Journal of Mathematical Inequalities
Volume 18, Number 4 (2024), 1547–1555 doi:10.7153/jmi-2024-18-88

REFINEMENTS AND APPLICATIONS OF SEVERAL HERMITE—HADAMARD TYPE INEQUALITIES

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This paper is dedicated to Professor Bo-Yan Xi retired from Inner Mongolia Minzu University.

(*Communicated by M. Klaričić Bakula*)

Abstract. In the work, the authors establish several Hermite–Hadamard type inequalities, which refine those inequalities in Theorems 1 and 2 in the paper "C. E. M. Pearce and J. Pečarić, *Inequalities for differentiable mappings with application to special means and quadrature formula*, Appl. Math. Lett. **13** (2000), no. 2, 51–55", and give some applications to special means.

1. Introduction

The following definition is well known in the literature.

DEFINITION 1. A function f on an interval $I \subseteq \mathbb{R}$ is said to be convex (or concave, respectively) if the inequality

$$f(\lambda x+(1-\lambda)y) \lesseqgtr \lambda f(x)+(1-\lambda)f(y) \tag{1}$$

holds for any $x,y \in I$ and $\lambda \in [0,1]$.

Let f be a convex function on an interval $I \subseteq \mathbb{R}$. For any $a,b \in I$ with $a<b$, the well-known Hermite–Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2} \tag{2}$$

holds. If f is concave on I, then all the inequalities in (2) are reversed.

In [4], Dragomir and Agarwal established the following Hermite–Hadamard type inequality, which refines the right hand side inequality in (2) under different conditions.

Mathematics subject classification (2020): Primary 26D15; Secondary 26A51, 26E60.
Keywords and phrases: Hermite–Hadamard type inequality, convex function, refinement, integral inequality, mean.
The first author was supported in part by the National Natural Science Foundation of China (Grant No. 62161044), by the Natural Science Foundation of Inner Mongolia Autonomous Region (Grant No. 2021LHMS01008), and by the Key Laboratory of Infinite-dimensional Hamiltonian System and its Algorithm Application (Inner Mongolia Normal University), Ministry of Education (Grant No. 2023KFZD01).
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THEOREM 1. ([4, Theorem 2.2]) *Let f be a differentiable function on an open interval $I^\circ \subseteq \mathbb{R}$ and $a,b \in I^\circ$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \leqslant \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}.$$

In the paper [8], Pearce and Pečarić established the following Hermite–Hadamard type inequalities, which refine both inequalities in (2) under different requirements.

THEOREM 2. ([8, Theorems 1 and 2]) *Let f be a differentiable function on an open interval $I^\circ \subseteq \mathbb{R}$ and $a,b \in I^\circ$ with $a<b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q \geqslant 1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \leqslant \frac{b-a}{4}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{1/q} \tag{3}$$

and

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \leqslant \frac{b-a}{4}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{1/q}. \tag{4}$$

So far, the investigation of the Hermite–Hadamard type inequalities has been being exceptionally active; see, for example, the papers [1, 2, 3, 5, 6, 7, 9, 10].

In this paper, we will first improve the upper bounds of the inequalities (3) and (4), then establish some analogous results for $0<q<1$, and finally apply our newly-established results to estimate some errors on special means of two positive real numbers.

2. Lemmas

For proving our main results, we need the following lemmas.

LEMMA 1. ([4, Lemma 2.1]) *Let a,b belong to an open interval $I^\circ \subseteq \mathbb{R}$ with $a<b$ and let $f: I^\circ \to \mathbb{R}$ be a differentiable function such that $f' \in L_1([a,b])$. Then*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)\,\mathrm{d}t.$$

Making use of [5, Lemma 2.1] or [8, Eq. (2.2)], we derive

LEMMA 2. *Let a,b belong to an open interval $I^\circ \subseteq \mathbb{R}$ with $a<b$ and let $f: I^\circ \to \mathbb{R}$ be a differentiable function and $f' \in L_1([a,b])$. Then*

$$f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x = (b-a)\int_0^1 M(t)f'(ta+(1-t)b)\,\mathrm{d}t,$$

where

$$M(t)=\begin{cases} -t, & 0 \leqslant t < \frac{1}{2}; \\ 1-t, & \frac{1}{2} \leqslant t \leqslant 1. \end{cases} \tag{5}$$

LEMMA 3. *Let* $\xi,\eta\geqslant 0$ *and* $r>0$. *Then*

$$\int_0^1 |1-2t|[t\xi+(1-t)\eta]^r\,\mathrm{d}t \leqslant \begin{cases} \frac{1}{4}\left[\left(\frac{\xi+3\eta}{4}\right)^r+\left(\frac{3\xi+\eta}{4}\right)^r\right], & 0<r\leqslant 1;\\ \frac{1}{6}\left[\xi^r+\eta^r+\left(\frac{\xi+\eta}{2}\right)^r\right], & r\geqslant 1. \end{cases} \tag{6}$$

