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WILKER'S, CUSA–HUYGENS', AND FINK–MORTICI'S TYPE INEQUALITIES FOR PARABOLIC TRIGONOMETRIC FUNCTIONS

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Abstract. Parabolic trigonometric functions (PTF) have been recently defined as functions of an area lying on a parabolic circle. In this paper, we begin with an investigation into PTF's properties as these functions still need to be sufficiently understood. We find a precise formulation of Wilker's inequality together with Fink-Mortici's type inequality for PTF. Furthermore, we conjecture the proper form of Cusa-Huygens' inequality for PTF and find both upper and lower bounds on this inequality.

1. Introduction

In recent years, several generalisations of classical trigonometric functions using various methods have been created. These methods include power series as well as tools of integral and differential calculus. We mention, for example, the integral

$$\int_0^x (1-t^q)^{-\frac{1}{p}}\,\mathrm{d}t$$

through which the definition of a generalized $\arcsin_{p,q} x$ is given. These generalised functions have been examined from a number of angles, and some of their properties have been found. Usually, they create a direct or indirect counterpart to the properties of the classical trigonometric functions. A number of recent results can be found in the papers [3, 5, 6, 9, 15, 16, 21].

Much attention is paid to deriving inequalities that involve the mentioned generalised trigonometric functions. Among studied inequalities belongs Wilker's inequality, which origin lies in the question posed in [24]. The proposed question was answered in [25], and Wilker's inequality has been studied ever since. For certain generalised trigonometric functions are Wilker's inequality and its variations are proved in [14, 18]. However, the mentioned inequalities continue to be studied with respect to the classical trigonometric functions as illustrate articles [2, 4, 26, 27]. Another extension of Wilker's inequality, this time related to Bessel functions, appears in [1].

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Another well-known inequality for trigonometric functions is Cusa-Huygens' inequality which origin goes back (according to [19], see also [14, 17]) to 15th century where Nicolaus de Cusa proved that

$$\frac{3\sin x}{2+\cos x} < x. \tag{1}$$

Another proof of (1) is due to Christian Huygens' from the 17th century. Consequently, inequality (1) is called Cusa-Huygens' in the literature. For certain generalised trigonometric functions is Cusa-Huygens' type inequality proved in [13, 22], see also [11]. Moreover, Fink-Mortici's type inequalities are investigated for generalised trigonometric functions in [8, 23, 28].

A recent article [7] (see also [12, 20]) introduced, as far as we know, a new generalisation of trigonometric functions through their link to an area of a generalised (parabolic) circle

$$x^2 + |y| = 1.$$

Due to their relation to the parabolic circle, they have been named parabolic trigonometric functions (abbreviated PTF). Throughout the article, we denote these functions as $\sin_p x$ and $\cos_p x$.

The article is organised as follows. In the second section, we summarise known definitions and facts about PTF and establish properties that

$$(\tan_p x)' = \frac{1}{\cos_p^2 x}, \quad \text{and} \quad \sin_p x < x \text{ for } x > 0.$$

In the third section, we examine Wilker's inequality for PTF and prove that

$$\left(\frac{\sin_p x}{x}\right)^2 + \frac{\tan_p x}{x} > 2, \quad x \in \left(0, \frac{\pi_p}{2}\right). \tag{2}$$

Here π_p denotes a parabolic Pi, which definition will be given in a subsequent section. Furthermore, we derive other properties of parabolic trigonometric functions based on Wilker's inequality. In the fourth section, we conjecture a version of Cusa-Huygens' inequality for PTF and find upper and lower bounds for the conjectured inequality. In the fifth section, we prove a version of Fink-Mortici's type inequality for PTF that is

$$\frac{12x}{11+\sqrt{1-x^2}} < \operatorname{arcsin}_p x < \frac{2x}{1+\sqrt{1-x^2}}, \quad x \in (0,1). \tag{3}$$

In the final section, we compare graphically obtained inequalities.

Most of our proofs are conducted in the spirit of Paul Erdős as we use simple methods to obtain substantial results.

