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ON THE β–MODIFICATION OF THE MELLIN–GAUSS–WEIERSTRASS KERNEL AND ITS RELATED INFORMATION POTENTIAL

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Abstract. In the current study, we investigate the behaviour of β-modification of the Mellin-Gauss-Weierstrass (MGW) type operators with respect to pointwise and uniform convergence. Moreover, we give a Voronovskaya approximation formula for the MGW type operators using the new kernel. This formula contains Mellin derivatives and a different notion of moment which was called the logarithmic moment. In the last part, we analyze the related information potential, the variance $V[logp(\cdot,\cdot)]$ and expected value $EV[logp(\cdot,\cdot)]$ using the modified MGW kernel $p(\cdot,\cdot)$.

1. Introduction

The importance of Mellin integral operators is well-known not only in approximation theory (see, e.g., [13], [17]), but also in the various applications for example in optical physics and engineering. Indeed, they can be successfully used in problems of signal reconstruction where the samples are not uniformly spaced, as in the classical Shannon Sampling Theorem, but exponentially spaced.

Starting from the paper [13], the approximation by Mellin convolution operators is evolved using a more direct and inherent way, totally unconnected from the Fourier theory, based on the concepts of Mellin derivatives, 'logarithmic'uniform continuity and 'logarithmic'moment of a kernel function, which give a different and powerful approach.

In the recent important papers, the pointwise approximation theory for nets of Mellin convolution operators, acting on functions defined on the multiplicative group $\mathbb{R}^+$ have been improved by Bardaro and Mantellini (see e.g. [8], [7]). Mellin convolution operators express a significant tool in the Mellin transform theory, applying the similar goal of the classical convolution operators in Fourier analysis ([13], [14]). For the pointwise convergence of Mellin type convolution operators, Voronovskaya theorems can be found in [7] and [10]. A similar result using the Taylor formula in terms of Mellin derivatives and considering a notion of the logarithmic moment of the kernel was obtained in [6].

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Additionally, many studies have been carried out for similar operators on the subject. For example, in [5], Aral et al. present a family of operators considering Kantorovich-type generalizations of the exponential sampling series based on MGW kernel. Very recently, in [4], a new modulus of continuity for locally integrable function spaces is introduced and the acquired outputs are performed to the Gauss-Weierstrass operators. Moreover, in [19], Ozsarac et al. state a modification of singular integral of Mellin convolution type, and the obtained results are mentioned for the MGW operator.

The approximation properties of these convolution operators are connected with the solutions of certain boundary value problems in wedge-shaped regions [16], [20]. Indeed, in one dimensional case, the solution is expressed in terms of the convolution integrals of type:

$$(T_w f)(s) = \int_0^\infty K_w(t) f(ts) \frac{dt}{t}, \tag{1.1}$$

where f represents a boundary data and $\{K_w\}_{w \geqslant w_0}$ is a suitable family of kernel functions. Therefore, it seems to be interesting to study the rate of convergence to given f of such integrals, in various sense. Since the beginning of the 21-st century, these topics were broadly developed, in case of classical convolution operators of Fourier analysis ([2], [3]).

The paper presents approximation properties of a modified version of the Mellin-Gauss-Weierstrass convolution operator, in the space of continuous functions. Indeed, the operator under consideration, is generated by a kernel that in the classical setting of Fourier analysis was introduced many years ago by Bui, Fedorov and Cervakov (see [12]; also [11]). The Mellin version of the above operator, is here obtained using the same method as for the Gauss-Weierstrass operator.

In the current paper, we research the behaviour of β-modification of the MGW type operators with respect to pointwise and uniform convergence for functions defined over the positive real axis, using a suitable modulus of continuity. Moreover, we express a Voronovskaya approximation formula for the MGW type operators using a new kernel. This formula includes Mellin derivatives and a different notion of moment which was called the logarithmic moment. The last section is devoted to applications to Information Theoretic Learning. Using the modified MGW kernel $p(\cdot,\cdot)$, we examine the related information potential, $EV[logp(\cdot,\cdot)]$ and $V[logp(\cdot,\cdot)]$.

2. Notations and preliminary results

Let $\mathbb{R}^+$ be the multiplicative topological group endowed with the logarithmic (Haar) measure

$$\mu(H) = \int_H \frac{dt}{t},$$

being dt the Lebesgue measure and H is any (Lebesgue) measurable set.

