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SHARP INEQUALITIES FOR ZALCMAN FUNCTIONAL OF LOGARITHMIC COEFFICIENTS OF INVERSE FUNCTIONS IN CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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Abstract. We study Hankel matrices whose entries are logarithmic coefficients of inverse functions in selected subclasses of analytic functions. Particularly, we give sharp bounds for the second Hankel determinant which reduces to Zalcman functional of logarithmic coefficients of inverse convex and starlike functions, as well as of functions of bounded turning.

1. Introduction

Given $r>0$, let $\mathscr{A}(\mathbb{D}_r)$ denote the class of analytic functions f in the disk $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ normalized by $f(0) = 0 = f'(0) - 1$. Then $f \in \mathscr{A}(\mathbb{D}_r)$ has the following representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n(f) z^n, \quad z \in \mathbb{D}_r. \tag{1}$$

Let $\mathscr{A} := \mathscr{A}(\mathbb{D})$, where $\mathbb{D} := \mathbb{D}_1$. Let $\mathscr{S}$ denote the subclass of all univalent (i.e., one-to-one) functions in $\mathscr{A}$.

Denote by $\mathscr{S}^*$ the subclass of $\mathscr{S}$ consisting of starlike functions, i.e., functions f which map $\mathbb{D}$ onto a set which is star-shaped with respect to the origin. Then it is well-known that if $f \in \mathscr{A}$, then $f \in \mathscr{S}^*$ if, and only if,

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \quad z \in \mathbb{D}, \tag{2}$$

and that $\mathscr{S}^* \subset \mathscr{S}$ (cf. [4, pp. 40–41]).

By $\mathscr{C}$ we denote the subclass of $\mathscr{S}$ consisting of convex functions, i.e., functions f which map $\mathbb{D}$ onto a convex set. It is well-known that if $f \in \mathscr{A}$, then $f \in \mathscr{C}$ if, and only if,

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0, \quad z \in \mathbb{D}, \tag{3}$$

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(cf. [4, pp. 42–43]).

If $f \in \mathscr{A}$, then $f \in \mathscr{P}'$ if, and only if,

$$\operatorname{Re} f'(z) > 0, \quad z \in \mathbb{D}. \tag{4}$$

Elements of the class $\mathscr{P}'$ are called functions of bounded turning (cf. [6, vol. I., p. 101]). The Alexander Theorem states that $\mathscr{P}' \subset \mathscr{S}$ (e.g. [6, vol. I, p. 88]).

If $f \in \mathscr{A}$, then $f \in \mathscr{T}$ if, and only if,

$$\operatorname{Re} \frac{f(z)}{z} > 0, \quad z \in \mathbb{D}. \tag{5}$$

The class $\mathscr{T}$ although their elements are functions which are not necessarily univalent, plays an important role in the theory of semigroups of analytic functions as a generator of one-parameter continuous semigroups studied by Berkson, Porta, Shoikhet, Elin and others (e.g., [18], [5]). For other classical results concerning the class $\mathscr{T}$ see e.g., [14], [17].

For $f \in \mathscr{S}$ define

$$F_f(z) := \frac{1}{2} \log \frac{f(z)}{z} = \sum_{n=1}^{\infty} \gamma_n z^n, \quad \log 1 := 0,\ z \in \mathbb{D},$$

a logarithmic function associated with f. The numbers $\gamma_n := a_n(F_f)$ are called the logarithmic coefficients of f. It is well known, that the logarithmic coefficients play a crucial role in Milin's conjecture (see [15]; see also [4, p. 155]).

Note that if $\mathscr{F}$ is a compact subclass of $\mathscr{A}$, then there is $r_0(\mathscr{F}) \in (0,1]$ such that every $f \in \mathscr{F}$ is invertible in $\mathbb{D}_{r_0}$ and has the following representation

$$F(w) = f^{-1}(w) = w + \sum_{n=1}^{\infty} A_n w^n, \quad w \in \mathbb{D}_{r_0(\mathscr{F})}, \tag{6}$$

where $A_n := a_n(F)$. Thus for $f \in \mathscr{F}$ there exists the unique function $F_{f^{-1}}$ analytic in $\mathbb{D}_{r(\mathscr{F})}$ such that

$$F_{f^{-1}}(w) := \frac{1}{2} \log \frac{f^{-1}(w)}{w} = \sum_{n=1}^{\infty} \Gamma_n w^n, \quad w \in \mathbb{D}_{r(\mathscr{F})}, \tag{7}$$

where $\Gamma_n := a_n(F_{f^{-1}})$ are logarithmic coefficients of the inverse function f^{-1}. From (6) it follows that (e.g. [6, vol. I, p. 57])

$$\begin{aligned} A_2 &= -a_2(f), \quad A_3 = -a_3(f) + 2a_2(f)^2, \\ A_4 &= -a_4(f) + 5a_2(f)a_3(f) - 5a_2(f)^3. \end{aligned} \tag{8}$$

Thus from (7) we derive that

$$\Gamma_1 = \frac{1}{2} A_2, \quad \Gamma_2 = \frac{1}{2} A_3 - \frac{1}{4} A_2^2, \quad \Gamma_3 = \frac{1}{2} A_4 - \frac{1}{2} A_2 A_3 + \frac{1}{6} A_2^3,$$

and next using (8) we obtain

$$\begin{aligned}\Gamma_1 &= -\frac{1}{2}a_2(f), \quad \Gamma_2 = -\frac{1}{2}a_3(f)+\frac{3}{4}a_2(f)^2,\\ \Gamma_3 &= -\frac{1}{2}a_4(f)+2a_2(f)a_3(f)-\frac{5}{3}a_2(f)^3.\end{aligned} \tag{9}$$

