clearly $\eta(1^-)=0$, and by l'Hôpital's rule,

$$\lim_{x\to 0^+} \eta(x) = \lim_{x\to 0^+} \frac{\eta_1'(x)}{\eta_2'(x)} = \lim_{x\to 0^+} \frac{\sqrt{x}(1+x) - \sqrt{1-x^2}\operatorname{arcsl}\sqrt{x}}{2x^{3/2}} = \frac{1}{2}.$$

Case 2. $p \neq 0$. Then rewrite the inequality (2.37) as

$$\alpha < 1 - F_p(x) \leqslant \beta, \quad x \in (0,1), \tag{2.42}$$

where $F_p(x)$ is defined in (2.4). Therefore, the best possible constant $\alpha^*(p)$ and $\beta^*(p)$ in (2.38) can be obtained immediately by Theorem 2.4 and (2.42) (see Table 1).

Table 1.

p	$F_p(x)$	$1-F_p(x)$
$(-\infty,0]$	$\left(0,\frac{1}{2}\right)$	$\left(\frac{1}{2},1\right)$
$\left(0,\frac{1}{2}\right]$	$\left(1-\left(\frac{1}{2\omega^2}\right)^p,\frac{1}{2}\right)$	$\left(\frac{1}{2},\left(\frac{1}{2\omega^2}\right)^p\right)$
$\left(\frac{1}{2},p_0\right)$	$\left[\sigma_0^*,\frac{1}{2}\right)$	$\left(\frac{1}{2},1-\sigma_0^*\right]$
$\left(p_0,\frac{3}{5}\right)$	$\left[\sigma_0^*,1-\left(\frac{1}{2\omega^2}\right)^p\right)$	$\left(\left(\frac{1}{2\omega^2}\right)^p,1-\sigma_0^*\right]$
$\left[\frac{3}{5},\infty\right)$	$\left(\frac{1}{2},1-\left(\frac{1}{2\omega^2}\right)^p\right)$	$\left(\left(\frac{1}{2\omega^2}\right)^p,\frac{1}{2}\right)$

The Table 1 also shows that the right-side equality of (2.38) holds only for $p \in (1/2,3/5)$, $\beta = \beta^*(p)$ and the pair (a,b) satisfying $F_p'(\frac{a-b}{a+b}) = 0$. $\square$

THEOREM 2.7. *Let* $\lambda,\mu \in (0,1)$, $p_1 = (\log 2)/\log(2\kappa^2) = 0.359\cdots$, *and* σ_1^* *be defined in Theorem* 2.5. *Then for each fixed* $p \in \mathbb{R}$, *the inequality*

$$M_p(a,b;\lambda) < LM_{G,A}(a,b) \leqslant M_p(a,b;\mu) \tag{2.43}$$

holds for all $a > b > 0$ *if and only if* $\lambda \leqslant \lambda^*(p)$ *and* $\mu \geqslant \mu^*(p)$, *where*

$$\lambda^*(p) = \begin{cases} 1/2, & p \in (-\infty, p_1], \\ \left[1/(2\kappa^2)\right]^p, & p \in (p_1, \infty), \end{cases}$$
$$\mu^*(p) = \begin{cases} 1, & p \in (-\infty, 0], \\ \left[1/(2\kappa^2)\right]^p, & p \in (0, 1/4], \\ 1-\sigma_1^*, & p \in (1/4, 2/5), \\ 1/2, & p \in [2/5, \infty). \end{cases} \tag{2.44}$$

In particular, the equality of (2.43) *holds only for* $p \in (1/4, 2/5)$, $\mu = \mu^*(p)$ *and some* (a,b) *satisfying* $G_p'(\frac{a-b}{a+b}) = 0$.

Proof. Without loss of generality, we assume that $a = 1+x > b = 1-x$ for $x \in (0,1)$, thus $LM_{G,A}(a,b) = x/(\operatorname{arctlh}\sqrt{x})^2$. The proof will be split into two cases.

Case 1 $p = 0$. In this case, the inequality (2.43) is equivalent to

$$\lambda < 1 - \frac{\log\left[\frac{x}{(1+x)(\operatorname{arctlh}\sqrt{x})^2}\right]}{\log\left(\frac{1-x}{1+x}\right)} \leqslant \mu, \quad x \in (0,1). \tag{2.45}$$