By L^∞, we denote the space comprising all the essentially bounded functions defined on $\mathbb{R}^+$, and endowed with the usual norm $\|f\|_\infty := ess\sup_{x>0} |f(x)|$.

In what follows, we say that $f \in C^k$ locally at the point $p \in \mathbb{R}^+$ if there is a neighbourhood U_p of the point p such that f is $(k-1)$-times continuously differentiable in U_p and the derivative of order k exists at the point p.

The kernel generating our modified MGW operator is defined by

$$\mathscr{K}(s,t) = \frac{\beta}{4} \frac{1}{\Gamma\left(\frac{1}{\beta}\right)} \exp\left(-\left(\left|\frac{1}{2}\log\frac{t}{s}\right|\right)^{\beta}\right), \quad \beta, t, s \in \mathbb{R}^+,$$

where Γ is the Euler Gamma function. The above kernel is homogeneous of degree 0, that is

$$\mathscr{K}(\gamma s, \gamma t) = \mathscr{K}(s,t).$$

Moreover, we have

$$\int_0^\infty \mathscr{K}(s,t) \frac{dt}{t} = 1. \tag{2.1}$$

According to the definitions of algebraic and absolute logarithmic moment given in [9], for the logarithmic moment of order j of the function $\mathscr{K}$, we obtain if j is odd $m_j(\mathscr{K}) = 0$. If j is even, it is obtained that

$$\begin{aligned}
m_j(\mathscr{K}) &= \int_0^\infty \mathscr{K}(s,t) \log^j\left(\frac{t}{s}\right) \frac{dt}{t} \\
&= \frac{\beta}{4} \frac{1}{\Gamma\left(\frac{1}{\beta}\right)} \int_0^\infty e^{\left(-\left(\left|\frac{1}{2}\log\frac{t}{s}\right|\right)^{\beta}\right)} \log^j\left(\frac{t}{s}\right) \frac{dt}{t} \\
&= \frac{\beta}{4} \frac{1}{\Gamma\left(\frac{1}{\beta}\right)} \int_0^\infty e^{\left(-\left(\left|\frac{1}{2}\log z\right|\right)^{\beta}\right)} \log^j z \frac{dz}{z} \\
&= \frac{\beta}{4} \frac{1}{\Gamma\left(\frac{1}{\beta}\right)} \int_{-\infty}^\infty e^{\left(-\left(\left|\frac{1}{2}u\right|\right)^{\beta}\right)} u^j du \\
&= \frac{\beta}{2\Gamma\left(\frac{1}{\beta}\right)} \int_0^\infty e^{-\left(\frac{u}{2}\right)^{\beta}} u^j du \\
&= 2^j \frac{\Gamma\left(\frac{j+1}{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)}.
\end{aligned}$$

Similarly, using certain symmetry properties, we obtain

$$\begin{aligned} M_j(\mathscr{K}) &= \int_0^\infty \mathscr{K}(s,t)\left|\log^j\left(\frac{t}{s}\right)\right|\frac{dt}{t} \\ &= 2^j\frac{\Gamma\left(\frac{j+1}{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)}. \end{aligned}$$

The kernel $\mathscr{K}$ generates the family $\left(\mathscr{K}_\rho\right)_{\rho>0}$ on setting

$$\mathscr{K}_\rho(s,t) := \rho\mathscr{K}(s^\rho,t^\rho), \quad \rho,s,t\in\mathbb{R}^+.$$

In that case, we can write

$$\mathscr{K}_\rho(s,t) = \frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\exp\left(-\left(\left|\frac{\rho}{2}\log\frac{t}{s}\right|^\beta\right)\right).$$

Obviously, every function $\mathscr{K}_\rho$ is homogeneous of degree 0.

Again, according to the definitions given in [9], the corresponding moments of order j of the kernel $\mathscr{K}_\rho$ are

$$\begin{aligned} m_j\left(\mathscr{K}_\rho\right) &:= \int_0^\infty \mathscr{K}_\rho(s,t)\log^j\left(\frac{t}{s}\right)\frac{dt}{t} \\ &= \frac{2^j}{\rho^j}\frac{\left(1+(-1)^j\right)\Gamma\left(\frac{j+1}{\beta}\right)}{2\Gamma\left(\frac{1}{\beta}\right)} \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} M_j\left(\mathscr{K}_\rho\right) &:= \int_0^\infty \mathscr{K}_\rho(s,t)\left|\log^j\left(\frac{t}{s}\right)\right|\frac{dt}{t} \\ &= \frac{2^j}{\rho^j}\frac{\Gamma\left(\frac{j+1}{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)}. \end{aligned}$$