From the very beginning of GFT, special attention has been focused on coefficient problems in the class $\mathscr{S}$ and its subclasses, and more broadly, in subclasses of the class $\mathscr{A}$. In the early 70s, Lawrence Zalcman posed the conjecture that if $f \in \mathscr{S}$, and is given by (1), then for $n \geqslant 2$,

$$|a_n(f)^2 - a_{2n-1}(f)| \leqslant (n-1)^2$$

with equality for the Koebe function $K(z) := z/(1-z)^2$ for $z \in \mathbb{D}$, or its rotations. This conjecture implies the celebrated Bieberbach conjecture $|a_n(f)| \leqslant n$ for $f \in \mathscr{S}$. Bieberbach Theorem shows that the Zalcman conjecture is true for $n = 2$ (see [6, p. 35]). Kruskal established the conjecture when $n = 3$ (see [12], and more recently for $n = 4,5,6$ (see [13]). For $n > 6$ the Zalcman conjecture remains an open problem. The functional $\mathscr{A} \ni f \mapsto a_n(f)^2 - a_{2n-1}(f)$ is called the Zalcman functional.

Another issue that has been intensively studied in recent years is the Hankel determinant over the class $\mathscr{A}$. For $q,n \in \mathbb{N}$, the Hankel determinant $H_{q,n}(f)$ of $f \in \mathscr{A}$ of the form (1) is defined as

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}, \tag{10}$$

where $a_n := a_n(f)$. In recent years there has been a great deal of attention devoted to finding bounds for the modulus of the second and third Hankel determinants $H_{2,2}(f)$ and $H_{3,1}(f)$, when f belongs to various subclasses of $\mathscr{A}$ (see [1, 9, 10, 11] for further references).

Based on these ideas, in [7] and [8] the authors started the study the Hankel determinant $H_{q,n}(F_f)$ whose entries are logarithmic coefficients of $f \in \mathscr{S}$, that is, $a_n(f)$ in (10) are replaced by γ_n. In this paper, we continue analogous research considering the Hankel determinant $H_{q,n}(F_{f^{-1}})$ whose entries are logarithmic coefficients of inverse functions, i.e., $a_n(f)$ in (10) are now replaced by Γ_n. We demonstrate the sharp estimates of modulus of

$$H_{2,1}(F_{f^{-1}}) = \Gamma_1\Gamma_3 - (\Gamma_2)^2 = \frac{1}{48}\left(13a_2^4 - 12a_3^2 + 12a_2a_4 - 12a_2^2a_3\right) \tag{11}$$

in the classes $\mathscr{S}^*$, $\mathscr{C}$, $\mathscr{P}'$ and $\mathscr{T}$. Observe that the functional $\mathscr{A} \ni f \mapsto H_{2,1}(F_{f^{-1}})$ is a transfer of the Zalcman functional for $n = 2$ to the logarithmic coefficients of the inverse functions.

2. Preliminary Lemmas

Denote by $\mathscr{P}$ the class of analytic functions $p:\mathbb{D}\to\mathbb{C}$ with positive real part given by

$$p(z)=1+\sum_{n=1}^{\infty}c_nz^n,\quad z\in\mathbb{D}, \tag{12}$$

where $c_n:=a_n(p)$.

In the proof of the main result we will use the following lemma which contains the formulas for c_2 (see e.g., [16, p. 166]) and c_3 from [2] with further remarks on extremal functions. Let $\overline{\mathbb{D}}:=\{z\in\mathbb{C}:|z|\leqslant 1\}$ and $\mathbb{T}:=\{z\in\mathbb{C}:|z|=1\}$.

LEMMA 2.1. *If $p\in\mathscr{P}$ is of the form (12), then*

$$c_1=2\zeta_1, \tag{13}$$

$$c_2=2\zeta_1^2+2(1-|\zeta_1|^2)\zeta_2 \tag{14}$$

and

$$c_3=2\zeta_1^3+2(1-|\zeta_1|^2)(2\zeta_1-\overline{\zeta_1}\zeta_2)\zeta_2+2(1-|\zeta_1|^2)(1-|\zeta_2|^2)\zeta_3 \tag{15}$$

for some $\zeta_1,\zeta_2,\zeta_3\in\overline{\mathbb{D}}$.

