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TENSORIAL NORM INEQUALITIES FOR TAYLOR'S EXPANSIONS OF FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR AND MUBARIZ TAPDIGOGLU GARAYEV

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Abstract. Let H be a Hilbert space. In this paper we show among others that, if f is of class C^{n+1} on the open interval I, P and Q are selfadjoint operators with $\operatorname{Sp}(P)$, $\operatorname{Sp}(Q) \subset I$ and if $\left\| f^{(n+1)} \right\|_{I,\infty} := \sup_{u \in I} \left| f^{(n+1)}(u) \right| < \infty$, then

$$\left\| f(P) \otimes 1 - \sum_{k=0}^{n} \frac{1}{k!} (P \otimes 1 - 1 \otimes Q)^k \left(1 \otimes f^{(k)}(Q)\right) \right\| \leqslant \frac{1}{(n+1)!} \|P \otimes 1 - 1 \otimes Q\|^{n+1} \left\| f^{(n+1)} \right\|_{I,\infty}.$$

If $\left| f^{(n+1)} \right|$ is convex on I, then also

$$\left\| f(P) \otimes 1 - \sum_{k=0}^{n} \frac{1}{k!} (P \otimes 1 - 1 \otimes Q)^k \left(1 \otimes f^{(k)}(Q)\right) \right\| \leqslant \frac{1}{(n+1)!} \|P \otimes 1 - 1 \otimes Q\|^{n+1} \left[\frac{\left\| f^{(n+1)}(P) \right\| + (n+1) \left\| f^{(n+1)}(Q) \right\|}{n+2} \right].$$

Several examples for fundamental functions such as the logarithm and exponential are also provided.

1. Introduction

The following theorem is well known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

THEOREM 1. *Let $I \subset \mathbb{R}$ be a closed interval, $c \in I$ and let n be a positive integer. If $f : I \longrightarrow \mathbb{C}$ is such that the n-derivative $f^{(n)}$ is absolutely continuous on I, then for each $y \in I$*

$$f(y) = T_n(f; c, y) + R_n(f; c, y), \tag{1.1}$$

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where $T_n(f;c,y)$ *is Taylor's polynomial, i.e.,*

$$T_n(f;c,y) := \sum_{k=0}^{n} \frac{(y-c)^k}{k!} f^{(k)}(c). \tag{1.2}$$

Note that $f^{(0)} := f$ *and* $0! := 1$ *and the remainder is given by*

$$R_n(f;c,y) := \frac{1}{n!} \int_c^y (y-t)^n f^{(n+1)}(t)\, dt. \tag{1.3}$$

A simple proof of this theorem can be achieved by mathematical induction using the integration by parts formula in the Lebesgue integral.

In order to extend this result for tensorial products of selfadjoint operators and norms, we need the following preparations.

Let $I_1,\ldots,I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times \ldots \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1,\ldots,A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1,\ldots,H_k$ such that the spectrum of A_i is contained in I_i for $i = 1,\ldots,k.$ We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i = 1,\ldots,k;$ by following [2], we define

$$f(A_1,\ldots,A_k) := \int_{I_1} \ldots \int_{I_k} f(\lambda_1,\ldots,\lambda_k)\, dE_1(\lambda_1) \otimes \ldots \otimes dE_k(\lambda_k) \tag{1.4}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes \ldots \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [8] for functions of two variables and have the property that

$$f(A_1,\ldots,A_k) = f_1(A_1) \otimes \ldots \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1,\ldots,t_k) = f_1(t_1)\ldots f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if* f *is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, namely

$$f(st) \geqslant (\leqslant) f(s) f(t) \text{ for all } s,t \in [0,\infty)$$

and if f is continuous on $[0,\infty)$, then [11, p. 173]

$$f(A \otimes B) \geqslant (\leqslant) f(A) \otimes f(B) \text{ for all } A,\ B \geqslant 0. \tag{1.5}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A\otimes B)=\int_{[0,\infty)}\int_{[0,\infty)} f(st)\,dE(t)\otimes dF(s) \tag{1.6}$$

for the continuous function f on $[0,\infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B>0$

$$A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t\in[0,1]$ and

$$A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\#B = B\#A \text{ and } (A\#B)\otimes(B\#A)=(A\otimes B)\#(B\otimes A).$$

In 2007, S. Wada [17] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned}(A\#B)\otimes(A\#B) &\leqslant \frac{1}{2}\left[(A\#_\alpha B)\otimes(A\#_{1-\alpha}B)+(A\#_{1-\alpha}B)\otimes(A\#_\alpha B)\right] \\ &\leqslant \frac{1}{2}(A\otimes B+B\otimes A)\end{aligned} \tag{1.7}$$

for $A,\ B>0$ and $\alpha\in[0,1]$. For other similar results, see [1], [3] and [9]–[12]. More recent results may be found in [14], [15] and [16].

