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SPECTRAL ANALYSIS AND INEQUALITY BOUNDS FOR HEIGHT AND DETERMINANT FUNCTIONS

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Abstract. Spectral height and determinant are not invariant under operator or metric deformations. The variations of spectral height and determinant of Laplacian under conformal change of metric in two dimensional Riemannian manifolds are computed explicitly using Polyakov formula. However, in dimensions higher than two, there are no such formula for computing the conformal or other such variations. In this work, we extend the Polyakov formula to study some generic and conformal variations of the height and determinant functions on closed Riemannian manifolds in higher dimensions and found their spectral inequality bounds.

1. Introduction

The concepts of the spectral height and spectral determinant functions are spectral invariants usually defined through the zeta function of differential operators, see e.g. [9, 12] and [25]. Bounds have been developed for such spectral functions using some specialised inequalities, see e.g. [8, 23] and [18]. For example, [13] showed the spectral zeta function is closely bounded by automorphic forms and the Riemann zeta function. Also, [5] constructed a rational approximation of the Riemann zeta function and its derivatives valid on every vertical line in the right half-planes. Recently, [8] proved Oppenheim type inequalities for normalised determinant of positive invertible operators on a Hilbert space and discussed Hadamard type inequalities for positive definite matrices.

Let (M,g) be a compact smooth Riemannian manifold and Δ the Laplace-Beltrami operator (also called "Laplacian" for short). The Laplacian is given by

$$\Delta : C^{\infty}(M) \to C^{\infty}(M)$$

where in local coordinates and for $f \in C^{\infty}(M)$ compactly supported,

$$\Delta f = -\frac{1}{\sqrt{\det(g_{ij})}} \sum_{ij} \partial_j \sqrt{\det(g_{ij})} g^{ij} \partial_i f \tag{1}$$

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with $\partial_j = \frac{\partial}{\partial x^j}$, $\partial_i = \frac{\partial}{\partial x^i}$ and $x^i, x^j \in \mathbb{R}$; c.f [25]. Let $0 = \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \nearrow \infty$ be the eigenvalues of M with respect to the Laplacian. The regularised spectral zeta function of the Laplacian is defined by

$$Z(s) = \sum_{\lambda_j \neq 0} \frac{1}{\lambda_j^s}; \quad \Re(s) > \frac{n}{2} \tag{2}$$

where n is the dimension of M. Then, the height of the manifold is

$$h(g) = Z'(0) \tag{3}$$

while the spectral determinant is given by

$$\det{}'\Delta = \prod_{\lambda_j \neq 0} \lambda_j = \mathrm{e}^{-Z'(0)}. \tag{4}$$

See e.g. [17, 24].

The spectral height and determinant are not invariant under change of metrics. To see this, consider a scaling of the metric of a Riemannian manifold M by a constant $c > 0$, i.e (M, cg). It follows from (1) and (2) that $\Delta \mapsto \frac{1}{c}\Delta$ and

$$Z_c(s) = \sum_{k=1}^{\infty} \frac{c^s}{\lambda_k^s} = c^s Z(s).$$

So, the spectral functions change as

$$h_c(g) = Z_c'(0) = \ln c\big(Z(0) + Z'(0)\big); \quad \lambda_j \neq 0$$

and

$$\det{}_c'\Delta = c\big(Z(0) + Z'(0)\big); \quad \lambda_j \neq 0$$

where the spectral zeta function, Z, can be meromorphically continued to the whole s-complex plane, see e.g. [16].

Consequently, one would ask "how does the spectral height and determinant vary under a more general deformation such as the conformal perturbation of the operator and metric of Riemannian manifold? Can one find inequality bounds for the spectral height and determinant under conformal perturbation of the operator and metric of the Riemannian manifold?"

In this study, we attempt these questions and construct spectral inequality bounds for deformed spectral height and determinant functions on closed Riemannian manifolds.

2. Inequality bounds for the Riemann zeta function

Following [24], we defined Riemann zeta function by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \begin{cases} \dfrac{1}{1-2^{-s}} \displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^s}; & \Re(s) > 1, \\ \dfrac{1}{1-2^{1-s}} \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}; & \Re(s) > 0. \end{cases} \tag{5}$$

A Dirichlet polynomial of the Riemann zeta function enables one find a bound for the spectral height and determinant on Riemannian manifold. We have the following technical lemma.

LEMMA 1. *On the upper half-plane, the Dirichlet polynomial of the Riemann zeta function is finite and*

$$\Psi(s) := \zeta(s) + \zeta'(s) \ll k \ln|t| \tag{6}$$

for $\mathbb{C} \ni s = \sigma + it;\ t > 2$ *and* $k \gg 1$ *is an integer.*

Proof. We begin by showing that the Dirichlet polynomial of the Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} + \gamma_N(s) \tag{7}$$

where

$$\gamma_N(s) = -s \int_N^\infty \frac{\tilde{u}}{u^{s+1}} du$$

with $\tilde{u} = u - [u]$, for all $\Re(s) > 0,\ s \neq 1$ and all integers $N \geqslant 1$. Here, $[\cdot]$ is the greatest integer value function.

To see this, write

$$\sum_{1 \leqslant n \leqslant N} \frac{1}{n^s} = \sum_{1 \leqslant n \leqslant N} \left(\frac{1}{N^s} - \left(\frac{1}{N^s} - \frac{1}{n^s} \right) \right) = \frac{N}{N^s} + \sum_{1 \leqslant n \leqslant N} \int_n^N \frac{s}{u^{s+1}} du$$

for $N \geqslant 1$.

Now by Fubini-Tonelli theorem, (see e.g. [16]), one can interchange sum and integral to have

$$\sum_{1 \leqslant n \leqslant N} \frac{1}{n^s} = \frac{N}{N^s} + s \int_1^N \sum_{1 \leqslant n \leqslant N} \frac{1}{u^{s+1}} du = \frac{N}{N^s} + s \int_1^N \frac{[u]}{u^{s+1}} du.$$

So,

$$\sum_{1 \leqslant n \leqslant N} \frac{1}{n^s} = \frac{N}{N^s} - s \int_1^N \frac{\tilde{u}}{u^{s+1}} du.$$

This leads to the following majorisation

$$\left| \frac{\tilde{u}}{u^{s+1}} \right| \leqslant \frac{1}{u^{\Re(s)+1}} \leqslant \frac{1}{u^k} \quad \text{for some constant} \quad k \gg 1.$$

Therefore, by Lebesgue dominated convergence theorem [16], we interchange limit and integral, and let $N \to \infty$ in

$$s \int_1^N \frac{\tilde{u}}{u^{s+1}} du$$

to see that it is holomorphic and hence, the integral converges.