2. Preliminaries

In literature, there are different proposed generalizations of trigonometric functions. Article [7] proposes one based on areas where parabolic sine $\sin_p x$ and cosine $\cos_p x$ satisfy two conditions. First condition specifies that the functions lie on a generalized unit parabola

$$\cos_p^2 x + |\sin_p x| = 1, \quad x \in \mathbb{R}. \tag{4}$$

The second condition links the area x under the parabola with $\sin_p x$, $\cos_p x$ through the equation

$$\frac{\cos_p x \sin_p x}{2} + \int_{\cos_p x}^{1} 1 - t^2 \,\mathrm{d}t = \frac{x}{2}, \quad x \in \left[0, \frac{8}{3}\right]. \tag{5}$$

Conditions (4), (5) define a pair of functions given on $\left[0, \frac{8}{3}\right]$ that are called parabolic trigonometric functions. Parabolic tangent is then defined in a natural way as

$$\tan_p x := \frac{\sin_p x}{\cos_p x}.$$

Using conditions (4), (5) allows deriving properties of parabolic trigonometric functions. The derivatives are given as

$$(\cos_p x)' = -\frac{1}{1+\cos_p^2 x}, \tag{6}$$

$$(\sin_p x)' = \frac{2\cos_p x}{1+\cos_p^2 x} \tag{7}$$

for $x \in \left(0, \frac{8}{3}\right)$. System of equations (4), (5) is solvable and it yields that

$$\cos_p x = -2\sinh\left(\frac{1}{3}\operatorname{arcsinh}\frac{3x-4}{2}\right),$$

$$\sin_p x = 3 - 2\cosh\left(\frac{2}{3}\operatorname{arcsinh}\frac{3x-4}{2}\right)$$

for $x \in \left[0, \frac{8}{3}\right]$. Another consequence of (4), (5) is the following identity

$$\cos_p x \sin_p x - 4\cos_p x = 3x - 4.$$

It is a natural fact that trigonometric functions are connected to the number π. For PTF we define parabolic π_p as the area under parabola $y = 1 - x^2$ over the interval $[-1, 1]$, which is therefore

$$\pi_p = \frac{8}{3}.$$

As a consequence, PTF behave in the similar manner to classical trigonometric functions as we see in Table 1.

Furthermore, Eq. (4) bounds both functions, i.e. $0 \leqslant \sin_p x \leqslant 1$, $-1 \leqslant \cos_p x \leqslant 1$ for $x \in [0, \pi_p]$. The functions are displayed in Figure 1.

x	0	$\left(0,\frac{\pi_p}{2}\right)$	$\frac{\pi_p}{2}$	$\left(\frac{\pi_p}{2},\pi_p\right)$	π_p
$\sin_p x$	0	>0	1	>0	0
$\cos_p x$	1	>0	0	<0	-1

Table 1: *Table of function values and signs for PTF.*

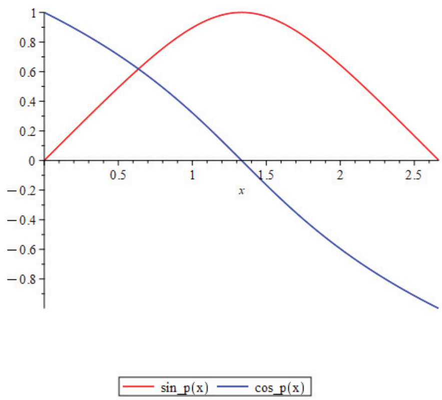


Figure 1: *Parabolic trigonometric functions on the interval* $[0,\pi_p]$

Let us emphasize, that condition (5) defines parabolic trigonometric functions solely over the interval $[0,\pi_p]$. Nevertheless, we can extend $\cos_p x$ as an even function over $[-\pi_p,\pi_p]$ and $\sin_p x$ as an odd function similarly. In this article, we will focus mainly on the interval $\left[0,\frac{\pi_p}{2}\right]$, where $\sin_p x$ is strictly increasing and $\cos_p x$ is strictly decreasing.

LEMMA 1. *It holds*

$$\lim_{x\to 0^+}\frac{\sin_p x}{x}=1.$$

Proof. As an application of L'hopital's rule we get

$$\lim_{x\to 0^+}\frac{\sin_p x}{x}=\lim_{x\to 0^+}\frac{2\cos_p x}{1+\cos_p^2 x}=\frac{2}{2}=1. \quad \square$$

LEMMA 2. *It holds*

$$(\tan_p x)'=\frac{1}{\cos_p^2 x} \quad \textit{and} \quad \tan_p x>x$$

for $x\in\left(0,\frac{4}{3}\right)$.