Proof. Denote

$$H(t)=[t\xi+(1-t)\eta]^r+[t\eta+(1-t)\xi]^r,\quad t\in[0,1]. \tag{7}$$

If $0<r\leqslant 1$, since the function $(1-2t)H(t)$ is concave on $\left[0,\frac{1}{2}\right]$, by virtue of the Hermite–Hadamard inequality (2), we derive

$$\int_0^1 |1-2t|[t\xi+(1-t)\eta]^r\,\mathrm{d}t=\int_0^{1/2}(1-2t)H(t)\,\mathrm{d}t\leqslant\frac{1}{4}H\left(\frac{1}{4}\right).$$

If $r\geqslant 1$, then $H(t)$ is convex on $[0,1]$. By the inequality (1), for any $t\in[0,1]$, we can induce

$$H\left(\frac{1-t}{2}\right)\leqslant tH(0)+(1-t)H\left(\frac{1}{2}\right)=t(\xi^r+\eta^r)+2(1-t)\left(\frac{\xi+\eta}{2}\right)^r. \tag{8}$$

Therefore, it follows that

$$\begin{aligned}\int_0^1 |1-2t|[t\xi+(1-t)\eta]^r\,\mathrm{d}t&=\int_0^{1/2}(1-2t)H(t)\,\mathrm{d}t\\&=\frac{1}{2}\int_0^1 tH\left(\frac{1-t}{2}\right)\mathrm{d}t\\&\leqslant\frac{\xi^r+\eta^r}{6}+\frac{1}{6}\left(\frac{\xi+\eta}{2}\right)^r.\end{aligned}$$

Hence, the inequality (6) holds. The proof of Lemma 3 is thus complete. □

LEMMA 4. *Let* $M(t)$ *be defined by* (5), $\xi,\eta\geqslant 0$, *and* $r>0$. *Then*

$$\int_0^1 |M(t)|[t\xi+(1-t)\eta]^r\,\mathrm{d}t \leqslant \begin{cases} \frac{1}{8}\left[\left(\frac{2\xi+3\eta}{5}\right)^r+\left(\frac{3\xi+2\eta}{5}\right)^r\right], & 0<r\leqslant 1;\\ \frac{1}{24}\left[\xi^r+\eta^r+4\left(\frac{\xi+\eta}{2}\right)^r\right], & r\geqslant 1. \end{cases} \tag{9}$$

Proof. Let $H(t)$ be defined by (7). If $0 < r \leqslant 1$, then $H(t)$ is increasing and concave on $\left[0, \frac{1}{2}\right]$, and so we have

$$\begin{aligned}
\int_0^1 |M(t)|[t\xi+(1-t)\eta]^r \,\mathrm{d}t &= \int_0^{3/10} tH(t)\,\mathrm{d}t + \int_{3/10}^{2/5} tH(t)\,\mathrm{d}t + \int_{2/5}^{1/2} tH(t)\,\mathrm{d}t \\
&\leqslant H\left(\frac{3}{10}\right)\int_0^{3/10} t\,\mathrm{d}t + H\left(\frac{2}{5}\right)\int_{3/10}^{2/5} t\,\mathrm{d}t + H\left(\frac{1}{2}\right)\int_{2/5}^{1/2} t\,\mathrm{d}t \\
&= \frac{9}{200}\left[H\left(\frac{3}{10}\right) + H\left(\frac{1}{2}\right)\right] + \frac{7}{200}H\left(\frac{2}{5}\right) \\
&\leqslant \frac{9}{100}H\left(\frac{2}{5}\right) + \frac{7}{200}H\left(\frac{2}{5}\right) \\
&= \frac{1}{8}H\left(\frac{2}{5}\right).
\end{aligned}$$

If $r \geqslant 1$, by the inequality (8), we acquire

$$\int_0^1 |M(t)|[t\xi+(1-t)\eta]^r \,\mathrm{d}t = \int_0^{1/2} tH(t)\,\mathrm{d}t \leqslant \frac{1}{24}\left[\xi^r + \eta^r + 4\left(\frac{\xi+\eta}{2}\right)^r\right].$$

Hence, the inequality (9) holds. The proof of Lemma 4 is thus complete. □

3. Main results and their proofs

We are now in a position to state and prove our main results.