For $\zeta_1\in\mathbb{T}$, there is a unique function $p\in\mathscr{P}$ with c_1 as in (13), namely,

$$p(z)=\frac{1+\zeta_1z}{1-\zeta_1z},\quad z\in\mathbb{D}.$$

For $\zeta_1\in\mathbb{D}$ and $\zeta_2\in\mathbb{T}$, there is a unique function $p\in\mathscr{P}$ with c_1 and c_2 as in (13) and (14), namely,

$$p(z)=\frac{1+(\overline{\zeta}_1\zeta_2+\zeta_1)z+\zeta_2z^2}{1+(\overline{\zeta}_1\zeta_2-\zeta_1)z-\zeta_2z^2},\quad z\in\mathbb{D}. \tag{16}$$

LEMMA 2.2. ([3]) *For real numbers A, B, C, let*

$$Y(A,B,C):=\max\left\{|A+Bz+Cz^2|+1-|z|^2:z\in\overline{\mathbb{D}}\right\}. \tag{17}$$

I. *If $AC\geqslant 0$, then*

$$Y(A,B,C)=\begin{cases}|A|+|B|+|C|, & |B|\geqslant 2(1-|C|),\\ 1+|A|+\dfrac{B^2}{4(1-|C|)}, & |B|<2(1-|C|).\end{cases}$$

II. *If $AC<0$, then*

$$Y(A,B,C)=\begin{cases}1-|A|+\dfrac{B^2}{4(1-|C|)}, & -4AC(C^{-2}-1)\leqslant B^2\wedge|B|<2(1-|C|),\\ 1+|A|+\dfrac{B^2}{4(1+|C|)}, & B^2<\min\left\{4(1+|C|)^2,-4AC(C^{-2}-1)\right\},\\ R(A,B,C), & \text{otherwise},\end{cases} \tag{18}$$

where

$$R(A,B,C) := \begin{cases} |A|+|B|-|C|, & |C|(|B|+4|A|) \leqslant |AB|, \\ -|A|+|B|+|C|, & |AB| \leqslant |C|(|B|-4|A|), \\ (|C|+|A|)\sqrt{1-\dfrac{B^2}{4AC}}, & \text{otherwise}. \end{cases} \tag{19}$$

3. The class $\mathscr{S}^*$ of starlike functions

THEOREM 3.1. *If $f \in \mathscr{S}^*$, then*

$$\left|H_{2,1}(F_{f^{-1}})\right| \leqslant \frac{13}{12}. \tag{20}$$

The inequality is sharp.

Proof. Let $f \in \mathscr{S}^*$ be of the form (1). Then by (2) there exists $p \in \mathscr{P}$ of the form (12) such that

$$zf'(z) = f(z)p(z), \quad z \in \mathbb{D}. \tag{21}$$

Putting the series (1) and (12) into (21), by equating the coefficients we get ([6, Vol I., p. 116])

$$a_2(f) = c_1, \quad a_3(f) = \frac{1}{2}(c_2+c_1^2), \quad a_3(f) = \frac{1}{6}(2c_3+3c_1c_2+c_1^3). \tag{22}$$

Hence and from (9) we obtain

$$\Gamma_1 = -\frac{1}{2}c_1, \quad \Gamma_2 = -\frac{1}{4}(c_2-2c_1^2), \quad \Gamma_3 = -\frac{1}{12}(2c_3-9c_1c_2+9c_1^3),$$

and therefore

$$\Gamma_1\Gamma_3-(\Gamma_2)^2 = \frac{1}{48}\left(6c_1^4-6c_1^2c_2+4c_1c_3-3c_2^2\right). \tag{23}$$

Since both the class $\mathscr{S}^*$ and $|H_{2,1}(F_{f^{-1}})|$ are rotationally invariant, without loss of generality we may assume that $a_2 \geqslant 0$, which in view of (22) yields $c_1 \in [0,2]$, i.e., by (13) that $\zeta_1 \in [0,1]$. Thus by (11) and Lemma 2.1 we obtain

$$\begin{aligned} H_{2,1}(F_{f^{-1}}) =& \Gamma_1\Gamma_3-(\Gamma_2)^2 \\ =& \frac{1}{12}\big(13\zeta_1^4-10(1-\zeta_1^2)\zeta_1^2\zeta_2-(1-\zeta_1^2)(\zeta_1^2+3)\zeta_2^2 \\ &+4\zeta_1(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_3\big) \end{aligned} \tag{24}$$

for some $\zeta_1 \in [0,1]$ and $\zeta_2, \zeta_3 \in \overline{\mathbb{D}}$.

A. Suppose that $\zeta_1 = 0$. Then from (24),

$$\left|H_{2,1}(F_{f^{-1}})\right| = \frac{1}{4}|\zeta_2|^2 \leqslant \frac{1}{4}.$$

B. Suppose that $\zeta_1 = 1$. Then from (24),

$$\left|H_{2,1}(F_{f^{-1}})\right| = \frac{13}{12}.$$

C. Suppose that $\zeta_1 \in (0,1)$. Since $\zeta_3 \in \overline{\mathbb{D}}$, from (24) we get

$$\left|H_{2,1}(F_{f^{-1}})\right| \leqslant \frac{1}{3}\zeta_1(1-\zeta_1^2)\Phi(A,B,C),$$

where

$$\Phi(A,B,C) := \left|A + B\zeta_2 + C\zeta_2^2\right| + 1 - |\zeta_2|^2,$$

with

$$A := \frac{13\zeta_1^3}{4(1-\zeta_1^2)}, \quad B := \frac{-5\zeta_1}{2}, \quad C := \frac{-(\zeta_1^2+3)}{4\zeta_1}.$$

Observe that $AC < 0$ and therefore we apply only the part II of Lemma 2.2.

C1. Let's consider the condition $|B| < 2(1-|C|)$, i.e.,

$$\frac{5\zeta_1}{2} < 2\left(1 - \frac{\zeta_1^2+3}{4\zeta_1}\right).$$

The above inequality is equivalent to

$$\frac{6\zeta_1^2 - 4\zeta_1 + 3}{2\zeta_1} < 0$$

which is false for all $0 < \zeta_1 < 1$.