Recently, see [5], the S. S. Dragomir proved among others that, if $f,\ g$ are synchronous and continuous on I and $A,\ B$ are selfadjoint with spectra $\operatorname{Sp}(A),\ \operatorname{Sp}(B)\subset I$, then

$$(f(A)g(A))\otimes 1+1\otimes(f(B)g(B))\geqslant f(A)\otimes g(B)+g(A)\otimes f(B).$$

Let either $p,q\in(0,\infty)$ or $p,q\in(-\infty,0)$. If $A,\ B>0$, then

$$A^{p+q}\otimes 1+1\otimes B^{p+q}\geqslant A^p\otimes B^q+A^q\otimes B^p.$$

In the recent paper [7], the first author have shown that, if f is continuous differentiable convex on the open interval I and $A,\ B$ are selfadjoint operators in $B(H)$ with spectra $\operatorname{Sp}(A),\ \operatorname{Sp}(B)\subset I$, then we have the tensorial inequality

$$\begin{aligned}\left(f'(A)\otimes 1\right)(A\otimes 1-1\otimes B) &\geqslant f(A)\otimes 1-1\otimes f(B)\\ &\geqslant (A\otimes 1-1\otimes B)\left(1\otimes f'(B)\right).\end{aligned}$$

For the power function $f(t)=t^p,\ p\geqslant 1$, we obtain for $A,\ B>0$ that

$$\begin{aligned}p\left(A^{p-1}\otimes 1\right)(A\otimes 1-1\otimes B) &\geqslant A^p\otimes 1-1\otimes B^p\\ &\geqslant p(A\otimes 1-1\otimes B)\left(1\otimes B^{p-1}\right).\end{aligned}$$

Also in [6], the first author proved that, if f is continuously differentiable on I with $\|f'\|_{I,\infty} := \sup_{t\in I}|f'(t)| < \infty$ and A, B are selfadjoint operators with spectra $\operatorname{Sp}(A)$, $\operatorname{Sp}(B) \subset I$, then the following norm inequality holds

$$\begin{aligned}&\left\|(1-\lambda)f(A)\otimes 1+\lambda 1\otimes f(B)-\int_0^1 f((1-u)A\otimes 1+u1\otimes B)du\right\|\\&\leqslant \|f'\|_{I,\infty}\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\|1\otimes B-A\otimes 1\|\end{aligned}$$

for $\lambda\in[0,1]$. In particular, we have the *trapezoid inequality*

$$\begin{aligned}&\left\|\frac{f(A)\otimes 1+1\otimes f(B)}{2}-\int_0^1 f((1-u)A\otimes 1+u1\otimes B)du\right\|\\&\leqslant \frac{1}{4}\|f'\|_{I,\infty}\|1\otimes B-A\otimes 1\|.\end{aligned}$$

Motivated by the above results, in this paper we show among others that, if f is of class C^{n+1} on the open interval I, A and B are selfadjoint operators with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B)\subset I$ and if $\left\|f^{(n+1)}\right\|_{I,\infty} := \sup_{u\in I}\left|f^{(n+1)}(u)\right| < \infty$, then

$$\begin{aligned}&\left\|f(A)\otimes 1-\sum_{k=0}^{n}\frac{1}{k!}(A\otimes 1-1\otimes B)^k\left(1\otimes f^{(k)}(B)\right)\right\|\\&\leqslant \frac{1}{(n+1)!}\|A\otimes 1-1\otimes B\|^{n+1}\left\|f^{(n+1)}\right\|_{I,\infty}.\end{aligned}$$

If $\left|f^{(n+1)}\right|$ is convex on I, then also

$$\begin{aligned}&\left\|f(A)\otimes 1-\sum_{k=0}^{n}\frac{1}{k!}(A\otimes 1-1\otimes B)^k\left(1\otimes f^{(k)}(B)\right)\right\|\\&\leqslant \frac{1}{(n+1)!}\|A\otimes 1-1\otimes B\|^{n+1}\left[\frac{\left\|f^{(n+1)}(A)\right\|+(n+1)\left\|f^{(n+1)}(B)\right\|}{n+2}\right].\end{aligned}$$

Several examples for fundamental functions such as the logarithm and exponential are also provided.

2. Main results

Recall the following property of the tensorial product

$$(AC)\otimes(BD)=(A\otimes B)(C\otimes D) \tag{2.1}$$

that holds for any $A,B,C,D\in B(H)$.

If we take $C = A$ and $D = B$, then we get

$$A^2 \otimes B^2 = (A \otimes B)^2 .$$

By induction and using (2.1) we derive that

$$A^n \otimes B^n = (A \otimes B)^n \text{ for natural } n \geqslant 0. \tag{2.2}$$

In particular

$$A^n \otimes 1 = (A \otimes 1)^n \text{ and } 1 \otimes B^n = (1 \otimes B)^n \tag{2.3}$$

for all $n \geqslant 0$.