THEOREM 3. *Let a, b belong to an open interval $I^\circ \subseteq \mathbb{R}$ with $a < b$ and let $f : I^\circ \to \mathbb{R}$ be a differentiable function. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q \geqslant 1$, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\
&\qquad \leqslant \frac{b-a}{8}\left[\left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4}\right)^{1/q} + \left(\frac{|3f'(a)|^q + |f'(b)|^q}{4}\right)^{1/q}\right]
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\
&\qquad \leqslant \frac{b-a}{8}\left[\left(\frac{2|f'(a)|^q + 3|f'(b)|^q}{5}\right)^{1/q} + \left(\frac{3|f'(a)|^q + 2|f'(b)|^q}{5}\right)^{1/q}\right].
\end{aligned} \tag{11}$$

Proof. By Lemma 1 and the convexity of $|f|^q$, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\
&\qquad \leqslant \frac{b-a}{2}\int_0^1 |1-2t||f'(ta+(1-t)b)|\,\mathrm{d}t \\
&\qquad \leqslant \frac{b-a}{2}\int_0^1 |1-2t|\left[t|f'(a)|^q+(1-t)|f'(b)|^q\right]^{1/q}\mathrm{d}t.
\end{aligned}$$

Letting $r=\frac{1}{q}$, $\xi=|f'(a)|^q$, and $\eta=|f'(b)|^q$ in the inequality (6) leads to

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\
&\qquad \leqslant \frac{b-a}{2}\int_0^1 |1-2t|\left[t|f'(a)|^q+(1-t)|f'(b)|^q\right]^{1/q}\mathrm{d}t \\
&\qquad \leqslant \frac{b-a}{8}\left[\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{1/q}+\left(\frac{|3f'(a)|^q+|f'(b)|^q}{4}\right)^{1/q}\right].
\end{aligned}$$

Similarly, by Lemma 2, the convexity of $|f|^q$, and the inequality (9), we can derive the inequality (11). The proof of Theorem 3 is thus finished. □

COROLLARY 1. *Under the conditions of Theorem 3, we have*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\
&\qquad \leqslant \frac{b-a}{8}\left[\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{1/q}+\left(\frac{|3f'(a)|^q+|f'(b)|^q}{4}\right)^{1/q}\right] \\
&\qquad \leqslant \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\
&\qquad \leqslant \frac{b-a}{8}\left[\left(\frac{2|f'(a)|^q+3|f'(b)|^q}{5}\right)^{1/q}+\left(\frac{3|f'(a)|^q+2|f'(b)|^q}{5}\right)^{1/q}\right] \\
&\qquad \leqslant \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}.
\end{aligned} \tag{13}$$

Proof. Employing the concavity of the function $x^{1/q}$ on $[0,\infty)$ for $q \geqslant 1$, we obtain the inequality

$$\frac{u^{1/q}+v^{1/q}}{2} \leqslant \left(\frac{u+v}{2}\right)^{1/q}, \quad u,v \geqslant 0.$$

Accordingly, the inequalities (12) and (13) hold. The proof of Corollary 1 is thus finished. □

REMARK 1. The last inequalities in (12) and (13) in Corollary 1 show that the upper bounds of the inequalities (10) and (11) in Theorem 3 are smaller than the corresponding ones in the inequalities (3) and (4) in Theorem 2. Consequently, we arrive at refinements of the inequalities (3) and (4) in Theorem 2.

To the best of our knowledge, there have been plenty of the Hermite–Hadamard type inequalities under different conditions and not including each other, but there have been seldom results under the same conditions and known inequalities being refined. Consequently, Theorem 3 is the highlight of this paper and also the highlight in the theory of convex functions and inequalities.

Similarly, in view of Lemmas 1 to 4, we can obtain the following result.

THEOREM 4. *Let* a,b *belong to an open interval* $I^\circ \subseteq \mathbb{R}$ *with* $a<b$ *and let* $f: I^\circ \to \mathbb{R}$ *be a differentiable function. If* $|f'|^q$ *is convex on* $[a,b]$ *for some fixed* $0<q<1$*, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \leqslant \frac{b-a}{12}\left[|f'(a)|+|f'(b)|+\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}\right] \tag{14}$$

and

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \leqslant \frac{b-a}{24}\left[|f'(a)|+|f'(b)|+4\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}\right]. \tag{15}$$

REMARK 2. It is easy to see that, if $q_1 \geqslant 1 > q_2 > 0$ and $|f'|^{q_2}$ is convex on I, then $|f'|^{q_1}$ is also convex on I. Accordingly, the upper bounds of the inequalities (10) and (11) in Theorem 3 are also upper bounds of the inequalities (14) and (15) in Theorem 4, respectively. However, by the power mean inequality, we have

$$\begin{aligned}
\frac{b-a}{12}&\left[|f'(a)|+|f'(b)|+\left(\frac{|f'(a)|^{q_2}+|f'(b)|^{q_2}}{2}\right)^{1/q_2}\right] \\
&\leqslant \frac{(b-a)(|f'(a)|+|f'(b)|)}{8} \\
&\leqslant \frac{b-a}{8}\left[\left(\frac{|f'(a)|^{q_1}+3|f'(b)|^{q_1}}{4}\right)^{1/q_1}+\left(\frac{|3f'(a)|^{q_1}+|f'(b)|^{q_1}}{4}\right)^{1/q_1}\right]
\end{aligned}$$