C2. Since

$$-4AC\left(\frac{1}{C^2} - 1\right) = -\frac{13\zeta_1^2(9-\zeta_1^2)}{4(\zeta_1^2+3)}$$

and

$$4(1+|C|)^2 = \frac{(\zeta_1+3)^2(\zeta_1+1)^2}{4\zeta_1^2},$$

it follows that the condition $B^2 < \min\{4(1+|C|)^2, -4AC(C^{-2}-1)\}$ is equivalent to

$$\frac{3\zeta_1^2(\zeta_1^2+16)}{(\zeta_1^2+3)} < 0.$$

The last inequality is false for all $0 < \zeta_1 < 1$.

C3. The inequality $|C|(|B|+4|A|) \leqslant |AB|$ is equivalent to

$$\frac{44\zeta_1^4 - 68\zeta_1^2 - 15}{8(1-\zeta_1^2)} \geqslant 0$$

which is true for $\zeta_1 \in \left(-\infty, -\sqrt{374+22\sqrt{454}}/22\right] \cup \left[\sqrt{374+22\sqrt{454}}/22, +\infty\right)$, and since $\sqrt{374+22\sqrt{454}}/22 \approx 1.31956$, it is false for all $0 < \zeta_1 < 1$.

C4. Observe that the condition $|C|(|B|-4|A|) \geqslant |AB|$ equivalently written as

$$\frac{96\zeta_1^4+88\zeta_1^2-15}{8(1-\zeta_1^2)} \leqslant 0$$

is true for $\zeta_1 \in \left(0, \zeta_1^0\right]$, where $\zeta_1^0 := \sqrt{-66+6\sqrt{211}}/12 \approx 0.383288$. Applying Lemma 2.2 for $0 < \zeta_1 \leqslant \zeta_1^0$ we get

$$\left|H_{2,1}(F_{f^{-1}})\right| \leqslant \frac{1}{3}\zeta_1(1-\zeta_1^2)\left(-|A|+|B|+|C|\right) = \rho(\zeta_1),$$

where

$$\rho(t) := \frac{1}{12}(-24t^4+8t^2+3), \quad t \in [0,1].$$

Since $\rho'(t) = 0$ for $t \in (0,1)$ holds only for $t_0 := \sqrt{6}/6 > \zeta_1^0$, we see that the function ρ is increasing in $[0, \zeta_1^0]$ and therefore

$$\rho(\zeta_1) \leqslant \rho(\zeta_1^0) = -\frac{29}{24} + \frac{5}{48}\sqrt{211} \approx 0.30477.$$

C5. Applying Lemma 2.2 for $\zeta_1^0 < \zeta_1 < 1$ we get

$$\left|H_{2,1}(F_{f^{-1}})\right| \leqslant \frac{1}{3}\zeta_1(1-\zeta_1^2)(|A|+|C|)\sqrt{1-\frac{B^2}{4AC}} = \psi(\zeta_1),$$

where

$$\psi(t) := \frac{12t^4-2t^2+3}{78}\sqrt{\frac{208-39t^2}{t^2+3}}, \quad t \in [0,1].$$

Since the equation

$$\psi'(t) = -\frac{1}{6}\frac{(144t^6-48t^4-2326t^2+267)t}{(t^2+3)^2\sqrt{\dfrac{208-39t^2}{t^2+3}}} = 0$$

has in $(0,1)$ a unique root $t_1 \approx 0.3385 < \zeta_1^0$, we deduce that ψ is increasing in $[\zeta_1^0, 1]$ and therefore

$$\psi(\zeta_1) \leqslant \psi(1) = \frac{13}{12}.$$

Summarizing, from Parts A-C it follows that the inequality (20) is true.

D. Note by (11) that equality in (20) holds for the Koebe function K with $a_2(f) = 2$, $a_3(f) = 3$ and $a_4(f) = 4$.

This ends the proof of the theorem. □

4. The class $\mathscr{C}$ of convex functions

THEOREM 4.1. *If $f \in \mathscr{C}$, then*

$$\left|H_{2,1}(F_{f^{-1}})\right| \leqslant \frac{1}{33}. \tag{25}$$

The inequality is sharp.

Proof. Let $f \in \mathscr{C}$ be of the form (1). Then by (3) there exists $p \in \mathscr{P}$ of the form (12) such that

$$1+\frac{zf''(z)}{f'(z)} = p(z), \quad z \in \mathbb{D}. \tag{26}$$

Putting the series (1) and (12) into (26), by equating the coefficients we get ([6, Vol I., p. 116–117])

$$a_2(f) = \frac{1}{2}c_1, \quad a_3(f) = \frac{1}{6}(c_2+c_1^2), \quad a_3(f) = \frac{1}{24}(2c_3+3c_1c_2+c_1^3). \tag{27}$$

Hence and from (9) we obtain

$$\Gamma_1 = -\frac{1}{4}c_1, \quad \Gamma_2 = -\frac{1}{48}(4c_2-5c_1^2), \quad \Gamma_3 = -\frac{1}{48}(2c_3-5c_1c_2+3c_1^3),$$

and therefore

$$\Gamma_1\Gamma_3-(\Gamma_2)^2 = \frac{1}{2304}\left(11c_1^4-20c_1^2c_2+24c_1c_3-16c_2^2\right). \tag{28}$$

