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# FURTHER DEVELOPMENTS OF BELLMAN AND ACZÉL INEQUALITIES FOR OPERATORS

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*Abstract.* In the present paper, we derive some operator Bellman and Aczél inequalities involving quasi $\lambda$-geometric and arithmetic means. Among other inequalities, it is shown that if $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{H})$ is a unital positive linear map and $A, B \in \mathbb{B}(\mathscr{H})$ are two contraction operators, then for any $p > 1$,

$$\Phi\left((I - A\nabla_\lambda B)^{\frac{1}{p}}\right) \leqslant \Phi(I-A)^{\frac{1}{p}} \nabla_\lambda \Phi(I-B)^{\frac{1}{p}}$$

holds, where $\lambda \notin [0,1]$.

## 1. Introduction

Let $\mathbb{B}(\mathscr{H})$ be the $C^*$ algebra of bounded linear operators on a complex Hilbert space $\mathscr{H}$ with the identity $I$. An operator $A$ is said to be positive (denoted by $0 \leqslant A$) if $0 \leqslant \langle Ax, x\rangle$ for all $x \in \mathscr{H}$, and also an operator $A$ is said to be strictly positive (denoted by $0 < A$) if $A$ is positive and invertible. For two self-ajoint operators $A, B \in \mathbb{B}(\mathscr{H})$, we write $A \leqslant B$ if $0 \leqslant B - A$. A linear map $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{H})$ is said to be positive if $0 \leqslant \Phi(A)$ when $0 \leqslant A$. If, in addition, $\Phi(I) = I$, it is said to be unital.

For any strictly positive operator $A, B \in \mathbb{B}(\mathscr{H})$ and $\lambda \in [0,1]$, we write

$$A\nabla_\lambda B := (1-\lambda)A + \lambda B \quad \text{and} \quad A\sharp_\lambda B := A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\lambda}A^{\frac{1}{2}}.$$

For the case $\lambda = \frac{1}{2}$, we write $\nabla$ and $\sharp$, respectively. We use the same notions for scalars. The weighted operator arithmetic-geometric mean inequality asserts that $A\sharp_\lambda B \leqslant A\nabla_\lambda B$, for any positive and invertible operators $A, B \in \mathbb{B}(\mathscr{H})$ and any $\lambda \in [0,1]$.

We use the notation $\natural_\lambda$ for the binary operation

$$A\natural_\lambda B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\lambda}A^{\frac{1}{2}}, \ (\lambda \notin [0,1]). \tag{1.1}$$

Though $A\natural_\lambda B\,(\lambda \notin [0,1])$ are not operator mean in the sense of Kubo-Ando theory [9], $A\natural_\lambda B$ have operator mean like properties for any positive invertible operators $A$ and $B$. Thus we call (1.1) the quasi $\lambda$-geometric mean for $\lambda \notin [0,1]$.

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A real-valued function $f$ defined on an interval $J \subseteq \mathbb{R}$ is said to be operator convex (resp. operator concave) if $f(A\nabla_\lambda B) \leqslant f(A)\nabla_\lambda f(B)$ (resp. $f(A)\nabla_\lambda f(B) \leqslant f(A\nabla_\lambda B)$) for all self-adjoint operators $A,B$ with spectra in $J$ and all $\lambda \in [0,1]$. A continuous real-valued function $f$ defined on an interval $J$ is called operator monotone (more precisely, operator monotone increasing) if $A \leqslant B$ implies that $f(A) \leqslant f(B)$, and operator monotone decreasing if $A \leqslant B$ implies $f(B) \leqslant f(A)$ for all self-adjoint operators $A,B$ with spectra in $J$.

The scalar Bellman inequality [2] says that if $p$ is a positive integer and $a,b,a_i,b_i$ $(1 \leqslant i \leqslant n)$ are positive real numbers such that $\sum_{i=1}^n a_i^p \leqslant a^p$ and $\sum_{i=1}^n b_i^p \leqslant b^p$, then

$$\left(a^p - \sum_{i=1}^n t_i^p\right)^{\frac{1}{p}} + \left(b^p - \sum_{i=1}^n s_i^p\right)^{\frac{1}{p}} \leqslant \left((a+b)^p - \sum_{k=1}^n (a_i+b_i)^p\right)^{\frac{1}{p}}.$$

A multiplicative analogue of this inequality is due to Aczél [1]. In 1956, he proved that

$$\left(a_1^2 - \sum_{i=2}^n a_i^2\right)\left(b_1^2 - \sum_{i=2}^n b_i^2\right) \leqslant \left(a_1 b_1 - \sum_{i=2}^n a_i b_i\right)^2$$

where $a_i, b_i$ $(1 \leqslant i \leqslant n)$ are positive real numbers such that $a_1^2 - \sum_{i=2}^n a_i^2 > 0$ or $b_1^2 - \sum_{i=2}^n b_i^2 > 0$.