We also observe that, by (2.1), the operators $A \otimes 1$ and $1 \otimes B$ are commutative and

$$(A \otimes 1)(1 \otimes B) = (1 \otimes B)(A \otimes 1) = A \otimes B. \tag{2.4}$$

Moreover, for two natural numbers $m,\ n$ we have

$$(A \otimes 1)^m (1 \otimes B)^n = (1 \otimes B)^n (A \otimes 1)^m = A^m \otimes B^n. \tag{2.5}$$

We have the following representation results for continuous functions:

LEMMA 1. *Assume P and Q are selfadjoint operators with* $\operatorname{Sp}(P) \subset I$ *and* $\operatorname{Sp}(Q) \subset J$. *Let $f,\ h$ be continuous on $I,\ g,\ k$ continuous on J and φ and ψ continuous on an interval K that contains the product of the intervals $f(I)g(J),\ k(I)k(J)$, then*

$$\begin{aligned} &\varphi(f(P) \otimes g(Q))\,\psi(h(P) \otimes k(Q)) \\ &= \int_I \int_J \varphi(f(t)g(s))\,\psi(h(t)k(s))\, dE_t \otimes dF_s \end{aligned} \tag{2.6}$$

where P and Q have the spectral resolutions

$$P = \int_I t dE(t) \text{ and } Q = \int_J s dF(s). \tag{2.7}$$

Proof. By Stone-Weierstrass, any continuous function can be approximated by a sequence of polynomials, therefore it suffices to prove the equality for the power function $\varphi(t) = t^m$ and $\psi(t) = t^m$ with n and m any natural numbers.

We have

$$\begin{aligned} &\int_I \int_J (f(t)g(s))^m (h(t)k(s))^n dE_t \otimes dF_s \\ &= \int_I \int_J [f(t)]^m [g(s)]^m [h(t)]^n [k(s)]^n dE_t \otimes dF_s \\ &= \int_I \int_J [f(t)]^m [h(t)]^n [g(s)]^m [k(s)]^n dE_t \otimes dF_s \\ &= ([f(P)]^m [h(P)]^n) \otimes ([g(Q)]^m [k(Q)]^n) \\ &= ([f(P)]^m \otimes [g(Q)]^m)([h(P)]^n \otimes [k(Q)]^n) \\ &= (f(P) \otimes g(Q))^m (h(P) \otimes k(Q))^n \end{aligned}$$

and the equality (2.6) is that proved. □ □

The additive version is as follows:

LEMMA 2. *Assume P and Q are selfadjoint operators with* $\operatorname{Sp}(P) \subset I$, $\operatorname{Sp}(Q) \subset J$ *and having the spectral resolutions* (2.7). *Let f, h be continuous on I, g, k continuous on J and φ and ψ continuous on an interval K that contains the sum of the intervals $f(I)+g(J)$, $k(I)+k(J)$, then*

$$\begin{aligned} &\varphi(f(P)\otimes 1+1\otimes g(Q))\,\psi(h(P)\otimes 1+1\otimes k(Q)) \\ &=\int_I\int_J \varphi(f(t)+g(s))\,\psi(h(t)+k(s))\,dE_t\otimes dF_s. \end{aligned} \tag{2.8}$$

Proof. Let a, b, c and d positive continuous functions such that $f(t)=\ln a(t)$, $h(t)=\ln c(t)$ for $t\in I$ and $g(s)=\ln b(s)$, $k(s)=\ln d(s)$ for $s\in J$. Then

$$\begin{aligned} &\int_I\int_J \varphi(f(t)+g(s))\,\psi(h(t)+k(s))\,dE_t\otimes dF_s \\ &=\int_I\int_J \varphi(\ln a(t)+\ln b(s))\,\psi(\ln c(t)+\ln d(s))\,dE_t\otimes dF_s \\ &=\int_I\int_J (\varphi\circ\ln)(a(t)\,b(s))\,(\psi\circ\ln)(c(t)\,d(s))\,dE_t\otimes dF_s. \end{aligned} \tag{2.9}$$

If we use Lemma 1 for the functions $\varphi\circ\ln$ and $(\psi\circ\ln)$, we get

$$\begin{aligned} &\int_I\int_J (\varphi\circ 2.8\ln)(a(t)\,b(s))\,(\psi\circ\ln)(c(t)\,d(s))\,dE_t\otimes dF_s \\ &=(\varphi\circ\ln)(a(P)\otimes b(Q))\,(\psi\circ\ln)(c(P)\otimes d(Q)) \\ &=\varphi[\ln(a(P)\otimes b(Q))]\,\psi[\ln(c(P)\otimes d(Q))]. \end{aligned} \tag{2.10}$$

Now, observe that, by the commutativity of the operators $a(P)\otimes 1$ and $1\otimes b(Q)$,

$$\begin{aligned} \ln(a(P)\otimes b(Q)) &= \ln[(a(P)\otimes 1)(1\otimes b(Q))] \\ &= \ln(a(P)\otimes 1)+\ln(1\otimes b(Q)) \\ &= [\ln a(P)]\otimes 1+1\otimes \ln b(Q) \text{ (by (2.6))} \\ &= f(P)\otimes 1+1\otimes g(Q) \end{aligned}$$

and, similarly

$$\ln(c(P)\otimes d(Q)) = h(P)\otimes 1+1\otimes k(Q).$$

By utilising (2.9) and (2.10) we then get the desired representation (2.8) □ □

Our first main result is as follows:

THEOREM 2. *Assume that f is of class C^{n+1} on the open interval I, P and Q are selfadjoint operators with* $\operatorname{Sp}(P)\subset I$ *and* $\operatorname{Sp}(Q)\subset I$, *then we have the representation*

$$\begin{aligned} &f(P)\otimes 1 \\ &=\sum_{k=0}^{n}\frac{1}{k!}(P\otimes 1-1\otimes Q)^k\left(1\otimes f^{(k)}(Q)\right) \\ &\quad+\frac{1}{n!}(P\otimes 1-1\otimes Q)^{n+1}\int_0^1(1-u)^n f^{(n+1)}((1-u)1\otimes Q+uP\otimes 1)\,du. \end{aligned} \tag{2.11}$$

Proof. Using Taylor's representation with the integral remainder (1.1) we can write the following two identities

$$f(x)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(a)(x-a)^{k}+\frac{1}{n!}\int_{a}^{x}f^{(n+1)}(y)(x-y)^{n}dy \tag{2.12}$$

for $x,a\in I$.