Proof. Directly from the quotient rule

$$(\tan_p x)' = \frac{\frac{2\cos_p^2 x}{1+\cos_p^2 x} + \frac{\sin_p x}{1+\cos_p^2 x}}{\cos_p^2 x} = \frac{\cos_p^2 x + \overbrace{\cos_p^2 x + \sin_p x}^{=1}}{\cos_p^2 x(1+\cos_p^2 x)} = \frac{1}{\cos_p^2 x}.$$

Additionally,

$$\tan_p x = \int_0^x \frac{1}{\cos_p^2 t}\,\mathrm{d}t > \int_0^x 1\,\mathrm{d}t = x. \quad \square$$

LEMMA 3. *It holds*

$$\sin_p x < x, \quad x > 0.$$

Proof. It is enough to prove the inequality for $x \in \left(0, \frac{\pi_p}{2}\right)$ because $\sin_p x \leqslant 1 < \frac{\pi_p}{2}$. Furthermore, using inverse function rule together with 4 and 7 we get

$$(\arcsin_p x)' = \frac{2-x}{2\sqrt{1-x}}, \tag{8}$$

where $\arcsin_p x$ denotes an inverse function to $\sin_p x$. We also conclude that for $x \in (0,1)$ is

$$\frac{2-x}{2\sqrt{1-x}} > 1.$$

Indeed, that is true because

$$\left(\frac{2-x}{2\sqrt{1-x}}\right)' = \frac{-2(1-x)+(2-x)}{8(1-x)\sqrt{1-x}} = \frac{x}{8(1-x)\sqrt{1-x}}$$

is a positive function. Hence, $\arcsin_p x$ increases faster than x and thus we know that $\arcsin_p x > x$. Function $\sin_p x$ is increasing for $x \in \left(0, \frac{\pi_p}{2}\right)$ and $x = \sin_p \arcsin_p x > \sin_p x$. $\square$

We stress that from (8) we can conclude that

$$\arcsin_p x = \int_0^t \frac{2-t}{2\sqrt{1-t}}\,\mathrm{d}t = \frac{4+\sqrt{1-x}(x-4)}{3}, \quad x \in [0,1].$$

The following lemma is a special version of L'Hopital's rule and appears, for example, in book [10, Theorem 1.25] (see also [2, 13, 22, 23] and many others).

LEMMA 4. *For $-\infty < a < b < \infty$ let functions $f,g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b) and let $g'(x) \neq 0$ on (a,b). If $\frac{f'(x)}{g'(x)}$ is increasing (decreasing) on (a,b) then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \textit{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $\frac{f'(x)}{g'(x)}$ is strictly monotone, then the monotonicity in the conclusion is also strict.

LEMMA 5. *Let there be nonnegative functions $f(x)$, $g(x)$ such that $f(x)$, $g(x)$ are increasing (decreasing) on $[a,b]$. Then the following holds*

1. *$f(x)+g(x)$ is also increasing (decreasing) on $[a,b]$.*
2. *$f(x)g(x)$ is increasing (decreasing) on $[a,b]$.*

If $f(x)$, $g(x)$ are strictly monotone, then the monotonicity in the conclusion is also strict.

3. Wilker's inequality

In this section, we prove the version of Wilker's inequality appropriate for PTF.

LEMMA 6. *It holds*

$$2\sin_p x + \tan_p x > 3x, \quad x \in \left(0, \frac{\pi_p}{2}\right).$$

Proof. Since function

$$f(x) = 2\sin_p x + \tan_p x - 3x$$

satisfies that $f(0)=0$, it is enough to show that $f(x)$ is increasing for $x \in \left(0, \frac{\pi_p}{2}\right)$. The derivative is

$$f'(x) = \frac{4\cos_p x}{1+\cos_p^2 x} + \frac{1}{\cos_p^2 x} - 3 = \frac{4\cos_p^3 x + 1 + \cos_p^2 x - 3\cos_p^2 x - 3\cos_p^4 x}{(1+\cos_p^2 x)\cos_p^2 x}.$$