The operator theory related to inequalities in Hilbert space is studied in many papers. In [11, Corollary 2.2], Morassaei et al. showed the following non-commutative version of classical Bellman inequality:

$$\Phi\left((I-A)^{\frac{1}{p}}\nabla_\lambda (I-B)^{\frac{1}{p}}\right) \leqslant (\Phi(I - A\nabla_\lambda B))^{\frac{1}{p}}, \quad (0 \leqslant \lambda \leqslant 1,\ p > 1) \tag{1.2}$$

where $A,B \in \mathbb{B}(\mathscr{H})$ are two contractions (i.e., $0 < A,B \leqslant I$) and $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{H})$ is a unital positive linear map. We refer the reader to [13, 16] for a fresh discussion of Bellman inequality.

In [12, Theorem 2.2], Moslehian noted the following inequalities for non-negative operator decreasing, and operator concave $f$ and $p,q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$:

$$f(A^p)\sharp_{\frac{1}{q}} f(B^q) \leqslant f\left(A^p \sharp_{\frac{1}{q}} B^q\right) \tag{1.3}$$

and

$$\langle f(A^p)x,x\rangle^{\frac{1}{p}} \langle f(B^q)x,x\rangle^{\frac{1}{q}} \leqslant \left\langle f\left(A^p \sharp_{\frac{1}{q}} B^q\right)x,x\right\rangle, \quad \text{for any } x \in \mathscr{H}. \tag{1.4}$$

These inequalities may be considered as operator versions of Aczél inequality. We refer the reader to [6, 8, 15] for an excellent discussion of these inequalities.

The paper proves operator Bellman and Aczél inequalities for quasi $\lambda$-geometric and arithmetic means. Some other related results are also presented.

## 2. Results

### 2.1. Extensions of Bellman and Aczél inequalities

This section contains our main results. We start with the following simple but helpful result (see, e.g., [5]).

LEMMA 2.1. *Let $A, B \in \mathbb{B}(\mathscr{H})$. Then for any $\lambda \notin [0,1]$,*

$$A\nabla_\lambda B \leqslant A\natural_\lambda B.$$

LEMMA 2.2. *Let $f : J \to \mathbb{R}$ be operator concave, $A, B \in \mathbb{B}(\mathscr{H})$ be two self-adjoint operators with spectra contained in $J$. Then for any $\lambda \notin [0,1]$*

$$f(A\nabla_\lambda B) \leqslant f(A)\nabla_\lambda f(B). \tag{2.1}$$

*If $f : J \to \mathbb{R}$ is operator convex the preceding inequality is reversed.*

*Proof.* Assume $\lambda < 0$. Then

$$A = \frac{1}{1-\lambda}((1-\lambda)A + \lambda B) - \frac{\lambda}{1-\lambda}B.$$

Since $f$ is an operator concave and $\frac{1}{1-\lambda} + \left(-\frac{\lambda}{1-\lambda}\right) = 1$, we have

$$\begin{aligned} f(A) &= f\left(\frac{1}{1-\lambda}((1-\lambda)A + \lambda B) - \frac{\lambda}{1-\lambda}B\right) \\ &\geqslant \frac{1}{1-\lambda}f((1-\lambda)A + \lambda B) - \frac{\lambda}{1-\lambda}f(B). \end{aligned}$$

Multiplying both sides by $1-\lambda$, we get (2.1). Now, assume that $\lambda > 1$. Then

$$B = \frac{1}{\lambda}((1-\lambda)A + \lambda B) - \frac{1-\lambda}{\lambda}A.$$

Since $f$ is an operator concave and $\frac{1}{\lambda} + \left(-\frac{1-\lambda}{\lambda}\right) = 1$, we have

$$\begin{aligned} f(B) &= f\left(\frac{1}{\lambda}((1-\lambda)A + vB) - \frac{1-\lambda}{\lambda}A\right) \\ &\geqslant \frac{1}{\lambda}f((1-\lambda)A + vB) - \frac{1-\lambda}{\lambda}f(A). \end{aligned}$$