Consequently,

$$\sum_{1\leqslant n\leqslant N}\frac{1}{n^s}=1+\frac{1}{s-1}+\frac{N^{1-s}}{1-s}-s\int_1^N\frac{\tilde{u}}{u^{s+1}}du$$

so that for $\Re(s)>0$,

$$\zeta(s)=1+\frac{1}{s+1}-s\int_1^\infty\frac{\tilde{u}}{u^{s+1}}du. \tag{8}$$

Thus, the Riemannian zeta function defined by (8) is holomorphic for $\Re(s)>1$ in the half-plane except a simple pole at $s=1$ with residue 1. This is a well-known result, see e.g. [24,26] and [17].

Moreover, the remainder term $\gamma_N(s)$ satisfies

$$|\gamma_N(s)|\leqslant|s|\int_N^\infty\frac{1}{u^{\sigma+1}}du=\frac{|s|}{\sigma N^\sigma}$$

since

$$\left|\int_1^\infty\frac{\tilde{u}}{u^{s+1}}du\right|\leqslant\int_1^\infty\frac{1}{u^{\sigma+1}}$$

where $\sigma=\Re(s)$.

We can now prove the bound for $\Psi(s)$. That is, for $\sigma\geqslant\frac{1}{2}$ and $t\geqslant 2$, we want to show that

$$|\Psi(s)|\leqslant e^k\ln(t);\ \text{ for }\ k\gg 1.$$

Choose $N=[t]$ such that $N\leqslant t<N+1$, then

$$\left|\sum_{n=1}^N\frac{1}{n^s}\right|\leqslant e^k\sum_{n=1}^N\frac{1}{n^\sigma} \tag{9}$$

since $t\geqslant 2$. Also,

$$|\gamma_N(s)|\leqslant\frac{|\sigma+it|}{\sigma N^\sigma}\leqslant k_1e^k\ln|t|$$

since $\sigma\geqslant\frac{1}{2}$; for $N\geqslant 2$ and $k_1\gg 1$.

Similarly,

$$-\sum_{n=1}^N\frac{\ln n}{n^s}=-\frac{1}{(1-s)^2}+\frac{-(1-s)N^{1-s}\ln N+N^{1-s}}{(1-S)^2}$$
$$-\int_1^N\frac{\tilde{u}}{u^{s+1}}du+s\int_1^N\tilde{u}\frac{\ln u}{u^{s+1}}du.$$

Thus, as $N\to\infty$, we have the Dirichlet series

$$\zeta'(s)=-\sum_{n=1}^\infty\frac{\ln n}{n^s}-\frac{N^{1-s}\ln N}{1-S}-\frac{N^{1-s}\ln N}{(1-S)^2}$$
$$-\int_N^\infty\frac{\tilde{u}}{u^{s+1}}du+s\int_N^\infty\tilde{u}\frac{\ln u}{u^{s+1}}du\leqslant k_2e^k\ln|t|. \tag{10}$$

Combining (9) and (10) gives the bound for $\Psi(s)$. $\square$

In fact, Srivastava, Mehrez and Tomovski in [23] on constructing a Turán-type inequality on Mathieu series proved that for $s \geqslant 1$, the following inequalities hold:

$$\zeta(2s) \leqslant \sqrt{\frac{3\pi}{2}} \frac{\Gamma(s+1)}{\Gamma(s+\frac{1}{2})} \tag{11}$$

and

$$\frac{[\zeta(2s+1)]^{\frac{3}{2}}}{[\zeta(2s)]^2 \zeta(2s+3)} \leqslant (s+1)\left(\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)}\right)^2. \tag{12}$$

COROLLARY 1. *For* $\Re(s) > 0$,

$$\zeta(s) \leqslant \sqrt{\frac{3\pi}{2}} \frac{\Gamma(\frac{s+2}{2})}{(\frac{s+1}{2})} \tag{13}$$

and

$$\zeta'(s) \leqslant \frac{1}{2}\sqrt{\frac{3\pi}{2}} \frac{\Gamma(\frac{s+2}{2})}{(\frac{s+1}{2})}\left[\psi\left(\frac{s+1}{2}\right) + \psi\left(\frac{s+2}{2}\right)\right] \tag{14}$$

where $\psi(s) := \frac{\Gamma'(s)}{\Gamma(s)}$ *is the so-called digamma function.*

Proof. Let $2s \mapsto s$ in (12) and then differentiate in s. The results follow as a consequence of Lemma 1 for $\Re(s) > 0$. $\square$

3. Analysis of deformed operators

We consider a simple example of the Schrödinger-type operator

$$H_c := \Delta + c, \tag{15}$$

where c a constant potential, on the unit circle S^1 with spectrum of the form $\lambda_k = k^2 + c;\ k, c \in \mathbb{R}$. The associated spectral zeta function of the operator is then

$$Z_c(s) = \sum_{k\in\mathbb{Z}} \frac{1}{(k^2+c)^s};\ \ \Re(s) \gg 0.$$

LEMMA 2. *The spectral zeta function* $Z_c(s)$ *is meromorphic in the whole of* s*-complex plane via the expansion*

$$f(t) = \sum_{k\in\mathbb{Z}} e^{-(k^2+c)t};\quad as\ \ t \searrow 0.$$

Proof. For an arbitrary c, by Mellin transformation we write

$$Z_c(s) = \frac{1}{\Gamma(s)} \int_0^\infty \sum_{k\in\mathbb{Z}} e^{-k^2 t} e^{-ct} t^{s-1} dt,$$

where sum and integrals can be interchanged following Fubini-Tonelli theorem, [16].

Separating zero modes of $\sum_{k\in\mathbb{Z}} e^{-k^2 t}$ from the non-zero modes, we have

$$Z_c(s) = \frac{1}{\Gamma(s)} \int_0^\infty \Big[2 \sum_{k\in\mathbb{Z}|_{k\neq 0}} e^{-k^2 t} + \sum_{k\in\mathbb{Z}|_{k=0}} e^{-k^2 t}\Big] e^{-ct} t^{s-1} dt$$

which gives

$$Z_c(s) = \frac{\Gamma[\frac{(2m+1)-s}{2}]}{\Gamma(\frac{s}{2})} \zeta(\alpha - s) + M_2 + c^{-s}$$

where

$$M_2 = \frac{2}{\Gamma(s)} \int_1^\infty w(t) \sum_{m=1}^\infty \frac{(-1)^m (ct)^m}{m!} t^{s-1} dt; \text{ with } w(t) = \sum_{k=1}^\infty e^{-k^2 t}$$

is holomorphic except at the points $s = -\alpha;\ \alpha = 1, 3, 5, \cdots,$ which are all simple poles.