It suffices to prove that the function $x \mapsto \log\left[\frac{x}{(1+x)(\operatorname{arctlh}\sqrt{x})^2}\right]/\log\left(\frac{1-x}{1+x}\right)$ is strictly decreasing from $(0,1)$ onto $(1/2,1)$. Indeed, for $x \in (0,1)$, if we let

$$\zeta(x) = \frac{\log\left[\frac{x}{(1+x)(\operatorname{arctlh}\sqrt{x})^2}\right]}{\log\left(\frac{1-x}{1+x}\right)}, \quad \zeta_1(x) = \log\left[\frac{x}{(1+x)(\operatorname{arctlh}\sqrt{x})^2}\right],$$
$$\zeta_2(x) = \log\left(\frac{1-x}{1+x}\right),$$

then

$$\zeta(x) = \frac{\zeta_1(x)}{\zeta_2(x)}, \quad \zeta_1(0^+) = \zeta_2(0) = 0, \tag{2.46}$$

$$\frac{\zeta_1'(x)}{\zeta_2'(x)} = \left[\frac{x}{(1-x)(\operatorname{arctlh}\sqrt{x})^2}\right]^{-1} \frac{\sqrt{x}(1+x) - (1-x^2)^{3/4}\operatorname{arctlh}\sqrt{x}}{2(1-x^2)^{3/4}(\operatorname{arctlh}\sqrt{x})^3}. \tag{2.47}$$

It follows from (2.47), (2.32) and (2.33) $\zeta_1'(x)/\zeta_2'(x)$ is strictly decreasing on $(0,1)$, so is $\zeta(x)$ by (2.46) and Lemma 2.1. For the limiting values, it is clear that $\zeta(1^-) = 0$, and by l'Hôpital's rule,

$$\lim_{x\to 0^+} \zeta(x) = \lim_{x\to 0^+} \frac{\zeta_1'(x)}{\zeta_2'(x)} = \lim_{x\to 0^+} \frac{\sqrt{x}(1+x) - (1-x^2)^{3/4}\operatorname{arctlh}\sqrt{x}}{2x^{3/2}} = \frac{1}{2}.$$

Case 2 $p \neq 0$. Then rewrite the inequality (2.43) as

$$\lambda < 1 - G_p(x) \leqslant \mu, \quad x \in (0,1), \tag{2.48}$$

where $G_p(x)$ is defined in (2.28). Therefore, the best possible constant $\lambda^*(p)$ and $\mu^*(p)$ in (2.44) can be obtained immediately by Theorem 2.5 and (2.48), for the detail see Table 2.

Table 2.

p	$G_p(x)$	$1-G_p(x)$
$(-\infty,0]$	$\left(0,\frac{1}{2}\right)$	$\left(\frac{1}{2},1\right)$
$\left(0,\frac{1}{4}\right]$	$\left(1-\left(\frac{1}{2\kappa^2}\right)^p,\frac{1}{2}\right)$	$\left(\frac{1}{2},\left(\frac{1}{2\kappa^2}\right)^p\right)$
$\left(\frac{1}{4},p_1\right]$	$\left[\sigma_1^*,\frac{1}{2}\right)$	$\left(\frac{1}{2},1-\sigma_1^*\right]$
$\left(p_1,\frac{2}{5}\right)$	$\left[\sigma_1^*,1-\left(\frac{1}{2\kappa^2}\right)^p\right)$	$\left(\left(\frac{1}{2\kappa^2}\right)^p,1-\sigma_1^*\right]$
$\left[\frac{2}{5},\infty\right)$	$\left(\frac{1}{2},1-\left(\frac{1}{2\kappa^2}\right)^p\right)$	$\left(\left(\frac{1}{2\kappa^2}\right)^p,\frac{1}{2}\right)$

Moreover, as in the proof of Theorem 2.6, the right-side equality of (2.48) holds only when $p \in (1/4,2/5)$ and $G_p'(x) = 0$. □

3. Proofs of Theorems 1.1 and 1.2

Since the proof of Theorem 1.2 can be completed by Theorem 2.7 and the same argument as in the proof of Theorem 1.1, then we omit the proof of Theorem 1.2.

Proof of Theorem 1.1. By substituting $(\alpha,p) = ([1/(2\omega)^2]^p, p_0)$ and $(\beta,p) = ([1/(2\omega^2)]^p, 1/2)$ into the inequality (1.15) and applying Theorem 2.6, we obtain the inequality (1.15) with $p = p_0$ and $q = 1/2$.