For the above MGW kernel, we have the following property:

PROPOSITION 1. *For every $\delta>1$, putting $U_\delta(s) = \left(\frac{s}{\delta}, s\delta\right)$, we have*

$$\lim_{\rho\to\infty}\int_{\mathbb{R}^+\setminus U_\delta(s)} \mathscr{K}_\rho(s,t)\frac{dt}{t} = 0$$

uniformly with respect to $s \in \mathbb{R}^+$. *Furthermore, if* $M_j(\mathscr{K})$ *is finite, then*

$$\lim_{\rho\to\infty} \rho^j \int\limits_{\mathbb{R}^+\setminus U_\delta(s)} \mathscr{K}_\rho(s,t) \left|\log^j\left(\frac{t}{s}\right)\right| \frac{dt}{t} = 0$$

and

$$\lim_{\rho\to\infty} \rho^j \int\limits_{\mathbb{R}^+\setminus U_\delta(s)} \mathscr{K}_\rho(s,t) \frac{dt}{t} = 0$$

uniformly with respect to $s \in \mathbb{R}^+$.

Proof. First, let us consider the second part. We obtain

$$\begin{aligned}\rho^j \int\limits_{\mathbb{R}^+\setminus U_\delta(s)} \mathscr{K}_\rho(s,t) \left|\log^j\left(\frac{t}{s}\right)\right| \frac{dt}{t} &= \rho^j \int\limits_{\mathbb{R}^+\setminus U_\delta(1)} \mathscr{K}_\rho(1,z) \left|\log^j z\right| \frac{dz}{z} \\ &= \rho^j \frac{\beta}{4} \frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)} \int\limits_{\mathbb{R}^+\setminus U_\delta(1)} e^{-\left|\frac{\rho}{2}\log z\right|^\beta} \left|\log^j z\right| \frac{dz}{z} \\ &= 2^j \frac{\beta}{\Gamma\left(\frac{1}{\beta}\right)} \int\limits_{\mathbb{R}^+\setminus U_{\delta\rho}(1)} e^{-|\log v|^\beta} \left|\log^j v\right| \frac{dv}{v}\end{aligned}$$

and since $M_j(\mathscr{K})$ is finite, the last integral tends to zero as $\rho \to \infty$.

Furthermore, for a given $\delta > 1$, we get

$$\begin{aligned}\rho^j \int\limits_{\mathbb{R}^+\setminus U_\delta(s)} \mathscr{K}_\rho(s,t) \frac{dt}{t} &= \rho^j \frac{\beta}{4} \frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)} \int\limits_{\mathbb{R}^+\setminus U_\delta(1)} e^{-\left|\frac{\rho}{2}\log z\right|^\beta} \frac{dz}{z} \\ &\leqslant \frac{\rho^j}{\left|\log^j \delta\right|} \frac{\beta}{4} \frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)} \int\limits_{\mathbb{R}^+\setminus U_\delta(1)} e^{-\left|\frac{\rho}{2}\log z\right|^\beta} \left|\log^j z\right| \frac{dz}{z} \\ &= \frac{2^j}{\left|\log^j \delta\right|} \frac{\beta}{\Gamma\left(\frac{1}{\beta}\right)} \int\limits_{\mathbb{R}^+\setminus U_{\delta\rho}(1)} e^{-|\log v|^\beta} \left|\log^j v\right| \frac{dv}{v}\end{aligned}$$

and the assertion follows as before. The first part of the theorem is obtained on putting $j = 0$. $\square$

3. Pointwise and uniform convergence

The modulus of continuity has been operated as the ordinary tool to measure the smoothness of approximated function f and to estimate errors in approximation.

Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function. We use the modulus of continuity of f by

$$\omega(f,\xi) = \sup\{|f(s_1) - f(s_2)| : |\log s_1 - \log s_2| \leqslant \xi\}$$

for $\xi > 0$. It is easy to see that the modulus of continuity ω satisfies all the classical properties of a modulus of continuity (see [15]). Especially, $\omega(f,\xi)$ is finite for every $\xi > 0$ and $\lim\limits_{\xi\to 0}\omega(f,\xi) = 0$ if and only if f is uniformly continuous wit respect to the metric $dis(s,t) = |\log s - \log t|$. We note that there are uniformly continuous functions in $\mathbb{R}^+$ in the usual sense but not in the log-sense and conversely. For instance, $h(x) = \sin x$ is clearly uniformly continuous function but not in the log-sense, whereas $k(x) = \sin(\log x)$ is log-uniformly continuous function but not in the classical sense. It is clear that these mentioned notions are equivalent on every compact interval in $\mathbb{R}^+$.