Multiplying both sides by $\lambda$, we get (2.1). □

THEOREM 2.1. *Let $f$ be a non-negative operator decreasing and operator concave on the interval $J \subseteq (0,\infty)$, and $A, B \in \mathbb{B}(\mathscr{H})$ be two positive operators whose spectra are contained in $J$. Then for any $\lambda \notin [0,1]$*

$$f(A\natural_\lambda B) \leqslant f(A)\natural_\lambda f(B).$$

*Proof.*

$$f(A)\natural_\lambda f(B) \geqslant f(A)\nabla_\lambda f(B) \tag{2.2}$$
$$\geqslant f(A\nabla_\lambda B) \tag{2.3}$$
$$\geqslant f(A\natural_\lambda B) \tag{2.4}$$

where the inequality (2.2) follows from Lemma 2.1, in the inequality (2.3) we used Lemma 2.2, and the inequality (2.4) follows directly from operator decresingness of $f$ together with Lemma 2.1. □

In the next result, we aim to improve Lemma 2.1.

PROPOSITION 2.1. *Let $A,B \in \mathbb{B}(\mathscr{H})$ be two positive operators, and let $\lambda \notin [0,1]$.*

*(i) If $0 < m'I \leqslant A \leqslant mI < MI \leqslant B \leqslant M'I$, then*

$$\frac{1}{m}(m\natural_\lambda M - m\nabla_\lambda M)A \leqslant A\natural_\lambda B - A\nabla_\lambda B$$
$$\leqslant \frac{1}{m'}\left(m'\natural_\lambda M' - m'\nabla_\lambda M'\right)A.$$

*(ii) If $0 < m'I \leqslant B \leqslant mI < MI \leqslant A \leqslant M'I$, then*

$$\frac{1}{M}(M\natural_\lambda m - M\nabla_\lambda m)A \leqslant A\natural_\lambda B - A\nabla_\lambda B$$
$$\leqslant \frac{1}{M'}\left(M'\natural_\lambda m' - M'\nabla_\lambda m'\right)A.$$

*Proof.* We prove (i). Define

$$g(x) \equiv x^\lambda - ((1-\lambda)+\lambda x), \qquad (s \leqslant x \leqslant t,\ \lambda \notin [0,1]).$$

Thus, $g'(x) = \lambda\left(1 - x^{\lambda-1}\right)$. We have two cases:

- $g'(x) > 0$ for $x > 1$, $\lambda \notin [0,1]$. So $g(x)$ is increasing on $[s,t]$. Hence $g(s) \leqslant g(x) \leqslant g(t)$.
- $g'(x) < 0$ for $0 < x \leqslant 1$, $\lambda \notin [0,1]$, i.e, $g(x)$ is decreasing on $[s,t]$. Hence $g(t) \leqslant g(x) \leqslant g(s)$.

The first case implies

$$s^\lambda - ((1-\lambda)+\lambda s) \leqslant x^\lambda - ((1-\lambda)+\lambda x) \leqslant t^\lambda - ((1-\lambda)+\lambda t).$$

From $0 < m'I \leqslant A \leqslant mI < MI \leqslant B \leqslant M'I$, we know that $I < \frac{M}{m}I \leqslant A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leqslant \frac{M'}{m'}I$. Now put $s = \frac{M}{m}$, $t = \frac{M'}{m'}$, and $x = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, we get

$$\left(\frac{M}{m}\right)^\lambda I - \left((1-\lambda)+\lambda\frac{M}{m}\right)I \leqslant \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\lambda - \left((1-\lambda)I + \lambda A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)$$
$$\leqslant \left(\frac{M'}{m'}\right)^\lambda I - \left((1-\lambda)+\lambda\frac{M'}{m'}\right)I.$$

Multiplying both sides by $A^{\frac{1}{2}}$, we get

$$\frac{1}{m}(m\natural_\lambda M - m\nabla_\lambda M)A \leqslant A\natural_\lambda B - A\nabla_\lambda B \leqslant \frac{1}{m'}\left(m'\natural_\lambda M' - m'\nabla_\lambda M'\right)A.$$

Hence, (i) is proved. □

It follows from Proposition 2.1 that

$$A\nabla_\lambda B \leqslant A\natural_\lambda B - \frac{m'}{m}(m\natural_\lambda M - m\nabla_\lambda M)I$$

whenever $0 < m'I \leqslant A \leqslant mI < MI \leqslant B \leqslant M'I$, and

$$A\nabla_\lambda B \leqslant A\natural_\lambda B - (M\natural_\lambda m - M\nabla_\lambda m)I$$

whenever $0 < m'I \leqslant B \leqslant mI < MI \leqslant A \leqslant M'I$.

Now, we can prove our promised refinement of Theorem 2.1.