The residues are $\frac{(-1)^m c^m}{m!}$, with $m = 0, 1, 2, \cdots$. □

We have the following result.

THEOREM 1. *Let H_c be as defined* (15) *above, then,*

$$\det(H_c) = c \cdot \exp[-2 \sum_{m=0}^\infty \frac{(-1)^m c^m (m)}{m!} \zeta(2m)]$$

and for $0 < c \ll 1,\ h(S^1) < \ln 2$.

Proof. From the definition of spectral zeta function (2), we have

$$\begin{aligned} Z_c(s) &= \frac{2}{\Gamma(s)} \sum_{k=1}^\infty \int_0^\infty e^{-k^2 t} e^{-ct} t^{s-1} dt + c^{-s} \\ &= \frac{2}{\Gamma(s)} \sum_{k=1}^\infty \int_0^\infty e^{-k^2 t} \sum_{m=0}^\infty \frac{(-1)^m (ct)^m}{m!} t^{s-1} dt + c^{-s} \\ &= 2\zeta(2s) + \frac{2}{\Gamma(s)} \sum_{m=1}^\infty \frac{(-1)^m c^m \zeta(2m+2s)}{m!} \Gamma(s+m) + c^{-s} \end{aligned}$$

which simplifies

$$Z_c(s) = \frac{2}{\Gamma(s)} \sum_{m=0}^\infty \frac{(-1)^m c^m \zeta(2m+2s) \Gamma(s+m)}{m!} + c^{-s}.$$

Notice that at $c = 0$, we get back $Z(s) = 2\zeta(2s)$ as in the previous case.

Similarly,

$$Z_c'(s) = 2\sum_{m=0}^{\infty} \frac{(-1)^m c^m \zeta(2m+2s)\Gamma(s+m)}{m!} - c^{-s}\log(c)$$

so that

$$Z_c'(0) = 2\sum_{m=0}^{\infty} \frac{(-1)^m c^m \zeta(2m)\Gamma(m)}{m!} - \log(c).$$

Therefore,

$$-\log\det_{H_c}(0) = 2\sum_{m=0}^{\infty} \frac{(-1)^m c^m \zeta(2m)\Gamma(m)}{m!} - \log(c).$$

Thus,

$$h(S') = 2\sum_{m=0}^{\infty} \frac{(-1)^m c^m \zeta(2m)\Gamma(m)}{m!} - \log(c) \geqslant \sum_{m=0}^{\infty} \frac{(-1)^m c^m \zeta(2m)\Gamma(m)}{m!} < \ln 2$$

since

$$\lim_{m\to\infty}\left(\frac{(-1)^m \zeta(2m)\Gamma(m)}{m!}\right) = \ln 2.$$

This completes the proof of the theorem. □

Next, consider a more general Schrödinger-type operator

$$H_\varepsilon = \Delta + \varepsilon V,$$

where $H_0 = \Delta$ is the unperturbed Laplacian, V is a smooth potential and $0 < \varepsilon \ll 1$ a perturbation parameter acting on functions on the n-torus

$$\mathbb{T}^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n\text{-times}}.$$

We obtain the next result.

THEOREM 2. *The height of the flat n-torus is bounded by $2\rho\zeta(2)$ where ρ is the Fourier coefficient of V on $\mathbb{T}^n$.*

That is,

$$h(\mathbb{T}^n) \leqslant 2\rho\zeta(2). \tag{16}$$

Proof. Let the associated spectral zeta function of H_ε on $\mathbb{T}^n$ be

$$Z_\varepsilon(s) = \sum_{k\in\mathbb{Z}} \frac{1}{\lambda_k^s}$$

where λ_k are the eigenvalues of H on $\mathbb{T}^n$. So,

$$Z_\varepsilon(s) = \frac{1}{\Gamma(s)} \int_0^\infty \sum_{k\in\mathbb{Z}} e^{-\lambda_k t} t^{s-1} dt = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}(e^{-(\Delta+\varepsilon V)t}) t^{s-1}$$

by Mellin transform.

Observe that if $V = 0$ then,

$$Z_\varepsilon(s) = \frac{1}{\Gamma(s)} \int_0^\infty \sum_{k\in\mathbb{Z}} e^{-\lambda_k t} t^{s-1} dt = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}(e^{-\Delta t}) t^{s-1} = 2\zeta(2s)$$

on S^1 as before.

To see (16), let $F_H(s) := \frac{\partial}{\partial \varepsilon} Z_H(s)|_{\varepsilon=0}$. Then,

$$F_H(s) = -\frac{1}{\Gamma(s)} \int_0^\infty t \cdot \mathrm{Tr}(Ve^{-\Delta t}) t^{s-1}.$$

Now, using that

$$\mathrm{Tr}(Ve^{-\Delta t}) = \rho \sum_{k\in\mathbb{Z}} e^{-k^2 t}, \text{ where } \rho = \frac{1}{2\pi} \int_{\mathbb{T}^n} \overline{e^{ikx}} V(x) e^{ikx} dx,$$

we have

$$F_H(s) = -\frac{\rho}{\Gamma(s)} \int_0^\infty t \cdot \sum_{k\in\mathbb{Z}} e^{-k^2 t} t^{s-1} dt.$$

Again, we remove the zero mode for F_H to be defined to have

$$F_H(s) = -\frac{2\rho}{\Gamma(s)} \sum_{k=1}^\infty \int_0^\infty t \cdot e^{-k^2 t} t^{s-1} dt. \tag{17}$$