Now, let us consider the following MGW operator

$$\begin{aligned}\left(\mathscr{G}_\rho f\right)(s) &= \int\limits_0^\infty \mathscr{K}_\rho(s,t) f(t)\frac{dt}{t}\\ &= \frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\int\limits_0^\infty \exp\left(-\left(\left|\frac{\rho}{2}\log\frac{t}{s}\right|^\beta\right)\right) f(t)\frac{dt}{t}, \qquad \rho > 0.\end{aligned}$$

The point to note here is that if $\beta = 2$ is selected, this operator is reduced to the classical MGW operator.

The primary result in this part is on pointwise convergence:

THEOREM 1. *Suppose that f is essentially bounded function. If $s \in \mathbb{R}^+$ is a continuity point of given f, then*

$$\lim_{\rho\to\infty}\left(\mathscr{G}_\rho f\right)(s) = f(s).$$

Proof. Let $s \in \mathbb{R}^+$ be a continuity point of f. For a fixed $\varepsilon > 0$, let $\delta > 1$ be such that $|f(t) - f(s)| < \varepsilon$ whenever $t \in U_\delta(s) := \left(\frac{s}{\delta}, s\delta\right)$. We can write

$$\begin{aligned}\left|\left(\mathscr{G}_\rho f\right)(s) - f(s)\right| &\leqslant \frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\int_0^\infty e^{-\left|\frac{\rho}{2}\log\frac{t}{s}\right|^\beta}|f(t) - f(s)|\frac{dt}{t}\\ &= \frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\int_{U_\delta(s)} e^{-\left|\frac{\rho}{2}\log\frac{t}{s}\right|^\beta}|f(t) - f(s)|\frac{dt}{t}\\ &\quad + \frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\int_{\mathbb{R}^+\setminus U_\delta(s)} e^{-\left|\frac{\rho}{2}\log\frac{t}{s}\right|^\beta}|f(t) - f(s)|\frac{dt}{t}\\ &= I_1 + I_2.\end{aligned}$$

Firstly, we take into account I_1.

$$\begin{aligned}I_1 &= \frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\int_{U_\delta(s)} e^{-\left|\frac{\rho}{2}\log\frac{t}{s}\right|^\beta}|f(t) - f(s)|\frac{dt}{t}\\ &\leqslant \varepsilon.\end{aligned}$$

Now, we evaluate

$$I_2 = \frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\int_{\mathbb{R}^+\setminus U_\delta(s)} e^{-\left|\frac{\rho}{2}\log\frac{t}{s}\right|^\beta}\left|f(t)-f(s)\right|\frac{dt}{t}$$
$$\leqslant 2\left\|f\right\|_\infty \frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\int_{\mathbb{R}^+\setminus U_\delta(s)} e^{-\left|\frac{\rho}{2}\log\frac{t}{s}\right|^\beta}\frac{dt}{t}.$$

From Proposition 1, we have

$$\lim_{w\to\infty} I_2 = 0.$$

This proves the theorem. □

We continue with the following:

THEOREM 2. *Suppose that f is log-uniformly continuous in $\mathbb{R}^+$. Then $\mathscr{G}_\rho f$ converges to given f uniformly and the inequality*

$$\left\|\mathscr{G}_\rho f - f\right\|_\infty \leqslant \left(1+2\frac{\Gamma\left(\frac{2}{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)}\right)\omega\left(f,\frac{1}{\rho}\right)$$

holds.

Proof. Using the property $\omega(f,\mu\xi) \leqslant (\mu+1)\,\omega(f,\xi)$ for every $\mu,\xi>0$, we have for every $s\in\mathbb{R}^+$ and $\xi>0$,