If we substitute $y = \cos_p x$ we get that

$$f'(y) = \frac{-3y^4+4y^3-2y^2+1}{(1+y^2)y^2} = \frac{(1-y)(3y^3-y^2+y+1)}{(1+y^2)y^2}.$$

But $f'(y) > 0$ for all $y \in (0,1)$, which is true if and only if $3y^3-y^2+y+1>0$ for all $y \in (0,1)$. We bound

$$3y^3-y^2+y+1 > 3y^3-y^2+y = y(3y^2-y+1)$$

where $3y^2-y+1$ has solely complex roots and moreover we see that it is positive. As a consequence, $f(x)$ is increasing for $x \in \left(0, \frac{\pi_p}{2}\right)$ and the statement is true. □

THEOREM 1. *It holds*

$$\left(\frac{\sin_p x}{x}\right)^2 + \frac{\tan_p x}{x} > 2, \quad x \in \left(0, \frac{\pi_p}{2}\right).$$

Proof. First of all, if $\frac{\tan_p x}{x} > 3$ then the statement holds trivially. Let us consider the cases where $\frac{\tan_p x}{x} \leqslant 3$. From Lemma 6 yields

$$\begin{aligned}
&2\sin_p x + \tan_p x > 3x \\
&\qquad\frac{\sin_p x}{x} > \frac{3}{2} - \frac{\tan_p x}{2x} \\
&\qquad\frac{\sin_p x}{x} > 1 + \frac{1}{2}\left(1 - \frac{\tan_p x}{x}\right) \\
&\left(\frac{\sin_p x}{x}\right)^2 > 1 + \left(1 - \frac{\tan_p x}{x}\right) + \frac{1}{4}\left(1 - \frac{\tan_p x}{x}\right)^2 > 1 + \left(1 - \frac{\tan_p x}{x}\right) \\
&\left(\frac{\sin_p x}{x}\right)^2 + \frac{\tan_p x}{x} > 2. \quad \square
\end{aligned}$$

COROLLARY 1. *Let there be real parameters α, β, r, s such that $\alpha > 0$, $\beta > 0$ and $r \leqslant \frac{2s\beta}{\alpha}$. Then for $s \geqslant \max\left\{\frac{\alpha}{\beta}, 1\right\}$ holds*

$$\frac{\alpha}{\alpha+\beta}\left(\frac{\sin_p x}{x}\right)^r + \frac{\beta}{\alpha+\beta}\left(\frac{\tan_p x}{x}\right)^s > 1, \quad x \in \left(0, \frac{\pi_p}{2}\right).$$

Proof. We first consider the case where $\frac{\tan_p x}{x} \leqslant 2$. We see that

$$\begin{aligned}
&\frac{\alpha}{\alpha+\beta}\left(\frac{\sin_p x}{x}\right)^r + \frac{\beta}{\alpha+\beta}\left(\frac{\tan_p x}{x}\right)^s \\
&\overset{\text{Lemma 3}}{\geqslant} \frac{\alpha}{\alpha+\beta}\left(\frac{\sin_p x}{x}\right)^{\frac{2s\beta}{\alpha}} + \frac{\beta}{\alpha+\beta}\left(\frac{\tan_p x}{x}\right)^s \\
&\overset{\text{Theorem 1}}{>} \frac{\alpha}{\alpha+\beta}\left(2 - \frac{\tan_p x}{x}\right)^{\frac{s\beta}{\alpha}} + \frac{\beta}{\alpha+\beta}\left(\frac{\tan_p x}{x}\right)^s.
\end{aligned} \tag{9}$$

Now (9) becomes a function

$$f(y) = \frac{\alpha}{\alpha+\beta}(2-y)^{\frac{s\beta}{\alpha}} + \frac{\beta}{\alpha+\beta}y^s$$

via a substitution $y = \frac{\tan_p x}{x}$. Lemma 2 together with the assumption give that $2 \geqslant y > 1$.