Since both the class $\mathscr{C}$ and $|H_{2,1}(F_{f^{-1}})|$ are rotationally invariant, without loss of generality we may assume that $a_2 \geqslant 0$, which in view of (27) yields $c_1 \in [0,2]$, i.e., by (13) that $\zeta_1 \in [0,1]$. By (11) and Lemma 2.1 we obtain

$$\begin{aligned} H_{2,1}(F_{f^{-1}}) =& \Gamma_1\Gamma_3-(\Gamma_2)^2 \\ =& \frac{1}{144}\left(3\zeta_1^4-6(1-\zeta_1^2)\zeta_1^2\zeta_2-2(\zeta_1^2+2)(1-\zeta_1^2)\zeta_2^2\right. \\ & \left.+6\zeta_1(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_3\right) \end{aligned} \tag{29}$$

for some $\zeta_1 \in [0,1]$ and $\zeta_2, \zeta_3 \in \overline{\mathbb{D}}$.

A. Suppose that $\zeta_1 = 0$. Then from (29),

$$\left|H_{2,1}(F_{f^{-1}})\right| = \frac{1}{36}|\zeta_2|^2 \leqslant \frac{1}{36}.$$

B. Suppose that $\zeta_1 = 1$. Then from (29),

$$\left|H_{2,1}(F_{f^{-1}})\right| = \frac{1}{48}.$$

C. Suppose that $\zeta_1 \in (0,1)$. Since $\zeta_3 \in \overline{\mathbb{D}}$, from (29) we obtain

$$\left|H_{2,1}(F_{f^{-1}})\right| \leqslant \frac{1}{24}\zeta_1(1-\zeta_1^2)\Phi(A,B,C),$$

where

$$\Phi(A,B,C) := \left|A + B\zeta_2 + C\zeta_2^2\right| + 1 - |\zeta_2|^2,$$

with

$$A := \frac{\zeta_1^3}{2(1-\zeta_1^2)}, \quad B := -\zeta_1, \quad C := -\frac{\zeta_1^2+2}{3\zeta_1}.$$

Observe that $AC < 0$ and therefore we apply only the part II of Lemma 2.2.

C1. Let's consider the condition $|B| < 2(1-|C|)$, i.e.,

$$\zeta_1 < 2\left(1 - \frac{\zeta_1^2+2}{3\zeta_1}\right).$$

The above inequality is equivalent to

$$\frac{5\zeta_1^2 - 6\zeta_1 + 4}{3\zeta_1} \leqslant 0$$

which is false for all $0 < \zeta_1 < 1$.

C2. Since

$$-4AC\left(\frac{1}{C^2} - 1\right) = -\frac{2\zeta_1^2(4-\zeta_1^2)}{3(\zeta_1^2+2)}$$

and

$$4(1+|C|)^2 = \frac{4(\zeta_1+2)^2(\zeta_1+1)^2}{9\zeta_1^2},$$

it follows that the condition $B^2 < \min\{4(1+|C|)^2, -4AC(C^{-2}-1)\}$ is equivalent to

$$\frac{\zeta_1^2(\zeta_1^2+14)}{3(\zeta_1^2+2)} < 0$$

which is false for all $0 < \zeta_1 < 1$.

C3. The inequality $|C|(|B|+4|A|) \leqslant |AB|$ is equivalent to

$$\frac{\zeta_1^4 - 6\zeta_1^2 - 4}{6(1-\zeta_1^2)} \geqslant 0$$

which is false for all $0 < \zeta_1 < 1$.

C4. Observe that the condition $|C|(|B|-4|A|) \geqslant |AB|$ equivalently written as

$$\frac{9\zeta_1^4 + 10\zeta_1^2 - 4}{6(1-\zeta_1^2)} \leqslant 0$$

is true for $\zeta_1 \in (0, \zeta_1^0]$, where $\zeta_1^0 := \sqrt{-5+\sqrt{61}}/3 \approx 0.558793$. Applying Lemma 2.2 for $0 < \zeta_1 < \zeta_1^0$ we get

$$\left|H_{2,1}(F_{f^{-1}})\right| \leqslant \frac{1}{24}\zeta_1(1-\zeta_1^2)(-|A|+|B|+|C|) = \rho(\zeta_1),$$

where

$$\rho(t) := \frac{1}{144}(-11t^4+4t^2+4), \quad t \in [0,1].$$

Note that the equation $\rho'(t) = 0$ has in $(0, \zeta_1^0)$ a unique solution $t_0 := \sqrt{22}/11 \approx 0.4264$, where the function ρ attains its maximum value

$$\rho(t_0) = \frac{1}{33}.$$

Therefore for $\zeta_1 \in (0, \zeta_1^0]$,

$$\rho(\zeta_1) \leqslant \rho(t_0) = \frac{1}{33}.$$

C5. Applying Lemma 2.2 for $\zeta_1^0 < \zeta_1 < 1$ we get

$$\left|H_{2,1}(F_{f^{-1}})\right| \leqslant \frac{1}{24}\zeta_1(1-\zeta_1^2)(|A|+|C|)\sqrt{1-\frac{B^2}{4AC}} = \psi(\zeta_1),$$

where

$$\psi(t) := \frac{1}{288}\sqrt{\frac{14-2t^2}{t^2+2}}(t^4-2t^2+4), \quad t \in [0,1].$$

Since the equation

$$\psi'(t) = -\frac{1}{144}\frac{(4t^6-15t^4-54t^2+92)t}{\sqrt{\frac{14-2t^2}{t^2+2}}(t^2+2)^2} = 0$$

has no root in $(0,1)$, we deduce that ψ is decreasing in $(\zeta_1^0, 1)$ and therefore

$$\psi(\zeta_1) \leqslant \psi(\zeta_1^0) = \rho(\zeta_1^0) \leqslant \rho(t_0) = \frac{1}{33}.$$

Summarizing, from Parts A-C it follows that the inequality (25) is true.