So,

$$F_H(s) = -\frac{2\rho}{\Gamma(s)} \zeta(2s+2)\Gamma(s+1)$$

and then

$$F_H'(s) = -2\rho\zeta(2s+2)\Gamma(s+1) \Rightarrow F_H'(0) = -2\rho\zeta(2).$$

This leads to write

$$\log \det_{S^1}(H) = 2\rho\zeta(2) = \frac{\rho\pi^3}{3}.$$

Now let $\psi_k : \mathbb{T}^n \to \mathbb{C}$ defined by

$$\psi_k(x) = (2\pi)^{-n} \prod_{j=1}^n e^{ik_j x_j}; \; k_j \in \mathbb{Z}^n$$

constitute an orthonormal basis in the Sobolev space $H^p(\mathbb{T}^n)$. Then,

$$\langle \psi_k, \psi_l \rangle = (2\pi)^{-n} \int_{\mathbb{T}^n} e^{i(k_j - l_j)x_j} dx_j = \begin{cases} 1; & k_j = l_j \\ 0; & k_j \neq l_j. \end{cases}$$

It follows that

$$\operatorname{Tr}(Ve^{-\Delta t}) = \alpha \sum_{k_j \in \mathbb{Z}^n} e^{-k_j^2 t}, \text{ where } \alpha = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \overline{\psi_k} V \psi_k.$$

Therefore,

$$\log \det_{\mathbb{T}^n}(H) = \alpha \frac{(\pi)^3}{3}.$$

So, using the bounds for $\zeta(s)$ given in (11) and (12); the definition of $h(g)$ in (3), and the fact that

$$\frac{\partial}{\partial \varepsilon}[\log \det_{\mathbb{T}^n}(H)|_{\varepsilon=0}] = \frac{\rho \pi^2}{3},$$

we conclude that the inequality (16) on the flat n-torus is valid. □

4. Conformal metric deformation

We further consider a conformal transformation of the Laplacian and ask how the height and determinant function vary. Afterwards, we find their spectral inequality bounds.

DEFINITION 1. [11]. Let (M,g) and (N,h) be two Riemannian manifolds and let $f: M \to N$ be a diffeomorphism. The diffeomorphism f is called conformal if there exists a positive function $\rho: M \to \mathbb{R}$ such that $g = \rho g$ where ρ is a real-valued smooth function.

LEMMA 3. *Let Δ be the Laplacian defined locally by (1) above. Then, a conformally defined Laplacian is given by*

$$\Delta_h = \rho^{-1}\Delta_g + (1 - n/2)\rho^{-2}\nabla_g(\rho), \tag{18}$$

where n is the dimension of M and $\nabla_g(\rho)$ is the gradient vector field of the function ρ.

Proof. A straightforward computation shows that $\det(h_{ij}) = \rho^{n/2}\det(g_{ij})$. So, one has for $f \in C^\infty(M)$ that

$$\begin{aligned}\Delta_h f &= \frac{1}{\rho^{n/2}} \frac{\partial}{\partial x_1}\Big(\rho^{n/2} \frac{1}{\rho} \frac{\partial f}{\partial x_1}\Big) + \cdots + \frac{1}{\rho^{n/2}} \frac{\partial}{\partial x_n}\Big(\rho^{n/2} \frac{1}{\rho} \frac{\partial f}{\partial x_n}\Big) \\ &= \rho^{-n/2}\Big[\frac{n-2}{2}\rho^{\frac{n-4}{2}} \rho_{x_1} \frac{\partial f}{\partial x_1} + \cdots + \frac{n-2}{2}\rho^{\frac{n-4}{2}} \rho_{x_n} \frac{\partial f}{\partial x_n} + \rho^{\frac{n-2}{2}} \frac{\partial^2 f}{x_n^2}\Big]. \end{aligned}$$

□

COROLLARY 2. *If in Theorem (3) above, $\rho = e^{2\psi}$ and $f, \psi \in C^\infty(M)$; then,*

$$\Delta_h f = e^{-2\psi}[\Delta_g f - (n-2)\nabla^k \psi \nabla_k f]; \quad k = 1, 2, \cdots, n. \tag{19}$$

Proof. Replace ρ with $e^{2\psi}$ in the proof of Lemma (3) to get the expression. $\square$

Now to compute the conformal variation of the spectral zeta function, we consider the heat kernel, $K(t,x,y) : (0,\infty)\times M\times M \to \mathbb{R}$, which is a continuous function on $(0,\infty)\times M\times M$. Let $\{\psi_k\}_{k=0}^{\infty}$ be orthonormal basis of eigenfunctions of Δ with corresponding eigenvalues $\{\lambda_k\}$ listed with multiplicities, [7, 19] and [14]. Then $\{\psi_k\}_{k=0}^{\infty}$ are also eigenfunctions of the heat operator with corresponding eigenvalues $\{e^{-\lambda_k t}\}$. In terms of these eigenfunctions, the Mercer's theorem [22] implies that $e^{-t\Delta}$ is trace-class for all $t>0$ and one can write the heat kernel as

$$K(t,x,y) = \sum_{k=0}^{\infty} e^{-\lambda_k t}\psi_k(x)\bar{\psi}_k(y).$$

The convergence for all $t>0$ is uniform on $M\times M$. In particular, the trace of the heat operator

$$\mathrm{Tr}(e^{-\Delta_g t}) = \sum_{k=0}^{\infty} e^{-\lambda_k t}|\psi_k(x)|^2 = \sum_{k=0}^{\infty} e^{-\lambda_k t} = \int_M K(t,x,x)dV_g(x) < \infty; \tag{20}$$

see e.g. [12].

In dimension two, there is an explicit formula for computing the variation of $\log\det\Delta$ under conformal change of metric. It is the Polyakov-Ray-Singer (Polyakov) variation formula given below. However, in dimensions higher than two, there are no such formula for computing the variation of the determinant of the Laplacian under conformal or any other variations of the metric, c.f: [15]. Hence, the less study in this area.

THEOREM 3. (Polyakov formula) *Suppose (M,g) is a compact 2-dimensional closed surface and $h=e^{2\psi}g$ is the metric conformal to g with $vol(M,h)=vol(M,g)$. Then,*

$$\zeta_g(s)-\zeta_h(s) = \frac{1}{\Gamma(s)}\int_0^{\infty}[Tr(e^{-\Delta_g t})-Tr(e^{-\Delta_h t})]t^{s-1}dt = -\frac{1}{12\pi}\int_M(2k_g\psi+|\nabla\psi|^2)dA_g \tag{21}$$

where A_g is the area of (M,g), dA_g its volume form and k_g is the Gaussian curvature. The notations A_h, dA_h and k_h are similarly defined for (M,h).

For proof, one can see [6] and [20].