It was proved in [20, Lemma 3.1] or [14, Lemma 5] that, for any fixed $a,b > 0$ and $\tau \in (0,1)$, the function $p \mapsto M_p(a,b;\tau^p)$ is strictly decreasing on $(0,\infty)$. This implies that, to show that $M_{p_0}(a,b;\alpha_{p_0})$ and $M_{1/2}(a,b;\alpha_{1/2})$ are the best possible lower and upper power mean bounds of $LM_{A,G}(a,b)$, it suffices to prove that both inequalities $M_p(a,b;\alpha_p) < LM_{A,G}(a,b)$ and $M_p(a,b;\alpha_p) > LM_{A,G}(a,b)$ don't hold for all $a > b > 0$ when $p \in (1/2,p_0)$. Indeed, for any given $p \in (1/2,p_0)$, it follows from Theorem 2.6 that the double inequality

$$M_p(a,b;1/2) < LM_{A,G}(a,b) \leqslant M_p(a,b;1-\sigma_0^*) \tag{3.1}$$

takes place for all $a > b > 0$ with the best weight $1/2$ and $1-\sigma_0^*$. From Theorem 2.4(3), it is easy to see that, $1/2 > 1-[1/(2\omega^2)]^p > \sigma_0^*$, that is, $1/2 < \alpha_p = [1/(2\omega^2)]^p < 1-\sigma_0^*$ for $p \in (1/2, p_0)$ and thereby

$$M_p(a,b;1/2) < M_p(a,b;\alpha_p) < M_p(a,b;1-\sigma_0^*)$$

for all $a > b > 0$. This together with (3.1), implies that there exist (a_1,b_1) and $(a_2.b_2)$ with $a_1 > b_1 > 0$, $a_2 > b_2 > 0$ such that

$$M_p(a_1,b_1;\alpha_p) > LM_{A,G}(a_1.b_1),$$
$$M_p(a_2,b_2;\alpha_p) < LM_{A,G}(a_2,b_2).$$

Hence the proof of Theorem 1.1 is completed. □

COROLLARY 3.1. *Let $q_1, q_2 \in \mathbb{R}$. then the double inequality*

$$M_{q_1}(a,b) = M_{q_1}\left(a,b;\frac{1}{2}\right) < LM_{A,G}(a,b) < M_{q_2}\left(a,b;\frac{1}{2}\right) = M_{q_2}(a,b) \tag{3.2}$$

holds for all $a,b > 0$ with $a \neq b$ with the best possible constants $q_1 = p_0 = (\log 2)/\log(2\omega^2) = 0.561\cdots$ and $q_2 = 3/5$.

Proof. Since both $LM_{A,G}(a,b)$ and $M_p(a,b)$ are the symmetric means of their variables a and b, without loss of generality, we may assume that $a > b > 0$. By substituting $(\alpha, p) = (1/2, p_0)$ and $(\beta, p) = (1/2, 3/5)$ into the inequality (2.37) and applying Theorem 2.6, we obtain the inequality (3.2) with $q_1 = p_0$ and $q_2 = 3/5$ immediately.

Now we show that $M_{p_0}(a,b)$ and $M_{3/5}(a,b)$ as the power mean bounds of $LM_{A,G}(a,b)$ are sharp. Indeed, for any given $p \in (p_0, 3/5)$, it follows from Theorem 2.6 that the double inequality

$$M_p(a,b;1/(2\omega^2)) < LM_{A,G}(a,b) \leqslant M_p(a,b;1-\sigma_0^*) \tag{3.3}$$

takes place for all $a > b > 0$ with the best weight $1-[1/(2\omega^2)]^p$ and $1-\sigma_0^*$. From Theorem 2.4(3), it is easy to see that $1-[1/(2\omega^2)]^p > 1/2 > \sigma_0^*$, that is, $[1/(2\omega^2)]^p < 1/2 < 1-\sigma_0^*$ for $p \in (p_0, 3/5)$ and thereby

$$M_p(a,b;1/(2\omega^2)) < M_p(a,b;1/2) < M_p(a,b;1-\sigma_0^*)$$

for all $a > b > 0$. This together with (3.3), implies that there exist (a_1^*, b_1^*) and (a_2^*, b_2^*) with $a_1^* > b_1^* > 0$, $a_2^* > b_2^* > 0$ such that

$$M_p(a_1^*, b_1^*; 1/2) > LM_{A,G}(a_1^*, b_1^*),$$
$$LM_{A,G}(a_2^*, b_2^*) > M_p(a_2^*, b_2^*; 1/2).$$

Hence the proof of Corollary 3.1 is completed. □

With the similar argument of Corollary 3.1, the following corollary can also be derived by Theorem 2.7, and its proof will be omitted.

COROLLARY 3.2. *Let $q_3, q_4 \in \mathbb{R}$. Then the double inequality*

$$M_{q_3}(a,b) = M_{q_3}(a,b;1/2) < LM_{G,A}(a,b) < M_{q_4}(a,b;1/2) = M_{q_4}(a,b) \tag{3.4}$$

holds for all $a,b>0$ with $a \neq b$ with the best possible constants $q_3 = (\log 2)/\log(2\kappa^2) = 0.359\cdots$ and $q_4 = 2/5$.

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