Applying the derivative permits

$$f'(y) = \frac{\beta s}{\alpha+\beta}\left(y^{s-1} - (2-y)^{\frac{s\beta}{\alpha}-1}\right),$$

and therefore we see that

$$y^{s-1} > 1 > (2-y)^{\frac{s\beta}{\alpha}-1} \geqslant 0.$$

Hence, $f(y)$ is increasing for all $2 \geqslant y > 1$ and thus $f(y) > f(1) = 1$. The statement is true for $\frac{\tan_p x}{x} \leqslant 2$.

In the next part, let us consider the case where $\frac{\tan_p x}{x} > 2$. We assert that

$$\frac{\beta}{\alpha+\beta}\left(\frac{\tan_p x}{x}\right)^s > \frac{\beta}{\alpha+\beta}2^s = \frac{1}{1+\frac{\alpha}{\beta}}2^s \geqslant \frac{1}{1+s}2^s. \tag{10}$$

Combining (10) with the fact that

$$\frac{2^s}{1+s} \geqslant 1 \tag{11}$$

for $s \geqslant 1$ finishes the proof. Inequality (11) follows naturally from the derivative, that is

$$\frac{2^s}{(1+s)^2}((\overbrace{1+s}^{\geqslant 2})\overbrace{\ln 2}^{>\frac{1}{2}}-1) > 0. \quad \square$$

COROLLARY 2. *For all $s \geqslant 1$ it holds*

$$\left(\frac{\sin_p x}{x}\right)^s > \frac{4\cos_p^s x}{1+\sqrt{1+8\cos_p^{2s} x}}, \quad x \in \left(0, \frac{\pi_p}{2}\right).$$

Proof. Considering the choice $r = 2s$, $\alpha = \beta = 1$ in Corollary 1 yields

$$\left(\frac{\sin_p x}{x}\right)^{2s} + \left(\frac{\tan_p x}{x}\right)^s > 2$$

$$\left(\frac{\sin_p x}{x}\right)^{2s} + \frac{1}{\cos_p^s x}\left(\frac{\sin_p x}{x}\right)^s - 2 > 0$$

$$\left(\left(\frac{\sin_p x}{x}\right)^s + \frac{\cos_p^{-s} x + \sqrt{\cos_p^{-2s} x + 8}}{2}\right)$$

$$\times\left(\left(\frac{\sin_p x}{x}\right)^s + \frac{\cos_p^{-s} x - \sqrt{\cos_p^{-2s} x + 8}}{2}\right) > 0.$$

Hence, we conclude that

$$\left(\frac{\sin_p x}{x}\right)^{2s} + \frac{\cos_p^{-s} x - \sqrt{\cos_p^{-2s} x + 8}}{2} > 0$$

$$\left(\frac{\sin_p x}{x}\right)^{2s} > \frac{\sqrt{\cos_p^{-2s} x + 8} - \cos_p^{-s} x}{2}$$

$$\left(\frac{\sin_p x}{x}\right)^{2s} > \frac{8}{2\left(\sqrt{\cos_p^{-2s} x + 8} + \cos_p^{-s} x\right)}$$

$$\left(\frac{\sin_p x}{x}\right)^{2s} > \frac{8\cos_p^s x}{2\left(\sqrt{1 + 8\cos_p^{2s} x} + 1\right)}. \quad \square$$

COROLLARY 3. *It holds*

$$\left(\frac{\sin_p x}{x}\right)^{s} > \frac{\cos_p^s x}{1 + \cos_p^s x}, \quad x \in \left(0, \frac{\pi_p}{2}\right).$$

Proof. As a modification of Corollary 2 we get

$$\left(\frac{\sin_p x}{x}\right)^{s} > \frac{4\cos_p^s x}{1 + \sqrt{1 + 8\cos_p^{2s} x}} > \frac{4\cos_p^s x}{2 + \sqrt{8}\cos_p^s x} > \frac{4\cos_p^s x}{4 + 4\cos_p^s x}. \quad \square$$

4. Bounds on Cusa-Huygens' type inequality for parabolic trigonometric functions

We conjecture (see the final section) that the Cusa-Huygens' inequality for parabolic trigonometric functions has the form

$$\frac{\cos_p x + \frac{x}{2} + 2}{3} > \frac{\sin_p x}{x}, \quad x \in \left(0, \frac{\pi_p}{2}\right).$$