To extend the Polyakov's variation formula to n-dimensional compact Riemannian manifold, we begin with variation of the spectral zeta function. Let

$$\phi : M\times(-c,c)\to\mathbb{R}$$

be a smooth family of functions $\phi_\varepsilon := \phi(\cdot,\varepsilon)$ on M with $\phi_0 = 0$, and define the corresponding family of conformal metrics g_ε given by $\{g_\varepsilon = e^{\phi_\varepsilon}g\}$, with the condition that $g_\varepsilon^{(1)} = \frac{\partial}{\partial\varepsilon}(g_\varepsilon)|_{\varepsilon=0} = \dot{\phi}_0 g$, $\dot{\phi}_0\in C^\infty(M)$; where $\dot{\phi}_\varepsilon = \frac{\partial}{\partial\varepsilon}(\phi_\varepsilon)$. Then, the corresponding family of Laplacians Δ_ε are defined as

$$\Delta_\varepsilon\psi = e^{-\phi_\varepsilon}\Delta\psi + \left(1-\frac{n}{2}\right)e^{-2\phi_\varepsilon}\mathrm{div}(e^{\phi_\varepsilon}\nabla)\psi.$$

It can be seen that Δ_ε varies in ε as follows:

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}\Big(\Delta_\varepsilon\psi\Big) &= \frac{\partial}{\partial\varepsilon}\Big(e^{-\phi_\varepsilon}\Delta\psi+\Big(1-\frac{n}{2}\Big)e^{-2\phi_\varepsilon}\mathrm{div}(e^{\phi_\varepsilon}\nabla)\psi\Big)\\ &= -\dot\phi_\varepsilon e^{-\phi_\varepsilon}\Delta\psi-\Big(1-\frac{n}{2}\Big)\dot\phi_\varepsilon e^{-\phi_\varepsilon}\mathrm{div}(\phi_\varepsilon\nabla)\psi\\ &\quad+\Big(1-\frac{n}{2}\Big)e^{-\phi_\varepsilon}\mathrm{div}(\dot\phi_\varepsilon\nabla)\psi.\end{aligned}$$

THEOREM 4. *Let (M,g) be smooth, compact and connected Riemannian manifold and Δ the Laplacian on it with eigenvalues $\{\lambda_k\}$ listed according to their multiplicities. Let*

$$\{g_\varepsilon = e^{\phi_\varepsilon}g\}$$

be a family of volume-preserving conformal metrics. Then the spectral zeta function of Δ_ε, given by

$$\zeta_{g_\varepsilon}(s)=\sum_{k=1}^{\infty}\frac{1}{\big(\Lambda_k(\varepsilon)\big)^s} \tag{22}$$

varies as

$$\zeta_g^{(1)}(s)=s\int_M\dot\phi_0(x)\zeta_g(s,x,x)dV_g+\frac{1}{2}\Big(\frac{n}{2}-1\Big)s\int_M(\Delta\dot\phi_0(x))\zeta(s+1,x,x)dV_g. \tag{23}$$

We denote this variation evaluated at $\varepsilon=0$ by $\zeta_g^{(1)}(s)$.

Proof. Since

$$\zeta_{g_\varepsilon}(s)=\frac{1}{\Gamma(s)}\int_0^\infty(\mathrm{Tr}(e^{-t\Delta_\varepsilon})-1)t^{s-1}dt,$$

it follows that

$$\zeta_g^{(1)}(s)=\frac{\partial}{\partial\varepsilon}\zeta_{g_\varepsilon}(s)|_{\varepsilon=0}=\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}\Big(\frac{1}{\Gamma(s)}\int_0^\infty(\mathrm{Tr}(e^{-t\Delta_\varepsilon})-1)t^{s-1}dt\Big).$$

In line with Ray and Singer in [20], one gets

$$\begin{aligned}\zeta_g^{(1)}(s) &= -\frac{1}{\Gamma(s)}\int_0^\infty\mathrm{Tr}\big(\Delta_\varepsilon^{(1)}e^{-t\Delta}\big)t^s dt\\ &= -\frac{1}{\Gamma(s)}\int_0^\infty\mathrm{Tr}\Big(\Big[-\dot\phi_0\Delta+\Big(1-\frac{n}{2}\Big)\mathrm{div}\Big]\Big(\dot\phi_0\nabla(e^{-t\Delta})\Big)\Big)t^s dt\end{aligned}$$

where we have used the variation of Δ_ε in (18). So,

$$\begin{aligned}\zeta_g^{(1)}(s) &= \frac{1}{\Gamma(s)}\int_0^\infty\mathrm{Tr}\big(\dot\phi_0\Delta e^{-t\Delta}\big)t^s dt-\Big(1-\frac{n}{2}\Big)\frac{1}{\Gamma(s)}\int_0^\infty\mathrm{Tr}\big(\mathrm{div}(\dot\phi_0\nabla e^{-t\Delta})\big)t^s dt\\ &= -\frac{1}{\Gamma(s)}\int_0^\infty\frac{\partial}{\partial t}\mathrm{Tr}\big(\dot\phi_0 e^{-t\Delta}\big)t^s dt+\Big(\frac{n}{2}-1\Big)\frac{1}{\Gamma(s)}\int_0^\infty\mathrm{Tr}\big(\mathrm{div}(\dot\phi_0\nabla e^{-t\Delta})\big)t^s dt.\end{aligned}$$

Integrating by parts in the first term, gives

$$\begin{aligned}\zeta_g^{(1)}(s) &= \frac{s}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\Big(\dot{\phi}_0\Big(e^{-t\Delta}-\frac{1}{V}\Big)\Big)t^{s-1}dt \\ &\quad +\Big(\frac{n}{2}-1\Big)\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\Big(\mathrm{div}\Big(\dot{\phi}_0\nabla\Big(e^{-t\Delta}-\frac{1}{V}\Big)\Big)\Big)t^{s}dt\end{aligned}$$

where $\frac{1}{V}$ denotes $f\mapsto\frac{1}{V}\int_M f dV$ and V is the volume of (M,g).