D. Note that equality in (25) holds for the function f defined by (26) with

$$p(z) = \frac{1+2\tau z+z^2}{1-z^2}, \quad z \in \mathbb{D},$$

where $\tau := \sqrt{22}/11$, for which $a_2(f) = \tau$, $a_3 = (1+2\tau^2)/3$ and $a_4 = (2\tau+\tau^3)/3$.

This ends the proof of the theorem. $\square$

5. The class $\mathscr{P}'$ of functions of bounded turning

THEOREM 5.1. *If $f \in \mathscr{P}'$, then*

$$\left|H_{2,1}(F_{f^{-1}})\right| \leqslant \frac{17}{144}. \tag{30}$$

The inequality is sharp.

Proof. Let $f \in \mathscr{P}'$ be of the form (1). Then by (4) there exists $p \in \mathscr{P}$ of the form (12) such that

$$f'(z) = p(z), \quad z \in \mathbb{D}. \tag{31}$$

Putting the series (1) and (12) into (31), by equating the coefficients we get

$$a_2(f) = \frac{1}{2}c_1, \quad a_3(f) = \frac{1}{3}c_2, \quad a_3(f) = \frac{1}{4}c_3. \tag{32}$$

Hence and from (9) we obtain

$$\Gamma_1 = -\frac{1}{4}c_1, \quad \Gamma_2 = -\frac{1}{48}(8c_2 - 9c_1^2), \quad \Gamma_3 = -\frac{1}{24}(3c_3 - 8c_1c_2 + 5c_1^3), \tag{33}$$

and therefore

$$\Gamma_1\Gamma_3 - (\Gamma_2)^2 = \frac{1}{2304}\left(39c_1^4 - 48c_1^2c_2 + 72c_1c_3 - 64c_2^2\right).$$

Since both the class $\mathscr{P}'$ and $|H_{2,1}(F_{f^{-1}})|$ are rotationally invariant, without loss of generality we may assume that $a_2 \geqslant 0$, which in view of (32) yields $c_1 \in [0,2]$, i.e., by (13) that $\zeta_1 \in [0,1]$. Thus by (11) and Lemma 2.1 we obtain

$$\begin{aligned} H_{2,1}(F_{f^{-1}}) =& \Gamma_1\Gamma_3 - (\Gamma_2)^2 \\ =& \frac{1}{144}\big(17\zeta_1^4 - 20(1-\zeta_1^2)\zeta_1^2\zeta_2 - 2(1-\zeta_1^2)(\zeta_1^2+8)\zeta_2^2 \\ &+ 18\zeta_1(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_3\big) \end{aligned} \tag{34}$$

for some $\zeta_1 \in [0,1]$ and $\zeta_2, \zeta_3 \in \overline{\mathbb{D}}$.

A. Suppose that $\zeta_1 = 0$. Then from (34),

$$\left|H_{2,1}(F_{f^{-1}})\right| = \frac{1}{9}|\zeta_2|^2 \leqslant \frac{1}{9}.$$

B. Suppose that $\zeta_1 = 1$. Then from (34),

$$\left|H_{2,1}(F_{f^{-1}})\right| = \frac{17}{144}.$$

C. Suppose that $\zeta_1 \in (0,1)$. Since $\zeta_3 \in \overline{\mathbb{D}}$, from (34) we get

$$\left|H_{2,1}(F_{f^{-1}})\right| \leqslant \frac{1}{8}\zeta_1(1-\zeta_1^2)\Phi(A,B,C),$$

where
$$\Phi(A,B,C) := \left|A + B\zeta_2 + C\zeta_2^2\right| + 1 - |\zeta_2|^2,$$
with
$$A := \frac{17\zeta_1^3}{18(1-\zeta_1^2)}, \quad B := \frac{-10\zeta_1}{9}, \quad C := \frac{-(\zeta_1^2+8)}{9\zeta_1}.$$
Observe that $AC < 0$ and therefore we apply only the part II of Lemma 2.2.

C1. Let's consider the condition $|B| < 2(1-|C|)$, i.e.,
$$\frac{10\zeta_1}{9} < 2\left(1 - \frac{\zeta_1^2+8}{9\zeta_1}\right).$$
The above inequality is equivalent to
$$\frac{12\zeta_1^2 - 18\zeta_1 + 16}{9\zeta_1} < 0$$
which is false for all $0 < \zeta_1 < 1$.

C2. Since
$$-4AC\left(\frac{1}{C^2} - 1\right) = \frac{-34\zeta_1^2(64-\zeta_1^2)}{81(\zeta_1^2+8)}$$
and
$$4(1+|C|)^2 = \frac{(\zeta_1+8)^2(\zeta_1+1)^2}{\zeta_1^2},$$
it follows that the condition $B^2 < \min\{4(1+|C|)^2, -4AC(C^{-2}-1)\}$ is equivalent to
$$\frac{2\zeta_1^2(11\zeta_1^2+496)}{27(\zeta_1^2+8)} < 0.$$
The last inequality is false for all $0 < \zeta_1 < 1$.