In this section, we prove a bound on the conjectured inequality in the form

$$\frac{\tan_p x}{x}\frac{\cos_p x + 2}{3} > \max\left\{\frac{\cos_p x + \frac{x}{2} + 2}{3}, \frac{\sin_p x}{x}\right\}$$
$$> \min\left\{\frac{\cos_p x + \frac{x}{2} + 2}{3}, \frac{\sin_p x}{x}\right\} > \frac{\cos_p x - \frac{3}{4}x + 2}{3}, \quad x \in \left(0, \frac{\pi_p}{2}\right).$$

LEMMA 7. *It holds*

$$\cos_p x > 1 - x, \quad x \in \left(0, \frac{\pi_p}{2}\right).$$

Proof. Function $f(x) = \cos_p x - 1 + x$ satisfies $f(0) = 0$ and its derivative is positive because

$$f'(x) = -\frac{1}{1+\cos_p^2 x} + 1 = \frac{\cos_p^2 x}{1+\cos_p^2 x} > 0.$$

The function $f(x)$ is increasing, therefore positive. $\square$

THEOREM 2. *It holds*

$$\min\left\{\frac{\cos_p x + \frac{x}{2} + 2}{3}, \frac{\sin_p x}{x}\right\} > \frac{\cos_p x - \frac{3}{4}x + 2}{3}, \quad x \in \left(0, \frac{\pi_p}{2}\right).$$

Proof. First of all, auxiliary function $f(x) = 3\sin_p x - x\cos_p x + \frac{3}{4}x^2 - 2x$ satisfies that $f(0) = 0$ and its derivative is

$$\begin{aligned} f'(x) &= \frac{6\cos_p x}{1+\cos_p^2 x} - \cos_p x + \frac{x}{1+\cos_p^2 x} + \frac{3}{2}x - 2 \\ &\overset{1+\cos_p^2 x<2}{>} \frac{6\cos_p x}{2} - \cos_p x + \frac{x}{2} + \frac{3}{2}x - 2 = 2\cos_p x + 2x - 2 \\ &\overset{\text{Lemma 7}}{>} 2(1-x) + 2x - 2 = 0. \end{aligned}$$

The function $f(x)$ is positive and thus

$$\begin{aligned} 3\sin_p x &> x\cos_p x - \frac{3}{4}x^2 + 2x \\ \frac{\sin_p x}{x} &> \frac{\cos_p x - \frac{3}{4}x + 2}{3}. \end{aligned}$$

The second part is obvious as

$$\frac{\cos_p x + \frac{x}{2} + 2}{3} > \frac{\cos_p x - \frac{3}{4}x + 2}{3}$$

for $x > 0$. $\square$

LEMMA 8. *It holds*

$$\frac{\tan_p x}{x} - 1 > 1 - \cos_p x > \frac{x}{2\cos_p x + 4}, \quad x \in \left(0, \frac{\pi_p}{2}\right).$$

Proof. Considering function $f(x) = 4 - 2\cos_p x - 2\cos_p^2 x - x$ that satisfies $f(0) = 0$ yields that

$$\begin{aligned} f'(x) &= \frac{2}{1+\cos_p^2 x} + \frac{4\cos_p x}{1+\cos_p^2 x} - 1 = \frac{2}{1+\cos_p^2 x}(1+2\cos_p x) - 1 \\ &> \frac{2}{2}(1+2\cos_p x) - 1 = 2\cos_p x > 0. \end{aligned}$$

Furthermore, it holds

$$(1-\cos_p x)(2\cos_p x+4) = 4-2\cos_p x-2\cos_p^2 x = f(x)+x > x.$$

For the second part, we proceed with a function $g(x) = \tan_p x + x\cos_p x - 2x$ which satisfies $g(0) = 0$. Moreover, its derivative is

$$\begin{aligned} g'(x) &= \frac{1}{\cos_p^2 x} + \cos_p x - \frac{x}{1+\cos_p^2 x} - 2 \\ (1+\cos_p^2 x)g'(x) &= \frac{1}{\cos_p^2 x} + \cos_p x + \cos_p^3 x - 2\cos_p^2 x - x - 1. \end{aligned}$$

Function $h(x) = (1+\cos_p^2 x)g'(x)$ then also satisfies that $h(0) = 2g'(0) = 0$ and additionally

$$h'(x) = \frac{2}{\cos_p^3 x(1+\cos_p^2 x)} + \frac{4\cos_p x - 3\cos_p^2 x - 1}{(1+\cos_p^2 x)} - 1.$$