$$\begin{aligned}
\left|\left(\mathscr{G}_\rho f\right)(s) - f(s)\right| &\leqslant \int_0^\infty \mathscr{K}_\rho(s,t)\left|f(t)-f(s)\right|\frac{dt}{t}\\
&\leqslant \int_0^\infty \mathscr{K}_\rho(s,t)\,\omega\left(f,\left|\log\left(\frac{t}{s}\right)\right|\right)\frac{dt}{t}\\
&\leqslant \int_0^\infty \mathscr{K}_\rho(s,t)\left(1+\frac{\left|\log(t/s)\right|}{\xi}\right)\omega(f,\xi)\frac{dt}{t}\\
&= \omega(f,\xi)\int_0^\infty \mathscr{K}_\rho(s,t)\frac{dt}{t} + \frac{\omega(f,\xi)}{\xi}\int_0^\infty \mathscr{K}_\rho(s,t)\left|\log(t/s)\right|\frac{dt}{t}\\
&= \omega(f,\xi) + \frac{\omega(f,\xi)}{\xi}M_1\left(\mathscr{K}_\rho\right)\\
&= \left(1+\frac{1}{\xi}\frac{2}{\rho}\frac{\Gamma\left(\frac{2}{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)}\right)\omega(f,\xi).
\end{aligned}$$

If we choose $\xi = \frac{1}{\rho}$, then the assertion follows by arbitrariness of $s\in\mathbb{R}^+$. □

4. A Voronovskaya-type formula for $\mathscr{G}_\rho f$

In this section, we will use the Mellin differential operator, as given in [13]. The Mellin differential operator Θ or the Mellin derivative Θf of a function $f:\mathbb{R}^+\to\mathbb{R}$ is defined by

$$\Theta f(s)=sf'(s),\qquad s\in\mathbb{R}^+,$$

provided the usual derivative $f'(s)$ exists. The Mellin differential operator of order $l\in\mathbb{N}$ is defined inductively by putting $\Theta^1=\Theta$, $\Theta^l=\Theta\circ\Theta^{l-1}$, $\Theta^0=I$, I being the identity operator. As mentioned in [13], we get the following representation result:

$$\Theta^l f(s)=\sum_{i=0}^{l}S(l,i)\,f^{(i)}(s)\,s^i,$$

where $S(l,i)$, $l\in\mathbb{N}$, $0\leqslant i\leqslant l$, denotes the Stirling numbers of the second kind.

Now, we give the following Taylor formula with the remainder in the form of Peano ([8], [17]).

PROPOSITION 2. *Let $f\in C^n$ locally at a point $s\in\mathbb{R}^+$. Then there exists $\delta>1$ such that for $t\in\left(\frac{1}{\delta},\delta\right)$*

$$f(t)=f(s)+\Theta f(s)\log\left(\frac{t}{s}\right)+\frac{\Theta^2 f(s)}{2!}\log^2\left(\frac{t}{s}\right)+\ldots+\frac{\Theta^n f(s)}{n!}\log^n\left(\frac{t}{s}\right)$$
$$+h_s(t)\log^n\left(\frac{t}{s}\right),$$

where $h_s(t)\to 0$ as $t\to s$. Furthermore, if $f\in L^\infty$, the above-mentioned formula holds for every $t\in\mathbb{R}^+$, and the function h_s is bounded on $\mathbb{R}^+$.

We are ready to show the primary theorem of this part.

THEOREM 3. *Let $f\in L^\infty$ and $s\in\mathbb{R}^+$ is fixed. Then, we have*

$$\lim_{\rho\to\infty}\rho^2\left(\left(\mathscr{G}_\rho f\right)(s)-f(s)\right)=\frac{2\Gamma\left(\frac{3}{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)}\Theta^2 f(s)$$

for $f\in C^2$ locally at $s\in\mathbb{R}^+$.

Proof. Let $s\in\mathbb{R}^+$ be fixed. By Proposition 2, we get

$$f(t)-f(s)=\Theta f(s)\log\left(\frac{t}{s}\right)+\frac{\Theta^2 f(s)}{2!}\log^2\left(\frac{t}{s}\right)+h_s(t)\log^2\left(\frac{t}{s}\right),$$

where $h_s(t)$ is a bounded function with $h_s(t) \to 0$ for $t \to s$. So, using (2.2), it is obtained that

$$\left(\mathscr{G}_\rho f\right)(s) - f(s) = \sum_{j=1}^{2} \frac{\Theta^j f(s)}{j!} m_j\left(\mathscr{K}_\rho\right) + \int_0^\infty \mathscr{K}_\rho(s,t)\, h_s(t) \log^2\left(\frac{t}{s}\right) \frac{dt}{t}$$
$$= \frac{\Theta^2 f(s)}{2!} m_2\left(\mathscr{K}_\rho\right) + J,$$
$$= \frac{\Theta^2 f(s)}{2!} \frac{4}{\rho^2} \frac{\Gamma\left(\frac{3}{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)} + J,$$

where

$$J := \int_0^\infty \mathscr{K}_\rho(s,t)\, h_s(t) \log^2\left(\frac{t}{s}\right) \frac{dt}{t}.$$