and

$$\frac{b-a}{24}\left[|f'(a)|+|f'(b)|+4\left(\frac{|f'(a)|^{q_2}+|f'(b)|^{q_2}}{2}\right)^{1/q_2}\right]$$
$$\leqslant \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}$$
$$\leqslant \frac{b-a}{8}\left[\left(\frac{2|f'(a)|^{q_1}+3|f'(b)|^{q_1}}{5}\right)^{1/q_1}+\left(\frac{3|f'(a)|^{q_1}+2|f'(b)|^{q_1}}{5}\right)^{1/q_1}\right].$$

This means that, if $0<q<1$, using Theorem 4, we could obtain a better upper bound.

4. Applications to special means

The arithmetic, geometric, logarithmic, and identic mean of two real numbers $\alpha,\beta>0$ are defined respectively by

$$A(\alpha,\beta)=\frac{\alpha+\beta}{2}, \qquad G(\alpha,\beta)=\sqrt{\alpha\beta},$$
$$L(\alpha,\beta)=\begin{cases}\frac{\beta-\alpha}{\ln\beta-\ln\alpha}, & \alpha\neq\beta;\\ \alpha, & \alpha=\beta,\end{cases} \qquad I(\alpha,\beta)=\begin{cases}\frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right)^{1/(\beta-\alpha)}, & \alpha\neq\beta;\\ \alpha, & \alpha=\beta.\end{cases}$$

PROPOSITION 1. *Let $b>a>0$. Then*

$$\left|A\left(\frac{1}{a},\frac{1}{b}\right)-\frac{1}{L(a,b)}\right|\leqslant\frac{b-a}{12}\left[2A\left(\frac{1}{a^2},\frac{1}{b^2}\right)+G\left(\frac{1}{a^2},\frac{1}{b^2}\right)\right] \tag{16}$$

and

$$\frac{1}{L(a,b)}-\frac{1}{A(a,b)}\leqslant\frac{b-a}{12}\left[A\left(\frac{1}{a^2},\frac{1}{b^2}\right)+2G\left(\frac{1}{a^2},\frac{1}{b^2}\right)\right]. \tag{17}$$

Proof. Let $f(x)=\frac{1}{x}$ for $x\in[a,b]$. Then, for any $q\in(0,1)$, by Theorem 4, we can derive

$$\left|A\left(\frac{1}{a},\frac{1}{b}\right)-\frac{1}{L(a,b)}\right|\leqslant\frac{b-a}{12}\left\{2A\left(\frac{1}{a^2},\frac{1}{b^2}\right)+\left[A\left(\frac{1}{a^{2q}},\frac{1}{b^{2q}}\right)\right]^{1/q}\right\}$$

and

$$\frac{1}{L(a,b)}-\frac{1}{A(a,b)}\leqslant\frac{b-a}{12}\left\{A\left(\frac{1}{a^2},\frac{1}{b^2}\right)+2\left[A\left(\frac{1}{a^{2q}},\frac{1}{b^{2q}}\right)\right]^{1/q}\right\}.$$

Due to the limit

$$\lim_{q\to0^+}\left[A\left(\frac{1}{a^{2q}},\frac{1}{b^{2q}}\right)\right]^{1/q}=G\left(\frac{1}{a^2},\frac{1}{b^2}\right),$$

we arrive at the inequalities (16) and (17). □

If letting $f(x)=\ln x$ for $x\in[a,b]$ with $b>a>0$, then we can similarly derive

PROPOSITION 2. *Let $b > a > 0$. Then*

$$\ln \frac{I(a,b)}{G(a,b)} \leqslant \frac{b-a}{12}\left[2A\left(\frac{1}{a},\frac{1}{b}\right)+G\left(\frac{1}{a},\frac{1}{b}\right)\right]$$

and

$$\ln \frac{A(a,b)}{I(a,b)} \leqslant \frac{b-a}{12}\left[A\left(\frac{1}{a},\frac{1}{b}\right)+2G\left(\frac{1}{a},\frac{1}{b}\right)\right].$$

REMARK 3. For any $q \geqslant 1$, we have

$$G\left(\frac{1}{a^2},\frac{1}{b^2}\right) \leqslant A\left(\frac{1}{a^2},\frac{1}{b^2}\right) \leqslant \left[A\left(\frac{1}{a^{2q}},\frac{1}{b^{2q}}\right)\right]^{1/q}.$$

Hence, for $b > a > 0$, we obtain a refinement of [8, Proposition 2] and a refinement of [4, Proposition 3.4].

Acknowledgements. The authors are grateful to anonymous referees for their valuable comments, careful corrections, and helpful suggestions to the original version of this paper.

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(Received July 22, 2024)

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