Substituting $y = \cos_p x$ gives for $0 < y < 1$ that

$$h'(y) = \frac{2}{y^3(1+y^2)} + \frac{4y-4y^2-2}{1+y^2} = \frac{2}{(1+y^2)}\left(\overbrace{\frac{1}{y^3}-1}^{>0} + \overbrace{4y(1-y)}^{>0}\right) > 0.$$

Therefore, $h'(x) > 0$ and both functions $g'(x)$, $g(x)$ are increasing, therefore positive. $\square$

THEOREM 3. *It holds*

$$\frac{\tan_p x}{x}\frac{\cos_p x+2}{3} > \max\left\{\frac{\cos_p x+\frac{x}{2}+2}{3}, \frac{\sin_p x}{x}\right\}, \quad x \in \left(0, \frac{\pi_p}{2}\right).$$

Proof. From Lemma 8 follows that

$$\begin{aligned} \frac{\tan_p x}{x} - 1 &> \frac{\frac{x}{2}}{\cos_p x+2} \\ \frac{\tan_p x}{x} &> \frac{\cos_p x+\frac{x}{2}+2}{\cos_p x+2} \\ \frac{\tan_p x}{x}\frac{\cos_p x+2}{3} &> \frac{\cos_p x+\frac{x}{2}+2}{3}. \end{aligned}$$

Furthermore, it is straightforward that

$$\frac{\tan_p x}{x}\frac{\cos_p x+2}{3} > \frac{\sin_p x}{x\cos_p x}\frac{3\cos_p x}{3} > \frac{\sin_p x}{x}. \quad \square$$

5. Fink-Mortici's inequality

In this section, we prove a version of Fink-Mortici's inequality for parabolic trigonometric functions

$$\frac{12x}{11+\sqrt{1-x^2}} < \operatorname{arcsin}_p x < \frac{2x}{1+\sqrt{1-x^2}}, \quad x \in (0,1).$$

Proof. First of all, let us start with the upper bound

$$\operatorname{arcsin}_p x < \frac{2x}{1+\sqrt{1-x^2}}.$$

Calculating

$$\lim_{x\to 0^+} \frac{\operatorname{arcsin}_p x}{\frac{2x}{1+\sqrt{1-x^2}}} \overset{8}{=} \lim_{x\to 0^+} \frac{\frac{2-x}{2\sqrt{1-x}}}{\frac{2}{(1+\sqrt{1-x^2})\sqrt{1-x^2}}} = \lim_{x\to 0^+} \frac{(2-x)(1+\sqrt{1-x^2})\sqrt{1+x}}{4} = 1.$$

In the next part, we show that the function

$$(2-x)(1+\sqrt{1-x^2})\sqrt{1+x} = (2-x)\sqrt{1+x} + (2-x)(1+x)\sqrt{1-x}$$

is strictly decreasing on $(0,1)$. By Lemma 5 it is sufficient to show that both $(2-x)\sqrt{1+x}$ and $(2-x)(1+x)\sqrt{1-x}$ are strictly decreasing. However, we have

$$\left((2-x)\sqrt{1+x}\right)' = -\frac{3x}{2\sqrt{1+x}}, \tag{12}$$

$$\left((2-x)(1+x)\sqrt{1-x}\right)' = \frac{x(5x-7)}{2\sqrt{1-x}}, \tag{13}$$

where both derivatives (12), (13) are clearly negative for $x \in (0,1)$. As a consequence, we know via Lemma 4 that

$$\frac{\operatorname{arcsin}_p x}{\frac{2x}{1+\sqrt{1-x^2}}} < 1$$

$$\operatorname{arcsin}_p x < \frac{2x}{1+\sqrt{1-x^2}}.$$

In the next part, we show the lower bound

$$\frac{12x}{11+\sqrt{1-x^2}} < \operatorname{arcsin}_p x.$$

For the second part, we calculate

$$\lim_{x\to 0^+} \frac{\operatorname{arcsin}_p x}{\frac{12x}{1+\sqrt{11-x^2}}} \overset{8}{=} \lim_{x\to 0^+} \frac{(2-x)(11+\sqrt{1-x^2})^2\sqrt{1+x}}{24+264\sqrt{1-x^2}} = 1.$$