So the variation of the zeta function is

$$\begin{aligned}\zeta_g^{(1)}(s) &= \frac{s}{\Gamma(s)}\int_0^\infty\int_M \dot{\phi}_0(x)\Big(K(t,x,x)-\frac{1}{V}\Big)dV_g t^{s-1}dt \\ &\quad +\frac{1}{2}\Big(\frac{n}{2}-1\Big)\frac{1}{\Gamma(s)}\int_0^\infty\int_M \mathrm{div}(\dot{\phi}_0\nabla)\Big(K(t,x,x)-\frac{1}{V}\Big)dV_g t^{s}dt.\end{aligned}$$

Since

$$\int_M\Big(\dot{\phi}_0(x)K(t,x,x)-\frac{1}{V}\Big)dV_g(x)\to 0$$

decays exponentially fast as $t\to\infty$. Also, recognizing that $\frac{1}{\Gamma(s)}=\frac{s}{\Gamma(s+1)}$, we have

$$\begin{aligned}\zeta_g^{(1)}(s) &= s\int_M\dot{\phi}_0(x)\Big\{\frac{1}{\Gamma(s)}\int_0^\infty\Big(K(t,x,x)-\frac{1}{V}\Big)t^{s-1}dt\Big\}dV_g \\ &\quad +\frac{1}{2}\Big(\frac{n}{2}-1\Big)s\int_M \mathrm{div}(\dot{\phi}_0\nabla)\Big\{\frac{1}{\Gamma(s+1)}\int_0^\infty\Big(K(t,x,x)-\frac{1}{V}\Big)t^{s}dt\Big\}dV_g.\end{aligned}$$

Therefore,

$$\zeta_g^{(1)}(s) = s\int_M\dot{\phi}_0(x)\zeta_g(s,x,x)dV_g+\frac{1}{2}\Big(\frac{n}{2}-1\Big)s\int_M \mathrm{div}(\dot{\phi}_0\nabla_g)\zeta(s+1,x,x)dV_g.$$

By Green's formula, we have

$$\zeta_g^{(1)}(s) = s\int_M\dot{\phi}_0(x)\zeta_g(s,x,x)dV_g+\frac{1}{2}\Big(\frac{n}{2}-1\Big)s\int_M (\Delta\dot{\phi}_0(x))\zeta_g(s+1,x,x)dV_g$$

which completes the proof. □

Again, note that the first-order variation $\zeta_g^{(1)}(s)$ given by the formula (23) is true for large s. The right-hand-side of equation (23) is meromorphic in s with simple poles.

Let us define the Finite Part function FP for the varied spectral zeta function by

$$\begin{aligned}\mathrm{FP}[\zeta_g^{(1)}(s)]\big|_{s_0} &= -\frac{1}{2}\int_M\dot{\phi}_0(x)\mathrm{FP}[\zeta_g(s,x,x)]\big|_{s_0}dV_g \\ &\quad -\frac{1}{4}\Big(\frac{n}{2}-1\Big)\int_M(\Delta\dot{\phi}_0(x))\mathrm{FP}[\zeta_g(s+1,x,x)]\big|_{s_0}dV_g \qquad (24)\end{aligned}$$

for an arbitrary point $s=s_0$.

DEFINITION 2. The metric g is called a critical point of a point $s = s_0$ with respect to all variations $\{g_\varepsilon = e^{\phi_\varepsilon} g\}$, if the variation $\zeta_g^{(1)}(s_0)$ vanishes for all g_ε.

Another result of this work is the following.

THEOREM 5. *Let Δ_ε be the Laplacian on (M, g_ε) with zeta kernel $\zeta_g(s,x,y)$. Then, g is a critical point of the height function $h_\rho(g)$ for all constant-volume conformal variations of the metric if* $\mathrm{FP}\left[\zeta_g^{(1)}(s,x,x)\Big|_{s=0}\right]$ *is constant in x.*

Moreover,

$$1 - \frac{1}{\det'_\rho \Delta} \leqslant h_\rho(g) \leqslant \det_\rho{}' \Delta - 1. \tag{25}$$

Proof. By the definition of critical point above and the variation of the spectral zeta function (24), consider the function

$$F_{\dot\phi_0}(x) := \left(-\frac{1}{2}\dot\phi_0(x)\mathrm{FP}[\zeta_g(s,x,x)] - \frac{1}{4}\left(\frac{n}{2} - 1\right)(\Delta\dot\phi_0)\mathrm{FP}[\zeta_g(s+1,x,x)]\right)\Big|_{s=0}$$

where of course,

$$\zeta_g(s,x,x)\big|_{s=0} = \frac{1}{\Gamma(s)} \int_0^\infty \left[K(t,x,x) - \frac{1}{V}\right] t^{s-1} dt\big|_{s=0}.$$

We have a critical point if

$$\int_M F_{\dot\phi_0}(x)\mathrm{d}V_x = 0 \;\forall\; \dot\phi_0 \in C^\infty(M) \;\text{ such that }\; \int_M \dot\phi_0(x)\mathrm{d}V_x = 0.$$

Now, suppose $\mathrm{FP}\left[\zeta_g(s,x,x)\Big|_{s=0}\right]$ is constant. Then, one gets

$$\begin{aligned}
\int_M F_{\dot\phi_0}\mathrm{d}V_x &= -\frac{1}{2}\mathrm{FP}[\zeta_g(s,x,x)\Big|_{s=0}] \int_M \dot\phi_0(x)\mathrm{d}V_x \\
&\quad -\frac{1}{4}\left(\frac{n}{2} - 1\right) \int_M (\Delta\dot\phi_0(x))\mathrm{FP}[\zeta_g(s,x,x)\Big|_{s=0}]\mathrm{d}V_x \\
&= -\frac{1}{4}\left(\frac{n}{2} - 1\right) \int_M (\Delta\dot\phi_0(x))\mathrm{FP}[\zeta_g(s,x,x)\Big|_{s=0}]\mathrm{d}V_x
\end{aligned}$$

since $\int_M \dot\phi_0(x)\mathrm{d}V_x = 0$. Now by the self-adjointness of Δ, we have

$$\int_M F_{\dot\phi_0}(x)\mathrm{d}V_x = -\frac{1}{4}\left(\frac{n}{2} - 1\right) \int_M \dot\phi_0(x)\Delta\mathrm{FP}[\zeta_g(s,x,x)\Big|_{s=0}]\mathrm{d}V_x = 0$$

since the Laplacian of a constant function is zero.