For any integrable function h on an interval and any distinct numbers $c,\ d$ in that interval, we have, by the change of variable $y=(1-u)c+ud,\ u\in[0,1]$ that

$$\int_{c}^{d}h(y)\,dy=(d-c)\int_{0}^{1}h((1-u)c+ud)\,du.$$

Therefore,

$$\begin{aligned}&\int_{a}^{x}f^{(n+1)}(y)(x-y)^{n}dy\\&=(x-a)\int_{0}^{1}f^{(n+1)}((1-u)a+ux)(x-(1-u)a-ux)^{n}du\\&=(x-a)^{n+1}\int_{0}^{1}f^{(n+1)}((1-u)a+ux)(1-u)^{n}du\end{aligned}$$

and the identity (2.12) becomes

$$\begin{aligned}f(t)=&\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(s)(t-s)^{k}\\&+\frac{1}{n!}(t-s)^{n+1}\int_{0}^{1}f^{(n+1)}((1-u)s+ut)(1-u)^{n}du,\end{aligned} \tag{2.13}$$

for all $t,\ s\in I$.

If P and Q have the spectral resolutions

$$P=\int_{I}t\,dE(t)\ \text{and}\ Q=\int_{I}s\,dF(s),$$

then by taking the integral $\int_I\int_I$ over $dE_t\otimes dF_s$, we get

$$\begin{aligned}&\int_{I}\int_{I}f(t)\,dE_t\otimes dF_s\\&=\int_{I}\int_{I}\left(\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(s)(t-s)^{k}\right)dE_t\otimes dF_s\\&\quad+\frac{1}{n!}\int_{I}\int_{I}(t-s)^{n+1}\left(\int_{0}^{1}f^{(n+1)}((1-u)s+ut)(1-u)^{n}du\right)dE_t\otimes dF_s.\end{aligned} \tag{2.14}$$

We have

$$\int_{I}\int_{I}f(t)\,dE_t\otimes dF_s=f(P)\otimes 1$$

and, by (2.8)

$$\begin{aligned}
&\int_I\int_I\left(\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(s)(t-s)^k\right)dE_t\otimes dF_s\\
&=\sum_{k=0}^{n}\frac{1}{k!}\int_I\int_I\left((t-s)^k f^{(k)}(s)\right)dE_t\otimes dF_s\\
&=\sum_{k=0}^{n}\frac{1}{k!}(P\otimes 1-1\otimes Q)^k\left(1\otimes f^{(k)}(Q)\right).
\end{aligned}$$

By Fubini's theorem and (2.8) we also get

$$\begin{aligned}
&\int_I\int_I(t-s)^{n+1}\left(\int_0^1 f^{(n+1)}((1-u)s+ut)(1-u)^n du\right)dE_t\otimes dF_s\\
&=\int_0^1(1-u)^n\left(\int_I\int_I(t-s)^{n+1}\left(f^{(n+1)}((1-u)s+ut)\right)dE_t\otimes dF_s\right)du\\
&=(P\otimes 1-1\otimes Q)^{n+1}\int_0^1(1-u)^n\left(f^{(n+1)}((1-u)1\otimes Q+uP\otimes 1)\right)du
\end{aligned}$$

and by (2.14) we obtain the desired result (2.11). □ □

COROLLARY 1. *With the assumptions of Theorem 2 and if*

$$\left\|f^{(n+1)}\right\|_{I,\infty}:=\sup_{u\in I}\left|f^{(n+1)}(u)\right|<\infty,$$

then

$$\begin{aligned}
&\left\|f(P)\otimes 1-\sum_{k=0}^{n}\frac{1}{k!}(P\otimes 1-1\otimes Q)^k\left(1\otimes f^{(k)}(Q)\right)\right\|\\
&\leqslant\frac{1}{(n+1)!}\|P\otimes 1-1\otimes Q\|^{n+1}\left\|f^{(n+1)}\right\|_{I,\infty}.
\end{aligned}\tag{2.15}$$

Proof. From (2.11) we derive

$$\begin{aligned}
&\left\|f(P)\otimes 1-\sum_{k=0}^{n}\frac{1}{k!}(P\otimes 1-1\otimes Q)^k\left(1\otimes f^{(k)}(Q)\right)\right\|\\
&\leqslant\frac{1}{n!}\int_0^1(1-u)^n\left\|(P\otimes 1-1\otimes Q)^{n+1}f^{(n+1)}((1-u)1\otimes Q+uP\otimes 1)\right\|du\\
&\leqslant\frac{1}{n!}\|P\otimes 1-1\otimes Q\|^{n+1}\int_0^1(1-u)^n\left\|f^{(n+1)}((1-u)1\otimes Q+uP\otimes 1)\right\|du\\
&=\frac{1}{n!}\|P\otimes 1-1\otimes Q\|^{n+1}\int_0^1(1-u)^n\left\|f^{(n+1)}((1-u)1\otimes Q+uP\otimes 1)\right\|du.
\end{aligned}\tag{2.16}$$