For a given $\varepsilon > 0$, let $\delta > 1$ be such that $|h_s(t)| < \varepsilon$ for $t \in \left(\frac{s}{\delta}, s\delta\right)$. By Proposition 1, we have

$$|J| \leqslant \left\{ \int_{U_\delta(s)} + \int_{\mathbb{R}^+ \setminus U_\delta(s)} \right\} \mathscr{K}_\rho(s,t)\, |h_s(t)| \left|\log^2\left(\frac{t}{s}\right)\right| \frac{dt}{t}$$
$$\leqslant \varepsilon M_2\left(\mathscr{K}_\rho\right) + o\left(\rho^{-2}\right). \quad \square$$

5. The related information potential

Principe [18] introduce The Information Theoretic Learning. Signal Processing, Detection, Estimation, Times Series Analysis can be given among the application areas of this subject. This part is concerned with the information theoretic learning method, whose aim is to measure quantitatively scalar descriptors (for instance, entropy) of a probability density function. The main notion is the information potential $IP(s)$ of a probability density function $p(t,s)$ depending on a parameter s. Detailed explanation on the subject can also be seen in [1].

It is considered that probability density functions as kernels of integral operators, in which case a specific relation exists between $IP(s)$ and $V[p(t,s)]$. As an application to Information Theoretic Learning, we determine the new notions used in this subject: $EV[\log p(t,s)]$ and $V[\log p(t,s)]$. We deal with probability density functions of $p(t,s)$, depending on a parameter s. The positive linear operator related to $p(t,s)$ with Haar measure is defined by

$$Lf(s) := \int_0^\infty f(t)\, p(t,s) \frac{dt}{t}.$$

Consider the variance corresponding to the density $p(t,s)$:

$$V(s) := \int_0^\infty (\log t)^2 \; p(t,s)\frac{dt}{t} - \left(\int_0^\infty \log t \; p(t,s)\frac{dt}{t}\right)^2.$$

The related information potential will be

$$IP(s) := \int_0^\infty p^2(t,s)\frac{dt}{t}, \quad s>0.$$

Using (2.1), it is confirmed that the density

$$p(t,s) = \frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\exp\left(-\left(\left|\frac{\rho}{2}\log\frac{t}{s}\right|^\beta\right)\right) \tag{5.1}$$

is a probability density. With respect to it, we have the operator

$$\left(\mathcal{G}_\rho f\right)(s) = \frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\int_0^\infty \exp\left(-\left(\left|\frac{\rho}{2}\log\frac{t}{s}\right|^\beta\right)\right) f(t)\frac{dt}{t}.$$

THEOREM 4. *For each $s\in\mathbb{R}^+$, the function $p(t,s)$ in (5.1) is a probability density such that the product $V(s)IP^2(s)$ is constant with respect to s. More precisely,*

$$V(s)IP^2(s) = \frac{\beta^2}{2^{2+2/\beta}}\frac{\Gamma\left(\frac{3}{\beta}\right)}{\left[\Gamma\left(\frac{1}{\beta}\right)\right]^3}$$

for all $s\in\mathbb{R}^+$.

Proof. Using (2.2), we obtain

$$\begin{aligned}
V(s) &= \frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\int_0^\infty (\log t)^2 \; \exp\left(-\left(\left|\frac{\rho}{2}\log\frac{t}{s}\right|^\beta\right)\right)\frac{dt}{t} \\
&\quad - \left(\frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\int_0^\infty \log t \; \exp\left(-\left(\left|\frac{\rho}{2}\log\frac{t}{s}\right|^\beta\right)\right)\frac{dt}{t}\right)^2 \\
&= \log^2 s + \frac{4}{\rho^2}\frac{\Gamma\left(\frac{3}{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)} - \log^2 s \\
&= \frac{4}{\rho^2}\frac{\Gamma\left(\frac{3}{\beta}\right)}{\Gamma\left(\frac{1}{\beta}\right)}
\end{aligned}$$

and

$$IP(s) = \frac{\beta^2}{16} \frac{\rho^2}{\left[\Gamma\left(\frac{1}{\beta}\right)\right]^2} \int_0^\infty e^{-2\left(\left|\frac{\rho}{2}\log\frac{t}{s}\right|\right)^\beta} \frac{dt}{t}$$
$$= \frac{\beta}{2^{2+1/\beta}} \frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}.$$

Thus, we get

$$V(s)\, IP^2(s) = \frac{\beta^2}{2^{2+2/\beta}} \frac{\Gamma\left(\frac{3}{\beta}\right)}{\left[\Gamma\left(\frac{1}{\beta}\right)\right]^3}. \qquad \square$$

Various examples in which formulas of this kind are satisfied can be seen in [1]. Let us remind two notions.