Moreover, the inequality (25) follows from that fact

$$\frac{z-1}{z} \leqslant \ln z = \int_1^z \frac{1}{t} dt \leqslant \int_1^z dz \leqslant z - 1 \;\; \forall z > 0.$$

Furthermore, let $\zeta_g(s) - \zeta_h(s)$ be the Polyakov's conformal variation difference of the spectral zeta functions of (M,g) and (M,h) in equation (21). It is not difficult to see that

$$\begin{aligned}\mathrm{FP}\Big[\big|\zeta_g(s) - \zeta_h(s)\big|\Big|_{s=0}\Big] &\leqslant \mathrm{FP}\Big[\int_{\zeta_g(s)}^{\zeta_h(s)} \frac{1}{t}dt\Big]\Big|_{s=0} \\ &\leqslant \frac{\zeta_h(s) - \zeta_g(s)}{\zeta_g(s)}\Big|_{s=0} \to 0, \quad \text{as} \quad \zeta_g(s)|_{s=0} \to \zeta_h(s)|_{s=0}.\end{aligned}$$

Thus, the spectral inequality bound (25) on the height function is justified. □

We now illustrate the conformal variation results on the n-sphere. Let the Laplacian restricted to the n-sphere be given by

$$\Delta_n u = \frac{1}{r^n}\frac{\partial}{\partial r}\Big(r^n \frac{\partial u}{\partial r}\Big) + \frac{1}{r^2}\Delta_n u$$

where $u(r;x)$ is a harmonic polynomial. Let $\mathscr{P}_k$ be the space of homogeneous polynomials of degree k. It is well-known, (see e.g. [22] and [17]), that the eigenvalues of the operator $-\Delta_n$ is given by

$$\lambda_k = k(k+n-1), \;\; k = 0,1,2,\cdots,$$

with multiplicities equal to the dimension of $\mathscr{P}_k$ given by

$$d_k = \binom{k+n}{n} - \binom{k+n-2}{n}.$$

So,

$$Z_n(s) = \sum_{k=1}^{\infty} \frac{d_k}{[k(k+n-1)]^s}.$$

The determinant of the Laplacian on n-sphere is expressible in terms of the Multiple Gamma Function $(\Gamma_n)_{n\geqslant 0}$ of Barnes. According to [2], the Multiple Gamma Function is defined uniquely as the positive real-valued n-times differentiable function on $\mathbb{R}^+$ with the property $(-1)^{n+1}\frac{d^n}{dx^n}\log\Gamma_n(x)$ is increasing and satisfies

$$\Gamma_n(x+1) = \frac{\Gamma_n(x)}{\Gamma_{n-1}(x)}; \;\; \Gamma_n(1) = 1;$$

[2] and [4]. At $n = 1$, Γ_1 is the usual gamma function. Double Gamma Function, (also called the Barnes G-function), Γ_2 is defined by

$$\Gamma_2(x+1) = G(x+1) = \Gamma(x)G(x) \;\; \text{where} \;\; G(1) = 1, \; G(x) = (x-2)!(x-3)!\cdots 1!.$$

In general, for $x = 2k+1$,

$$G(\frac{1}{2}(2k+1)) = c_k A^{-3/2}\pi^{-(2k-3)/4}e^{1/8}2^{1/24}2^{-[(k-1)(k-2)]/2};$$

where $A = \exp(\frac{1}{12} - \zeta'(-1))$ is the so-called Glaisher or Kinkelin constant,

$$c_k = \prod_{i=1}^{k-2} \frac{2^i \Gamma(1/2+i)}{\sqrt{\pi}}; \;\; k > 1.$$

This is the generalization of the classical formula $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Thus, one can easily read off the height and determinant functions from the $Z_{S^n}(s)$ using the Finite Part scheme (24). Note, of course that

$$\mathrm{Res}_{s=\frac{n}{2}-j} Z_{S^n}(s) = \lim_{s \to \frac{n}{2}-j} Z_{S^n}(s) \cdot \left(s - \frac{n}{2} + j\right);$$

c.f: [4] and [3]. Vardi [25] computed the determinant of the n-sphere to be

$$\det \Delta_n = c_n e^{A_n} \prod_{l=1}^{n} \Gamma_l(\frac{1}{2})^{A_{n,l}}$$

for certain computable rational numbers A_n, $A_{n,l}$ and some algebraic number c_n. See also [10] and [21] for similar results. For example, using that

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^s} = (1-2^{-s})\zeta(s); \;\; \Re(s) > 1,$$

where $\zeta(s)$ is the Riemann zeta function defined by (5), we readily have that

$$Z_1(s) = \sum_{k=1}^{\infty} \frac{2}{k^{2s}} = 2\zeta(2s) \Rightarrow \mathrm{FP}[-Z_1(s)]\Big|_{s=0} = -1 - 2\ln 2\pi \quad \text{and}$$

$$\begin{aligned} Z_2(s) &= \sum_{k=1}^{\infty} \frac{2k+1}{(k^2+k)^s} = \sum_{k=1}^{\infty} \frac{2k+1}{[(2k+1)^2 - 4^{-1}]^s} \\ &= (2^{2s}-2)\zeta(2s-1) - 4^s \Rightarrow \mathrm{FP}[-Z_2(s)]\Big|_{s=0} = -\frac{11}{12}. \end{aligned}$$

Also,

$$\begin{aligned} Z_3(s) &= \sum_{k=1}^{\infty} \frac{(k+1)^2}{(k^2+2k)^s} = \sum_{k=1}^{\infty} \frac{(k+1)^2}{[(k+1)^2-1]^s} = \zeta(2s-2) - 1 \\ &\Rightarrow \mathrm{FP}[-Z_3(s)]\Big|_{s=0} = -1.0609. \end{aligned}$$

Furthermore,

$$\begin{aligned} Z_{S^4}(s) &= \frac{1}{6} \sum_{k=1}^{\infty} \frac{(k+1)(k+2)(2k+3)}{(k+\frac{3}{2})^{2s}} \\ &= \frac{1}{3}(2^{2s-3}-1)\zeta_R(2s-3) - \frac{1}{3}\left(2^{2s-3} - \frac{1}{4}\right)\zeta_R(2s-1) \\ &\quad -\frac{1}{3}\left(\frac{2}{3}\right)^{2s-3} + \frac{1}{8}\left(\frac{2}{3}\right)^{2s} \Rightarrow \mathrm{FP}[-Z_4(s)]\Big|_{s=0} = -\frac{2897}{2880}; \quad \text{and so on.} \end{aligned}$$