Now, observe that
$$\left|f^{(n+1)}\left((1-u)s+ut\right)\right| \leqslant \left\|f^{(n+1)}\right\|_{I,\infty}$$
by taking the integral $\int_I \int_I$ over $dE_t \otimes dF_s$, we get
$$\int_I \int_I \left|f^{(n+1)}\left((1-u)s+ut\right)\right| dE_t \otimes dF_s \leqslant \left\|f^{(n+1)}\right\|_{I,\infty} \int_I \int_I dE_t \otimes dF_s,$$
namely
$$\left|f^{(n+1)}\left((1-u)1\otimes Q+uP\otimes 1\right)\right| \leqslant \left\|f^{(n+1)}\right\|_{I,\infty} \tag{2.17}$$
for all $u \in [0,1]$.

If we take the norm in (2.17), then we get
$$\left\|f^{(n+1)}\left((1-u)1\otimes Q+uP\otimes 1\right)\right\| \leqslant \left\|f^{(n+1)}\right\|_{I,\infty},$$
which implies that
$$\begin{aligned}&\int_0^1 (1-u)^n \left\|f^{(n+1)}\left((1-u)1\otimes Q+uP\otimes 1\right)\right\| du \\ &\leqslant \left\|f^{(n+1)}\right\|_{I,\infty} \int_0^1 (1-u)^n du = \frac{1}{n+1}\left\|f^{(n+1)}\right\|_{I,\infty}.\end{aligned}$$
By utilising (2.16) we get the desired result (2.15). □ □

COROLLARY 2. *With the assumptions of Theorem 2 and if* $\left|f^{(n+1)}\right|$ *is continuous convex on* I, *then*
$$\begin{aligned}&\left\|f(P)\otimes 1-\sum_{k=0}^{n}\frac{1}{k!}(P\otimes 1-1\otimes Q)^k\left(1\otimes f^{(k)}(Q)\right)\right\| \\ &\leqslant \frac{1}{(n+1)!}\|P\otimes 1-1\otimes Q\|^{n+1}\left[\frac{\left\|f^{(n+1)}(P)\right\|+(n+1)\left\|f^{(n+1)}(Q)\right\|}{n+2}\right].\end{aligned} \tag{2.18}$$

Proof. By the convexity of $\left|f^{(n+1)}\right|$ we have
$$\left|f^{(n+1)}\left((1-u)s+ut\right)\right| \leqslant (1-u)\left|f^{(n+1)}(s)\right|+u\left|f^{(n+1)}(t)\right| \tag{2.19}$$
for all $t,s \in I$ and $u \in [0,1]$.

By taking the integral $\int_I \int_I$ over $dE_t \otimes dF_s$, we get
$$\begin{aligned}&\int_I \int_I \left|f^{(n+1)}\left((1-u)s+ut\right)\right| dE_t \otimes dF_s \\ &\leqslant \int_I \int_I \left[(1-u)\left|f^{(n+1)}(s)\right|+u\left|f^{(n+1)}(t)\right|\right] dE_t \otimes dF_s \\ &= (1-u)\int_I \int_I \left|f^{(n+1)}(s)\right| dE_t \otimes dF_s + u\int_I \int_I \left|f^{(n+1)}(t)\right| dE_t \otimes dF_s\end{aligned}$$

namely

$$\begin{aligned}&\left|f^{(n+1)}\left((1-u)1\otimes Q+uP\otimes 1\right)\right|\\&\leqslant(1-u)\left|f^{(n+1)}\left(1\otimes Q\right)\right|+u\left|f^{(n+1)}\left(P\otimes 1\right)\right|\\&=(1-u)\left|1\otimes f^{(n+1)}\left(Q\right)\right|+u\left|f^{(n+1)}\left(P\right)\otimes 1\right|\end{aligned}$$

for all $u\in[0,1]$.

If we take the norm, then we get

$$\begin{aligned}&\left\|f^{(n+1)}\left((1-u)1\otimes Q+uP\otimes 1\right)\right\|\\&\leqslant\left\|(1-u)\left|1\otimes f^{(n+1)}\left(Q\right)\right|+u\left|f^{(n+1)}\left(P\right)\otimes 1\right|\right\|\\&\leqslant(1-u)\left\|1\otimes f^{(n+1)}\left(Q\right)\right\|+u\left\|f^{(n+1)}\left(P\right)\otimes 1\right\|\\&=(1-u)\left\|f^{(n+1)}\left(Q\right)\right\|+u\left\|f^{(n+1)}\left(P\right)\right\|\end{aligned}$$

for all $u\in[0,1]$.