THEOREM 2.2. *Let $f$ be a non-negative operator decreasing and operator concave on the interval $J \subseteq (0,\infty)$, $A,B \in \mathbb{B}(\mathscr{H})$ be two positive operators whose spectra are contained in $J$, and let $\lambda \notin [0,1]$.*

*(i) If $0 < m'I \leqslant f(A) \leqslant mI < MI \leqslant f(B) \leqslant M'I$, then*

$$f(A\natural_\lambda B) \leqslant f(A)\natural_\lambda f(B) - \frac{m'}{m}(m\natural_\lambda M - m\nabla_\lambda M)I.$$

*(ii) If $0 < m'I \leqslant f(B) \leqslant mI < MI \leqslant f(A) \leqslant M'I$, then*

$$f(A\natural_\lambda B) \leqslant f(A)\natural_\lambda f(B) - (M\natural_\lambda m - M\nabla_\lambda m)I.$$

Before proceeding, we recall the well-known Choi-Davis-Jensen's inequality [3, 4]: If $f : J \to \mathbb{R}$ is operator convex, then for any unital positive linear map $\Phi$, we have

$$f(\Phi(A)) \leqslant \Phi(f(A)), \tag{2.5}$$

while we have the reversed inequality if $f$ is operator concave, for any self-adjoint operator $A$ with spectrum in $J$.

Now we generalize the operator Bellman inequality (1.2).

THEOREM 2.3. *Let $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{H})$ be a unital positive linear map, $f$ be a non-negative operator concave function on the interval $J \subseteq (0,\infty)$, and $A,B \in \mathbb{B}(\mathscr{H})$ be two positive operators whose spectra are contained in $J$. Then for any $\lambda \notin [0,1]$*

$$\Phi(f(A\nabla_\lambda B)) \leqslant f(\Phi(A))\nabla_\lambda f(\Phi(B)).$$

*If $f$ is operator convex, the preceding inequality is reversed.*

*Proof.* We have

$$\begin{aligned}\Phi(f(A\nabla_\lambda B)) &\leqslant f(\Phi(A\nabla_\lambda B))\\ &= f(\Phi(A)\nabla_\lambda\Phi(B))\\ &\leqslant f(\Phi(A))\nabla_\lambda f(\Phi(B))\end{aligned}$$

where for the first and the second inequalities we used (2.5), and Lemma 2.2, respectively. □

COROLLARY 2.1. *Let $\Phi:\mathbb{B}(\mathscr{H})\to\mathbb{B}(\mathscr{H})$ be a unital positive linear map and $A,B\in\mathbb{B}(\mathscr{H})$ be two contraction operators. Then for any $p>1$,*

$$\Phi\left((I-A\nabla_\lambda B)^{\frac{1}{p}}\right)\leqslant\Phi(I-A)^{\frac{1}{p}}\nabla_\lambda\Phi(I-B)^{\frac{1}{p}}$$

*holds, where $\lambda\notin[0,1]$.*

*Proof.* Since the function $g(t)=t^{\frac{1}{p}}$ $(p>1)$ is operator concave on $(0,\infty)$ (see, e.g., [7, Corollary 1.16]), so the function $f(t)=(1-t)^{\frac{1}{p}}$ $(p>1)$ is operator concave on $(0,1)$. Applying Theorem 2.3 for the function $f(t)=(1-t)^{\frac{1}{p}}$ we get the desired result. □

REMARK 2.1. Applying a same approach as in Corollary 2.1, we get

$$\Phi(I-A)^{\frac{1}{p}}\nabla_\lambda\Phi(I-B)^{\frac{1}{p}}\leqslant\Phi\left((I-A\nabla_\lambda B)^{\frac{1}{p}}\right),\ (p<-1 \text{ and } \lambda\notin[0,1]).$$

### 2.2. An inequality related to the operator geometric mean

This section presents an inequality related to the operator geometric mean. To this end, we need the following lemma.

LEMMA 2.3. [14, Lemma 2.1] *Let $p_1,p_2,\ldots,p_n$ be a probability vector such that $\sum_{j=1}^m p_j=1$, and $a_1,a_2,\ldots,a_n$ be a non-negative real numbers. Define the map $F:\mathbb{R}^m\to\mathbb{R}$ by*

$$F(x_1,x_2,\ldots,x_m):=\frac{\sum_{j=1}^m p_ja_jx_j}{\prod_{j=1}^m x_j^{p_j}}.$$

*Then,*

$$\inf_{x_1,x_2,\ldots,x_m>0}F(x_1,x_2,\ldots,x_m)=\prod_{j=1}^m a_j^{p_j}.$$

Now we give our main result.