DEFINITION 1. The variance connected with $\log p(t,s)$ is

$$V[\log p(t,s)] = \int_0^\infty p(t,s)\log^2 p(t,s)\frac{dt}{t} - \left(\int_0^\infty p(t,s)\log p(t,s)\frac{dt}{t}\right)^2$$

and the expected value connected with $\log p(t,s)$ with Haar measure is stated by

$$EV[\log p(t,s)] = \int_0^\infty p(t,s)\log p(t,s)\frac{dt}{t}.$$

The importance of the two concepts for the Information Theoretic Learning is underlined in Principe [18]. In particular, $V[\log p(t,s)]$ is an index of the intrinsic shape of $p(t,s)$ having more statistical power than kurtosis, and can be used as a partial order for the tails of distributions. We remark that for our specific family of densities the concept $V[\log p(t,s)]$ is constant with respect to s.

THEOREM 5. *For the density* $p(t,s)$ *in* (5.1), *we obtain*

$$EV[\log p(t,s)] = \left(\log \frac{\beta}{4} \frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\right) - \frac{1}{\beta}$$

and

$$V[\log p(t,s)] = \frac{1}{\beta}.$$

Proof. From the previous definition, for the density $p(t,s)$, we get

$$
\begin{aligned}
EV\left[\log p(t,s)\right] &= \int_0^\infty p(t,s)\log p(t,s)\frac{dt}{t}\\
&= \frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\int_0^\infty \exp\left(-\left(\left|\frac{\rho}{2}\log\frac{t}{s}\right|^\beta\right)\right)\\
&\quad\times\log\left[\frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\exp\left(-\left(\left|\frac{\rho}{2}\log\frac{t}{s}\right|^\beta\right)\right)\right]\frac{dt}{t}\\
&= \left(\log\frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\right)-\frac{1}{\beta}.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\int_0^\infty p(t,s)\log^2 p(t,s)\frac{dt}{t}\\
&= \frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\int_0^\infty \exp\left(-\left(\left|\frac{\rho}{2}\log\frac{t}{s}\right|^\beta\right)\right)\\
&\quad\times\left(\log\left[\frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\exp\left(-\left(\left|\frac{\rho}{2}\log\frac{t}{s}\right|^\beta\right)\right)\right]\right)^2\frac{dt}{t}\\
&= \frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\int_0^\infty \exp\left(-\left(\left|\frac{\rho}{2}\log\frac{t}{s}\right|^\beta\right)\right)\left(\log\frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}+\log e^{-\left(\left|\frac{\rho}{2}\log\frac{t}{s}\right|^\beta\right)}\right)^2\frac{dt}{t}\\
&= \left(\log\frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\right)^2\frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\int_0^\infty \exp\left(-\left(\left|\frac{\rho}{2}\log\frac{t}{s}\right|^\beta\right)\right)\frac{dt}{t}\\
&\quad+2\left(\log\frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\right)\frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\int_0^\infty \exp\left(-\left(\left|\frac{\rho}{2}\log\frac{t}{s}\right|^\beta\right)\right)\log e^{-\left(\left|\frac{\rho}{2}\log\frac{t}{s}\right|^\beta\right)}\frac{dt}{t}\\
&\quad+\frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\int_0^\infty \exp\left(-\left(\left|\frac{\rho}{2}\log\frac{t}{s}\right|^\beta\right)\right)\left(\left|\frac{\rho}{2}\log\frac{t}{s}\right|\right)^{2\beta}\frac{dt}{t}\\
&= \left(\log\frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\right)^2-\frac{2}{\beta}\left(\log\frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\right)+\frac{1+\beta}{\beta^2},
\end{aligned}
$$

we have

$$\begin{aligned}
V[\log p(t,s)] &= \int_0^\infty p(t,s)\log^2 p(t,s)\frac{dt}{t} - \left(\int_0^\infty p(t,s)\log p(t,s)\frac{dt}{t}\right)^2 \\
&= \left(\log\frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\right)^2 - \frac{2}{\beta}\left(\log\frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\right) \\
&\quad + \frac{1+\beta}{\beta^2} - \left(\left(\log\frac{\beta}{4}\frac{\rho}{\Gamma\left(\frac{1}{\beta}\right)}\right) - \frac{1}{\beta}\right)^2 \\
&= \frac{1}{\beta}. \quad \square
\end{aligned}$$