If we multiply by $(1-u)^n$ and integrate on $[0,1]$, then we get

$$\begin{aligned}&\int_0^1(1-u)^n\left\|f^{(n+1)}\left((1-u)1\otimes Q+uP\otimes 1\right)\right\|du\\&\leqslant\int_0^1(1-u)^n\left[(1-u)\left\|f^{(n+1)}\left(Q\right)\right\|+u\left\|f^{(n+1)}\left(P\right)\right\|\right]du\\&=\left\|f^{(n+1)}\left(Q\right)\right\|\int_0^1(1-u)^{n+1}du+\left\|f^{(n+1)}\left(P\right)\right\|\int_0^1(1-u)^n udu\\&=\frac{1}{n+2}\left\|f^{(n+1)}\left(Q\right)\right\|+\frac{1}{(n+1)(n+2)}\left\|f^{(n+1)}\left(P\right)\right\|\end{aligned}$$

and by utilising (2.16) we derive the desired result (2.18). □ □

We recall that the function $g:I\to\mathbb{R}$ is *quasi-convex*, if

$$\begin{aligned}g\left((1-\lambda)t+\lambda s\right)&\leqslant\max\left\{g\left(t\right),g\left(s\right)\right\}\\&=\frac{1}{2}\left(g\left(t\right)+g\left(s\right)+\left|g\left(t\right)-g\left(s\right)\right|\right)\end{aligned}$$

for all $t,s\in I$ and $\lambda\in[0,1]$.

COROLLARY 3. *With the assumptions of Theorem 2 and if* $\left|f^{(n+1)}\right|$ *is continuous* *quasi-convex on* I, *then*

$$\begin{aligned} &\left\| f(P) \otimes 1 - \sum_{k=0}^{n} \frac{1}{k!} (P \otimes 1 - 1 \otimes Q)^k \left(1 \otimes f^{(k)}(Q)\right) \right\| \\ &\leqslant \frac{1}{(n+1)!} \|P \otimes 1 - 1 \otimes Q\|^{n+1} \\ &\quad \times \left[\left\| \frac{\left|f^{(n+1)}(P)\right| \otimes 1 + 1 \otimes \left|f^{(n+1)}(Q)\right|}{2} \right\| \right. \\ &\quad \left. + \left\| \frac{\left|f^{(n+1)}(P)\right| \otimes 1 - 1 \otimes \left|f^{(n+1)}(Q)\right|}{2} \right\| \right]. \end{aligned} \tag{2.20}$$

Proof. By the quasi-convexity of $\left|f^{(n+1)}\right|$ we have

$$\begin{aligned} &\left|f^{(n+1)}((1-u)s+ut)\right| \\ &\leqslant \frac{1}{2}\left(\left|f^{(n+1)}(t)\right| + \left|f^{(n+1)}(s)\right| + \left|\left|f^{(n+1)}(t)\right| - \left|f^{(n+1)}(s)\right|\right|\right) \end{aligned}$$

for all $t, s \in I$ and $u \in [0,1]$.

By taking the integral $\int_I \int_I$ over $dE_t \otimes dF_s$, we get

$$\begin{aligned} &\int_I \int_I \left|f^{(n+1)}((1-u)s+ut)\right| dE_t \otimes dF_s \\ &\leqslant \frac{1}{2} \int_I \int_I \left(\left|f^{(n+1)}(t)\right| + \left|f^{(n+1)}(s)\right| + \left|\left|f^{(n+1)}(t)\right| - \left|f^{(n+1)}(s)\right|\right|\right) \\ &\quad \times dE_t \otimes dF_s \\ &= \frac{1}{2}\left[\int_I \int_I \left|f^{(n+1)}(t)\right| dE_t \otimes dF_s + \int_I \int_I \left|f^{(n+1)}(s)\right| dE_t \otimes dF_s \right. \\ &\quad \left. + \int_I \int_I \left|\left|f^{(n+1)}(t)\right| - \left|f^{(n+1)}(s)\right|\right| dE_t \otimes dF_s \right], \end{aligned}$$

namely

$$\begin{aligned} &\left|f^{(n+1)}((1-u)1 \otimes Q + uP \otimes 1)\right| \\ &\leqslant \frac{1}{2}\left[\left|f^{(n+1)}(P)\right| \otimes 1 + 1 \otimes \left|f^{(n+1)}(Q)\right| \right. \\ &\quad \left. + \left|\left|f^{(n+1)}(P)\right| \otimes 1 - 1 \otimes \left|f^{(n+1)}(Q)\right|\right|\right] \end{aligned}$$

for all $u \in [0,1]$.

If we take the norm, then we get

$$\begin{aligned}
&\left\|f^{(n+1)}\left((1-u)1\otimes Q+uP\otimes 1\right)\right\| \\
&\leqslant \frac{1}{2}\left\|\left|f^{(n+1)}(P)\right|\otimes 1+1\otimes\left|f^{(n+1)}(Q)\right| \right.\\
&\qquad\left.+\left|\left|f^{(n+1)}(P)\right|\otimes 1-1\otimes\left|f^{(n+1)}(Q)\right|\right|\right\| \\
&\leqslant \frac{1}{2}\left\|\left|f^{(n+1)}(P)\right|\otimes 1+1\otimes\left|f^{(n+1)}(Q)\right|\right\| \\
&\qquad+\frac{1}{2}\left\|\left|f^{(n+1)}(P)\right|\otimes 1-1\otimes\left|f^{(n+1)}(Q)\right|\right\|
\end{aligned}$$

for all $u\in[0,1]$.