Furthermore,

$$
\begin{aligned}
\det\Delta_1 &= 4\Big[\Gamma\Big(\frac{1}{2}\Big)\Big]^4 = 13.8174,\\
\det\Delta_2 &= 2^{1/9}e^{1/2}\pi^{-2/3}\Gamma_2\Big(\frac{1}{2}\Big)^{8/3} = 3.19531149\cdots,\\
\det\Delta_3 &= \pi^{8/7}\Gamma_3\Big(\frac{3}{2}\Big)^{16/7} = 3.33885121\cdots;
\end{aligned}
$$

where

$$
\begin{aligned}
\Gamma_n\Big(\frac{k}{2}\Big) &= \frac{\Gamma_n[(k-2)/2]}{\Gamma_{n-1}[(k-2)/2]}\\
&= \frac{\Gamma_n(\frac{1}{2})}{\Gamma_{n-1}[(k-2)/2]\Gamma_{n-1}[(k-4)/2]\cdots\Gamma_{n-1}\Big(\frac{1}{2}\Big)}.
\end{aligned}
$$

Thus from the relation (4) and the spectral inequality bound (25), we have that the spectral height values for the n-sphere satisfies

$$
h(S^n) \leqslant \ln\Big(\Gamma_n(\frac{k}{2})\Big); \quad n \geqslant 1.
$$

5. Conclusion

We have discussed different variations of the spectral height and determinant functions on closed Riemannian manifolds. Specifically we found that for a simple Schrödinger-type operator $H_c := \Delta + c$, where c a constant potential that the height and determinant functions can be computed explicitly and that for $0 < c \ll 1$, the height function $h(g) < \ln 2$.

Furthermore, for conformally deformed Laplacian, Δ_h, we found Polyakov's variation type formulas for the height and determinant functions on n-dimensional compact Riemannian manifolds and spectral inequality bounds on them. We illustrated our results with the circle, n-dimensional torus and spheres. These results are generalisations of the classical Polyakov-Ray-Singer variation formula for spectral determinants on compact Riemannian manifolds.

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REFERENCES

[1] W. ARENDT, *A characterisation of generators of positive semigroups*, Proceedings of the Royal Irish Academy, Section A: Mathematical and Physical Sciences **84A**, 1 (1984), 155–174.

[2] E. W. BARNES, *The genesis of the double gamma functions*, Proceedings of the London Mathematical Society **1**, 1 (1899), 358–81.

[3] JUNESANG CHOI, *Determinants of the Laplacians on the n-dimensional unit sphere S^n*, Advances in Difference Equations **2013**, 1 (2013), 1–12.

[4] JUNESANG CHOI AND H. M. SRIVASTAVA, *An application of the theory of the double Gamma function*, Kyushu Journal of Mathematics **53**, 1 (1999), 209–222.

[5] LAHOUCINE ELAISSAOUI, *On expansions involving the Riemann zeta function and its derivatives*, Journal of Mathematical Analysis and Applications **517**, 1 (2023), 126569.

[6] PETER B. GILKEY, *Invariance theory: the heat equation and the Atiyah-Singer index theorem*, vol. 16, CRC press, 2018.

[7] ALEXANDER GRIGORYAN, *Heat kernel and analysis on manifolds*, vol. 47, American Mathematical Soc., 2009.

[8] SORA HIRAMATSU AND YUKI SEO, *Determinant for positive operators and Oppenheim's inequality Journal of Mathematical Inequalities*, **15**, 4 (2021), 1637–1645.

[9] K. KIRSTEN AND ALAN MCKANE, *Functional determinants in the presence of zero modes. Quantum Field Theory under the Influence of External Conditions*, Rinton Press Incorporated, 2004.

[10] H. KUMAGAI, *The determinant of the Laplacian on the n-sphere*, Acta Arithmetica **91**, 3 (1999), 199–208.

[11] JOHN M. LEE, *Smooth manifolds*, Springer, New York, 2012.

[12] SUBBARAMIAH MINAKSHISUNDARAM AND ÅKE PLEIJEL, *Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds*, Canadian Journal of Mathematics **1**, 3 (1949), 242–256.

[13] YOICHI MOTOHASHI, *Spectral theory of the Riemann zeta-function*, no. 127, Cambridge University Press, 1997.

[14] WERNER MÜLLER, *Relative zeta functions, relative determinants and scattering theory*, Communications in mathematical physics **192**, 1 (1998), 309–347.

[15] KATE OKIKIOLU, *Critical metrics for the determinant of the Laplacian in odd dimensions*, Annals of Mathematics **153**, 1 (2001), 471–531.

[16] LOUIS OMENYI, *Meromorphic continuation of the spectral zeta kernel*, General Mathematics Notes **36**, 2 (2016), 26–37.

[17] L. OMENYI AND M. UCHENNA, *Global analysis on Riemannian manifolds*, The Australian Journal of Mathematical Analysis and Applications **16**, 2 (2019), 1–17.

[18] LOUIS OMENYI AND MCSYLVESTER OMABA, *Some domination inequalities for spectral zeta kernels on closed Riemannian manifolds*, Journal of Mathematical Inequalities **16**, 4 (2022), 1525–1539.

[19] BRAD OSGOOD, RALPH PHILLIPS AND PETER SARNAK, *Extremals of determinants of Laplacians*, Journal of functional analysis **80**, 1 (1988), 148–211.

[20] DANIEL B. RAY AND ISADORE M. SINGER, *R-torsion and the Laplacian on Riemannian manifolds*, Advances in Mathematics **7**, 2 (1971), 145–210.

[21] KEN RICHARDSON, *Critical points of the determinant of the Laplace operator*, Journal of Functional Analysis **122**, 1 (1994), 52–83.

[22] STEVEN ROSENBERG, *The Laplacian on a Riemannian manifold: an introduction to analysis on manifolds*, no. 31, Cambridge University Press, 1997.

[23] H. M. SRIVASTAVA, KHALED MEHREZ AND ZIVORAD TOMOVSKI, *New inequalities for some generalized Mathieu type series and the Riemann zeta function*, Journal of Mathematical Inequalities **12**, 1 (2018), 163–174.

[24] EDWARD CHARLES TITCHMARSH AND DAVID RODNEY HEATH-BROWN, *The theory of the Riemann zeta-function*, Oxford university press, 1986.

[25] ILAN VARDI, *Determinants of Laplacians and multiple gamma functions*, SIAM journal on mathematical analysis **19**, 2 (1988), 493–507.

[26] ANDRÉ VOROS, *Zeta functions over zeros of zeta functions*, vol. 8, Springer Science & Business Media, 2009.

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