C3. The inequality $|C|(|B|+4|A|) \leqslant |AB|$ is equivalent to
$$\frac{61\zeta_1^4 - 202\zeta_1^2 - 80}{81(1-\zeta_1^2)} \geqslant 0$$
which is false for all $0 < \zeta_1 < 1$.

C4. Observe that the condition $|C|(|B|-4|A|) \geqslant |AB|$ equivalently written as
$$\frac{129\zeta_1^4 + 342\zeta_1^2 - 80}{81(1-\zeta_1^2)} \leqslant 0$$
is true for $\zeta_1 \in \left(0, \zeta_1^0\right]$, where $\zeta_1^0 := \sqrt{-22059 + 129\sqrt{39561}}/129 \approx 0.46505$. Applying Lemma 2.2 for $0 < \zeta_1 \leqslant \zeta_1^0$ we get
$$\left|H_{2,1}(F_{f^{-1}})\right| \leqslant \frac{1}{8}\zeta_1(1-\zeta_1^2)\left(-|A| + |B| + |C|\right) = \rho(\zeta_1),$$

where

$$\rho(t) := \frac{1}{144}(-39t^4 + 6t^2 + 16), \quad t \in [0,1].$$

Note that the equation $\rho'(t) = 0$ has in $(0, \zeta_1^0)$ a unique solution $t_0 := \sqrt{13}/13$, where the function ρ attains its maximum value

$$\rho(t_0) = \frac{211}{1872}.$$

Therefore for $\zeta_1 \in \left(0, \zeta_1^0\right]$,

$$\rho(\zeta_1) \leqslant \rho(t_0) = \frac{211}{1872}.$$

C5. Applying Lemma 2.2 for $\zeta_1^0 < \zeta_1 < 1$ we get

$$\left|H_{2,1}(F_{f^{-1}})\right| \leqslant \frac{1}{8}\zeta_1(1-\zeta_1^2)(|A|+|C|)\sqrt{1-\frac{B^2}{4AC}} = \psi(\zeta_1),$$

where

$$\psi(t) := \frac{15t^4 - 14t^2 + 16}{2448}\sqrt{\frac{3162 - 561t^2}{t^2+8}}, \quad t \in [0,1].$$

Since the equation

$$\psi'(t) = -\frac{1}{24}\frac{(330t^6 + 1751t^4 - 16294t^2 + 8144)t}{(t^2+8)^2\sqrt{\dfrac{3162-561t^2}{t^2+8}}} = 0$$

has in $(0,1)$ a unique root $t_1 \approx 0.73039 > \zeta_1^0$, we deduce that ψ is decreasing in $[\zeta_1^0, t_1]$ and is increasing in $[t_1, 1]$. Therefore for $\zeta_1 \in (\zeta_1^0, 1)$ we have

$$\psi(\zeta_1) \leqslant \max\{\psi(\zeta_0^1), \psi(1)\} = \max\{\rho(\zeta_0^1), \psi(1)\} \leqslant \max\{\rho(t_0), \psi(1)\} = \psi(1) = \frac{17}{144}.$$

Summarizing, from Parts A-C it follows that the inequality (30) is true.

D. Note by (11) that equality in (30) holds for the function f defined by (31) with

$$p(z) = \frac{1+z}{1-z}, \quad z \in \mathbb{D},$$

for which $a_2(f) = 1$, $a_3(f) = 2/3$ and $a_4(f) = 1/2$.

This ends the proof of the theorem. □

6. The class $\mathscr{T}$

THEOREM 6.1. *If $f \in \mathscr{T}$, then*

$$\left|H_{2,1}(F_{f^{-1}})\right| \leqslant \frac{7}{3}. \tag{35}$$

The inequality is sharp.

Proof. Let $f \in \mathcal{T}$ be of the form (1). Then by (5) there exists $p \in \mathcal{P}$ of the form (12) such that

$$f(z) = zp(z), \quad z \in \mathbb{D}. \tag{36}$$

Putting the series (1) and (12) into (36), by equating the coefficients we get

$$a_2(f) = c_1, \quad a_3(f) = c_2, \quad a_4(f) = c_3. \tag{37}$$

Hence and from (9) we obtain

$$\Gamma_1 = -\frac{1}{2}c_1, \quad \Gamma_2 = -\frac{1}{4}(2c_2 + 3c_1^2), \quad \Gamma_3 = -\frac{1}{6}(3c_3 - 12c_1c_2 + 10c_1^3),$$

and therefore

$$\Gamma_1\Gamma_3 - (\Gamma_2)^2 = \frac{1}{48}\left(13c_1^4 - 12c_1^2c_2 + 12c_1c_3 - 12c_2^2\right). \tag{38}$$

Since both the class $\mathcal{T}$ and $|H_{2,1}(F_{f^{-1}})|$ are rotationally invariant, without loss of generality we may assume that $a_2 \geqslant 0$, which in view of (37) yields $c_1 \in [0,2]$, i.e., by (13) that $\zeta_1 \in [0,1]$. By (11) and Lemma 2.1 we obtain

$$\begin{aligned} H_{2,1}(F_{f^{-1}}) =& \Gamma_1\Gamma_3 - (\Gamma_2)^2 \\ =& \frac{1}{3}\big(7\zeta_1^4 - 6(1-\zeta_1^2)\zeta_1^2\zeta_2 - 3(1-\zeta_1^2)\zeta_2^2 \\ & + 3\zeta_1(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_3\big) \end{aligned} \tag{39}$$

for some $\zeta_1 \in [0,1]$ and $\zeta_2, \zeta_3 \in \overline{\mathbb{D}}$.