Additionally, the function

$$\frac{(2-x)(11+\sqrt{1-x^2})^2\sqrt{1+x}}{24+264\sqrt{1-x^2}}$$

is increasing. Indeed its derivative is

$$\begin{aligned}
&\left(\frac{(2-x)(11+\sqrt{1-x^2})^2\sqrt{1+x}}{24+264\sqrt{1-x^2}}\right)'\\
&=\frac{12x(11+\sqrt{1-x^2})\left(128x^2+238x+110+(55x^2-22x-110)\sqrt{1-x^2}\right)}{(1+x)\sqrt{1-x}\left(24+264\sqrt{1-x^2}\right)^2}\\
&>\frac{12x(11+\sqrt{1-x^2})(128x^2+238x+110+55x^2-22x-110)}{(1+x)\sqrt{1-x}\left(24+264\sqrt{1-x^2}\right)^2}\\
&=\frac{12x^2(11+\sqrt{1-x^2})(183x+216)}{(1+x)\sqrt{1-x}\left(24+264\sqrt{1-x^2}\right)^2}>0
\end{aligned}$$

for $x\in(0,1)$. Hence, Lemma 4 shows that

$$\frac{\operatorname{arcsin}_p x}{\frac{12x}{1+\sqrt{11-x^2}}}>1$$

$$\operatorname{arcsin}_p x>\frac{12x}{1+\sqrt{11-x^2}}. \qquad \square$$

6. Conclusion and final remarks

We conjecture that Cusa-Huygens' type inequality for parabolic functions has the following form.

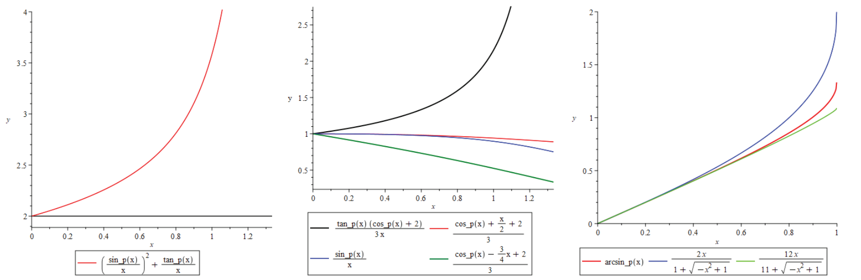
CONJECTURE 1. It holds

$$\frac{\sin_p x}{x}<\frac{\cos_p x+\frac{x}{2}+2}{3}, \quad x\in\left(0,\frac{\pi_p}{2}\right).$$

In Figure 2 part (b) we see, that the inequality in Conjecture 1 seems to hold and it is quite tight. As a consequence, we could bound the integral

$$\int_0^x \frac{\sin_p t}{t}\,\mathrm{d}t<\frac{1}{3}\int_0^x \cos_p t\,\mathrm{d}t+\frac{x^2}{12}+\frac{2x}{3}, \tag{14}$$

where the right-hand side integral in (14) seems easier to solve. Moreover, the bound in Figure 2 part (b) seems quite tight and we could use the right-hand side of (14) as



(a) Wilker's inequality (b) Cusa-Huygens' bounds (c) Fink-Mortici's inequality

Figure 2: *Representation of studied inequalities. Based on (b) we hypothesize that Conjecture 1 is indeed true. In (c) we see that the bounds in Fink-Mortici's type inequality are quite tight.*

an approximation. Nevertheless, the proof of Conjecture 1 is yet missing and we do not know what is the solution of

$$\int_0^x \cos_p t \, \mathrm{d}t .$$

Furthermore, let us highlight that PTF are not the only functions that satisfy Eq. (4). Functions $\sin_{2,1} x$, $\cos_{2,1} x$ (see [12] or the definition in [9,15,16]) also satisfy Eq. (4). It remains to show what is the relationship between PTF and $\sin_{2,1} x$, $\cos_{2,1} x$ and whether this connection can be used to derive further properties.

Competing interests. The author declares that he is also associated with Masaryk University in Brno, Czech Republic.

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