If we multiply by $(1-u)^n$, $u\in[0,1]$ and take the integral, then we get

$$\begin{aligned}
&\int_0^1(1-u)^n\left\|f^{(n+1)}\left((1-u)1\otimes Q+uP\otimes 1\right)\right\|du \\
&\leqslant \frac{1}{2(n+1)}\left[\left\|\left|f^{(n+1)}(P)\right|\otimes 1+1\otimes\left|f^{(n+1)}(Q)\right|\right\|\right.\\
&\qquad\left.+\left\|\left|f^{(n+1)}(P)\right|\otimes 1-1\otimes\left|f^{(n+1)}(Q)\right|\right\|\right]
\end{aligned}$$

and by utilising (2.16) we derive the desired result (2.20). □ □

3. Some examples

We consider the function $f(t)=\ln t$, $t>0$. Then

$$f^{(k)}(t)=\frac{(-1)^{k-1}(k-1)!}{t^k},\quad k\geqslant 1,\quad t>0.$$

From (2.11) we then get

$$\begin{aligned}
\ln(P)\otimes 1&=1\otimes\ln Q+\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}(P\otimes 1-1\otimes Q)^k\left(1\otimes Q^{-k}\right) \\
&\quad+(-1)^n(P\otimes 1-1\otimes Q)^{n+1} \\
&\quad\times\int_0^1(1-s)^n\left[(1-u)1\otimes Q+uP\otimes 1\right]^{-n-1}ds
\end{aligned}\tag{3.1}$$

for all $P,\ Q>0$.

For $n=0$ we have

$$\begin{aligned}
\ln(P)\otimes 1&=1\otimes\ln Q+(P\otimes 1-1\otimes Q) \\
&\quad\times\int_0^1\left[(1-u)1\otimes Q+uP\otimes 1\right]^{-1}ds,
\end{aligned}\tag{3.2}$$

while for $n=1$ we derive

$$\begin{aligned}\ln(P)\otimes 1 = 1\otimes \ln Q + P\otimes Q^{-1} - 1 \\ -(P\otimes 1 - 1\otimes Q)^2 \\ \times \int_0^1 (1-s)\left[(1-u)1\otimes Q + uP\otimes 1\right]^{-2} ds\end{aligned} \tag{3.3}$$

for all $P,\ Q>0$.

Now, if $P,\ Q>m>0$ for some constant m, then by (2.15) we get

$$\begin{aligned}&\left\|\ln(P)\otimes 1 - 1\otimes \ln Q - \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}(P\otimes 1 - 1\otimes Q)^k\left(1\otimes Q^{-k}\right)\right\| \\ &\leqslant \frac{1}{(n+1)m^{n+1}}\|P\otimes 1 - 1\otimes Q\|^{n+1}.\end{aligned} \tag{3.4}$$

For $n=0$ we get

$$\|\ln(P)\otimes 1 - 1\otimes \ln Q\| \leqslant \frac{1}{m}\|P\otimes 1 - 1\otimes Q\|, \tag{3.5}$$

while for $n=1$ we obtain

$$\left\|\ln(P)\otimes 1 - 1\otimes \ln Q - P\otimes Q^{-1} + 1\right\| \leqslant \frac{1}{2m^2}\|P\otimes 1 - 1\otimes Q\|^2$$

provided that if $P,\ Q>m>0$.

Since $\left|f^{(n+1)}(t)\right| = \frac{n!}{t^{n+1}}$ which is convex on $(0,\infty)$, then by (2.18) we get

$$\begin{aligned}&\left\|\ln(P)\otimes 1 - 1\otimes \ln Q - \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}(P\otimes 1 - 1\otimes Q)^k\left(1\otimes Q^{-k}\right)\right\| \\ &\leqslant \frac{1}{n+1}\|P\otimes 1 - 1\otimes Q\|^{n+1}\left[\frac{\left\|P^{-n-1}\right\| + (n+1)\left\|Q^{-n-1}\right\|}{n+2}\right]\end{aligned} \tag{3.6}$$

for all $P,\ Q>0$.

For $n=0$ we obtain

$$\|\ln(P)\otimes 1 - 1\otimes \ln Q\| \leqslant \|P\otimes 1 - 1\otimes Q\|\left[\frac{\left\|P^{-1}\right\| + \left\|Q^{-1}\right\|}{2}\right] \tag{3.7}$$

while for $n=1$ we derive

$$\begin{aligned}&\left\|\ln(P)\otimes 1 - 1\otimes \ln Q - P\otimes Q^{-1} + 1\right\| \\ &\leqslant \frac{1}{2}\|P\otimes 1 - 1\otimes Q\|^2\left[\frac{\left\|P^{-2}\right\| + 2\left\|Q^{-2}\right\|}{3}\right]\end{aligned} \tag{3.8}$$

Consider the exponential function $f(t) = \exp(t)$, then by (2.11) we get

$$\exp(P) \otimes 1 = \sum_{k=0}^{n} \frac{1}{k!} (P \otimes 1 - 1 \otimes Q)^k (1 \otimes \exp Q) + \frac{1}{n!} (P \otimes 1 - 1 \otimes Q)^{n+1} \times \int_0^1 (1-s)^n \exp[(1-u) 1 \otimes Q + uP \otimes 1] ds \tag{3.9}$$

for any selfadjoint operators P, Q.