A. Suppose that $\zeta_1 = 0$. Then from (39),

$$\left|H_{2,1}(F_{f^{-1}})\right| = |\zeta_2|^2 \leqslant 1.$$

B. Suppose that $\zeta_1 = 1$. Then from(39),

$$\left|H_{2,1}(F_{f^{-1}})\right| = \frac{7}{3}.$$

C. Suppose that $\zeta_1 \in (0,1)$. Since $\zeta_3 \in \overline{\mathbb{D}}$, from (39) we obtain

$$\left|H_{2,1}(F_{f^{-1}})\right| \leqslant \zeta_1(1-\zeta_1^2)\Phi(A,B,C),$$

where

$$\Phi(A,B,C) := \left|A + B\zeta_2 + C\zeta_2^2\right| + 1 - |\zeta_2|^2,$$

with

$$A := \frac{7\zeta_1^3}{3(1-\zeta_1^2)}, \quad B := -2\zeta_1, \quad C := -\frac{1}{\zeta_1}.$$

Observe that $AC < 0$ and therefore we apply only the part II of Lemma 2.2.

C1. Let's consider the condition $|B| < 2(1-|C|)$, i.e.,

$$2\zeta_1 < 2\left(1-\frac{1}{\zeta_1}\right).$$

The above inequality is equivalent to

$$\frac{2\zeta_1^2-2\zeta_1+2}{\zeta_1} \leqslant 0$$

which is false for all $0<\zeta_1<1$.

C2. Since

$$-4AC\left(\frac{1}{C^2}-1\right) = -\frac{28}{3}\zeta_1^2$$

and

$$4(1+|C|)^2 = \frac{4(\zeta_1+1)^2}{\zeta_1^2},$$

it follows that the condition $B^2 < \min\{4(1+|C|)^2, -4AC(C^{-2}-1)\}$ is equivalent to

$$\frac{40}{3}\zeta_1^2 < 0$$

which is false for all $0<\zeta_1<1$.

C3. The inequality $|C|(|B|+4|A|) \leqslant |AB|$ is equivalent to

$$-7\zeta_1^4+11\zeta_1^2+3 \leqslant 0$$

which is false for all $0<\zeta_1<1$.

C4. Observe that the condition $|C|(|B|-4|A|) \geqslant |AB|$ equivalently written as

$$-7\zeta_1^4-17\zeta_1^2+3 \geqslant 0$$

is true for $\zeta_1 \in (0,\zeta_1^0]$, where $\zeta_1^0 := \sqrt{-238+14\sqrt{373}}/14 \approx 0.4064838709$. Applying Lemma 2.2 for $0<\zeta_1\leqslant\zeta_1^0$ we get

$$\left|H_{2,1}(F_{f^{-1}})\right| \leqslant \zeta_1(1-\zeta_1^2)(-|A|+|B|+|C|) = \rho(\zeta_1),$$

where

$$\rho(t) := \frac{1}{3}(-13t^4+3t^2+3), \quad t\in[0,1].$$

Note that the equation $\rho'(t)=0$ has in $(0,\zeta_1^0)$ a unique solution $t_0 := \sqrt{78}/26 \approx 0.3396831102$, where the function ρ attains its maximum value

$$\rho(t_0) = \frac{55}{52}.$$

Therefore

$$\rho(\zeta_1) \leqslant \rho(t_0) = \frac{55}{52}.$$

C5. Applying Lemma 2.2 for $\zeta_1^0 < \zeta_1 < 1$ we get

$$\left|H_{2,1}(F_{f^{-1}})\right| \leqslant \zeta_1(1-\zeta_1^2)\,(|A|+|C|)\sqrt{1-\frac{B^2}{4AC}} = \psi(\zeta_1),$$

where

$$\psi(t) := \frac{1}{21}\sqrt{70-21t^2}(7t^4-3t^2+3), \quad t\in[0,1].$$

Since the equation

$$\psi'(t) = -\frac{1}{3}\frac{(105t^4-307t^2+69)t}{\sqrt{70-21t^2}} = 0$$

has in $(0,1)$ a unique root $t_1 := \sqrt{64470-210\sqrt{65269}}/210 \approx 0.4953210438 > \zeta_1^0$, we deduce that ψ is decreasing in $[\zeta_1^0, t_1]$ and is increasing in $[t_1, 1]$. Therefore

$$\psi(\zeta_1) \leqslant \max\{\psi(\zeta_0^1), \psi(1)\} = \max\{\rho(\zeta_0^1), \psi(1)\} \leqslant \max\{\rho(t_0), \psi(1)\} = \psi(1) = \frac{7}{3}.$$

Summarizing, from Parts A-C it follows that the inequality (35) is true.

D. Note that equality in (35) holds for the function $f \in \mathscr{A}$ of the form

$$f(z) = \frac{z+z^2}{1-z}, \quad z \in \mathbb{D},$$

for which $a_2(f) = a_3(f) = a_4(f) = 2$.

This ends the proof of the theorem. □

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