6. Conclusions

In the current study, we research the behaviour of β-modification of the MGW type operators with respect to pointwise and uniform convergence. Also, it is expressed a Voronovskaya approximation formula for the MGW type operators using the new kernel. This formula includes Mellin derivatives and a different notion of moment which was called the logarithmic moment. In the last section, using the modified MGW kernel $p(\cdot,\cdot)$, we analyze the related information potential, $EV[logp(\cdot,\cdot)]$ and $V[logp(\cdot,\cdot)]$. The results expressed here may lead to further research on the modification of MGW type operators.

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REFERENCES

[1] A. M. ACU, G. BASCANBAZ-TUNCA, I. RASA, *Information potential for some probability density functions*, Applied Mathematics and Computation **389** (2021), 125578.

[2] G. A. ANASTASSIOU, R. A. MEZEI, *Global smoothness and uniform convergence of smooth Gauss–Weierstrass singular operators*, Math. Comput. Model. **50** (2009), 984–998.

[3] G. A. ANASTASSIOU, R. A. MEZEI, *A Voronovskaya type theorem for Poisson–Cauchy type singular operators*, J. Math. Anal. Appl. **366** (2010), 525–529.

[4] A. ARAL, *On a new approach in the space of measurable functions*, Constr. Math. Anal. **6** (2023), 237–248.

[5] A. ARAL, T. ACAR, S. KURSUN, *Generalized Kantorovich forms of exponential sampling series*, Anal. Math. Phys. **12** (2022), 1–19.

[6] C. BARDARO, I. MANTELLINI, *A note on theVoronovskaja theorem for Mellin-Fejer convolution operators*, Appl. Math. Lett. **24** (2011), 2064–2067.

[7] C. BARDARO, I. MANTELLINI, *A quantitative Voronovskaya formula for Mellin convolution operators*, Mediterr. J. Math. **7** (2010), 483–501.
[8] C. BARDARO, I. MANTELLINI, *Asymptotic behaviour of Mellin-Fejer convolution operators*, East J. Approx. **17** (2011), 181–201.
[9] C. BARDARO, I. MANTELLINI, *On the iterates of Mellin-Fejer convolution operators*, Acta Appl. Math. **121** (2012), 213–229.
[10] C. BARDARO, I. MANTELLINI, *Voronovskaya-type estimates for Mellin convolution operators*, Res. Math. **50** (2007), 1–16.
[11] R. BOJANIC, O. SHISHA, *On the precision of uniform approximation of continuous functions by certain linear positive operators of convolution type*, J. Approx. Theory **8** (1973), 101–113.
[12] V. P. BUI, I. G. FEDOROV, N. A. CERVAKOV, *On a sequence of positive linear operators* (Russian), Trudy Moskov. Vish. Tschn. uc.im. N.E. Baumana **139** (1970), 562–566.
[13] P. L. BUTZER, S. JANSCHE, *A direct approach to the Mellin transform*, J. Fourier Anal. Appl. **3** (1997), 325–375.
[14] P. L. BUTZER, R. J. NESSEL, *Fourier Analysis and Approximation I*, New York-London: Academic Press, 1971.
[15] R. A. DEVORE, G. G. LORENTZ, *Constructive Approximation*, Grundlehren der Mathematischen Wissenschaften, p. 303, Springer, Berlin, 1993.
[16] W. KOLBE, R. J. NESSEL, *Saturation theory in connection with Mellin transform methods*, SIAM J. Math. Anal. **3** (1972), 246–262.
[17] R. G. MAMEDOV, *The Mellin Transform and Approximation Theory*, Elm, Baku: (in Russian), 1991.
[18] J. C. PRINCIPE, *Information Theoretic Learning. Renyi's Entropy and Kernel Perspectives*, Springer, New York, 2010.
[19] C. TOPUZ, F. OZSARAC, A. ARAL, *On the generalized Mellin integral operators*, Demonstratio Mathematica **57** (2024), pp. 20230133.
[20] A. H. ZEMANIAN, *Generalized Integral Transformation*, Interscience, New York, 1968.

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