If $P, Q \leqslant M$ for some constant M, then by (2.15) we get

$$\left\| \exp(P) \otimes 1 - \sum_{k=0}^{n} \frac{1}{k!} (P \otimes 1 - 1 \otimes Q)^k (1 \otimes \exp Q) \right\| \leqslant \frac{1}{(n+1)!} \exp(M) \|P \otimes 1 - 1 \otimes Q\|^{n+1}. \tag{3.10}$$

Since $\left| f^{(n+1)}(t) \right| = \exp t$ is convex on $\mathbb{R}$, then by (2.18) we get

$$\left\| \exp(P) \otimes 1 - \sum_{k=0}^{n} \frac{1}{k!} (P \otimes 1 - 1 \otimes Q)^k (1 \otimes \exp Q) \right\| \leqslant \frac{1}{(n+1)!} \|P \otimes 1 - 1 \otimes Q\|^{n+1} \left[\frac{\|\exp P\| + (n+1) \|\exp Q\|}{n+2} \right] \tag{3.11}$$

for any selfadjoint operators P, Q.

If $C, D > 0$ and if we take in (3.9) $P = \ln C$, $Q = \ln D$, then we get

$$C \otimes 1 = \sum_{k=0}^{n} \frac{1}{k!} (\ln(C) \otimes 1 - 1 \otimes \ln D)^k (1 \otimes D) + \frac{1}{n!} (\ln(C) \otimes 1 - 1 \otimes \ln D)^{n+1} \times \int_0^1 (1-s)^n \exp[(1-u) 1 \otimes \ln D + u \ln(C) \otimes 1] ds. \tag{3.12}$$

For $n = 0$ we get

$$C \otimes 1 = (1 \otimes D) + (\ln(C) \otimes 1 - 1 \otimes \ln D) \times \int_0^1 \exp[(1-u) 1 \otimes \ln D + u \ln(C) \otimes 1] ds, \tag{3.13}$$

while for $n = 1$ we derive

$$C \otimes 1 = 1 \otimes D + (\ln(C) \otimes 1 - 1 \otimes \ln D)(1 \otimes D) + (\ln(C) \otimes 1 - 1 \otimes \ln D)^2 \times \int_0^1 (1-s) \exp[(1-u) 1 \otimes \ln D + u \ln(C) \otimes 1] ds. \tag{3.14}$$

If $0 < C,\ D < N$ for some constant $N > 0,$ then by (3.10)

$$\left\| C \otimes 1 - \sum_{k=0}^{n} \frac{1}{k!} \left(\ln(C) \otimes 1 - 1 \otimes \ln D \right)^{k} (1 \otimes D) \right\| \leqslant \frac{1}{(n+1)!} N \left\| \ln(C) \otimes 1 - 1 \otimes \ln D \right\|^{n+1}. \tag{3.15}$$

For $n = 0$ we get

$$\| C \otimes 1 - (1 \otimes D) \| \leqslant N \left\| \ln(C) \otimes 1 - 1 \otimes \ln D \right\|, \tag{3.16}$$

provided that $0 < C,\ D < N,$ while for $n = 1$

$$\| C \otimes 1 - 1 \otimes D - \left(\ln(C) \otimes 1 - 1 \otimes \ln D \right) (1 \otimes D) \| \leqslant \frac{1}{(n+1)!} N \left\| \ln(C) \otimes 1 - 1 \otimes \ln D \right\|^{2}$$

From (3.11) we get

$$\left\| C \otimes 1 - \sum_{k=0}^{n} \frac{1}{k!} \left(\ln(C) \otimes 1 - 1 \otimes \ln D \right)^{k} (1 \otimes D) \right\| \leqslant \frac{1}{(n+1)!} \left\| \ln(C) \otimes 1 - 1 \otimes \ln D \right\|^{n+1} \left[\frac{\|C\| + (n+1)\|D\|}{n+2} \right] \tag{3.17}$$

for $C,\ D > 0.$

For $n = 0,$ we get

$$\| C \otimes 1 - (1 \otimes D) \| \leqslant \frac{\|C\| + \|D\|}{2} \left\| \ln(C) \otimes 1 - 1 \otimes \ln D \right\|, \tag{3.18}$$

while for $n = 1,$

$$\| C \otimes 1 - 1 \otimes D - \left(\ln(C) \otimes 1 - 1 \otimes \ln D \right) (1 \otimes D) \| \leqslant \frac{\|C\| + 2\|D\|}{6} \left\| \ln(C) \otimes 1 - 1 \otimes \ln D \right\|^{2} \tag{3.19}$$

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Silvestru Sever Dragomir
Applied Mathematics Research Group, ISILC
Victoria University
P.O. Box 14428, Melbourne City, MC 8001, Australia
e-mail: sever.dragomir@vu.edu.au
http://rgmia.org/dragomir

Mubariz Tapdigoglu Garayev
Department of Mathematics, College of Science
King Saud University
P.O. Box 2455, Riyadh 11451, Saudi Arabia
e-mail: mgarayev@ksu.edu.sa