THEOREM 2.4. *Let* $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{H})$ *be a positive linear map and* $A, B \in \mathbb{B}(\mathscr{H})$ *with* $0 < mI \leqslant A, B \leqslant MI$. *Then*

$$\begin{aligned}\langle \Phi(A)x,x\rangle^{1+\lambda}\left\langle \Phi\left(AB^{-1}A\right)x,x\right\rangle^{-\lambda} &\leqslant \langle \Phi(A\sharp_\lambda B)x,x\rangle\\ &\leqslant \langle \Phi(A)x,x\rangle^{1-\lambda}\langle \Phi(B)x,x\rangle^{\lambda},\end{aligned}$$

*for any* $0 \leqslant \lambda \leqslant 1$.

*Proof.* First, we prove the right side of inequality (2.4). By the operator arithmetic-geometric mean inequality, we have

$$A\sharp_\lambda B \leqslant (1-\lambda)A + \lambda B.$$

Since $\Phi$ is a positive linear map, we get

$$\Phi(A\sharp_\lambda B) \leqslant (1-\lambda)\Phi(A) + \lambda\Phi(B). \tag{2.6}$$

Now, replacing $A$ and $B$ with $sA$ and $tB$ $(s,t > 0)$, in (2.6), respectively, then we deduce

$$\Phi(A\sharp_\lambda B) \leqslant \frac{(1-\lambda)s\Phi(A) + \lambda t\Phi(B)}{s^{1-\lambda}t^{\lambda}}.$$

Then we have for any $x \in \mathscr{H}$ with $\|x\| = 1$ that

$$\langle \Phi(A\sharp_\lambda B)x,x\rangle \leqslant \frac{(1-\lambda)s\langle \Phi(A)x,x\rangle + \lambda t\langle \Phi(B)x,x\rangle}{s^{1-\lambda}t^{\lambda}}.$$

Therefore

$$\langle \Phi(A\sharp_\lambda B)x,x\rangle \leqslant \inf_{s,t>0}\left(\frac{(1-\lambda)s\langle \Phi(A)x,x\rangle + \lambda\langle \Phi(B)x,x\rangle}{s^{1-\lambda}t^{\lambda}}\right)$$

By Lemma 2.3, with $m = 2$, $p_1 = 1-\lambda$, $p_2 = \lambda$, $a_1 = \langle \Phi(A)x,x\rangle$, $a_2 = \langle \Phi(B)x,x\rangle$ and

$$F(s,t) = \frac{(1-\lambda)s\langle \Phi(A)x,x\rangle + \lambda\langle \Phi(B)x,x\rangle}{s^{1-\lambda}t^{\lambda}},$$

we obtain the desired inequality.

For the second inequality, we have

$$A\sharp_\lambda B + \lambda\left(AB^{-1}A\right) \geqslant (1+\lambda)A.$$

Since $\Phi$ is positive linear map we get

$$\Phi(A\sharp_\lambda B) + \lambda\Phi\left(AB^{-1}A\right) \geqslant (1+\lambda)\Phi(A). \tag{2.7}$$

Replacing $A$ and $B$ with $sA$ and $tB$ which $(s,t > 0)$, in (2.7), respectively, then we deduce

$$\Phi(A\sharp_\lambda B) \geqslant (1+\lambda)s^{\lambda}t^{-\lambda}\Phi(A) - \lambda s^{1+\lambda}t^{-(1+\lambda)}\Phi\left(AB^{-1}A\right).$$

Then we have for each $x \in \mathscr{H}$ with $\|x\| = 1$ that

$$\begin{aligned}&\langle \Phi(A \sharp_\lambda B)x,x\rangle \\ &\geqslant (1+\lambda)s^\lambda t^{-\lambda}\langle \Phi(A)x,x\rangle - \lambda s^{1+\lambda}t^{-(1+\lambda)}\left\langle \Phi\left(AB^{-1}A\right)x,x\right\rangle.\end{aligned}$$

Therefore

$$\begin{aligned}&\langle \Phi(A \sharp_\lambda B)x,x\rangle \\ &\geqslant \sup_{s,t>0}\left((1+\lambda)s^\lambda t^{-\lambda}\langle \Phi(A)x,x\rangle - \lambda s^{1+\lambda}t^{-(1+\lambda)}\left\langle \Phi\left(AB^{-1}A\right)x,x\right\rangle\right).\end{aligned}$$

It is not hard to see that the right side of this inequality is equal to

$$\langle \Phi(A)x,x\rangle^{1+\lambda}\left\langle \Phi\left(AB^{-1}A\right)x,x\right\rangle^{-\lambda},$$

so the proof is complete. □

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