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INEQUALITIES FOR TWO POWER SERIES OF NONCOMMUTATIVE OPERATORS IN HILBERT SPACES WITH APPLICATIONS TO NUMERICAL RADIUS

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Abstract. Let H be a complex Hilbert space. We consider the power series with complex coefficients $f(z) := \sum_{k=0}^{\infty} a_k z^k$ with $a_k \in \mathbb{C}$ for $k \in \mathbb{N} := \{0,1,\dots\}$. Suppose that this power series is convergent on the open disk $D(0,R) := \{z \in \mathbb{C} | z < R\}$. We define $f_a(z) := \sum_{k=0}^{\infty} |a_k| z^k$ which has the same radius of convergence R. Assume that the power series $f(z) = \sum_{i=0}^{\infty} a_i z^i$ is convergent on the open disk $D(0,R_1)$ and $g(z) = \sum_{i=0}^{\infty} b_i z^i$ is convergent on $D(0,R_2)$ and A, B, C, D be operators in $B(H)$ with $\|A\|^{1/2}, \|A\| < R_1$ and $\|B\|^{1/2}$, $\|B\| < R_2$. In this paper we show among others that

$$|\langle D^* A f(A) g(B) BCx, y\rangle| \leqslant \|A\|^{\alpha} \|B\|^{1-\alpha} f_a\left(\|A\|^{1/2}\right) g_a\left(\|B\|^{1/2}\right) \times \left\langle \left||B|^{\alpha} C\right|^2 x,x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} D\right|^2 y,y\right\rangle^{1/2}$$

for all $\alpha \in [0,1]$ and x, $y \in H$. Application for norm and numerical radius inequalities for the composite operator $D^* A f(A) g(B) BC$ are provided. Some examples for fundamental power series are also given.

1. Introduction

The study of numerical range and the inequalities for numerical radius is useful in investigating many properties of linear operators and has various applications in numerous fields of sciences such as, see [1], application in quantum information theory, in particular, quantum error correction, additive uncertainty relations, multi-observable quantum uncertainty relations etc... By making use of the numerical radius inequalities one can also estimate the roots of polynomials using the notion of the Frobenius companion matrix, see [4] and [3].

The main aim of the work is to develop vector and numerical radius inequalities for a product of functions defined by power series of two noncommutative bounded linear operators on a complex Hilbert space H.

Let start by recalling some fundaments notions and present some facts that will be used in the sequel.

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The *numerical radius* $w(X)$ of an operator X on H is given by

$$\omega(X) = \sup\{|\langle Xx, x\rangle|, \ \|x\| = 1\}. \tag{1.1}$$

Obviously, by (1.1), for any $x \in H$ one has

$$|\langle Xx, x\rangle| \leqslant w(X)\|x\|^2. \tag{1.2}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $\mathscr{B}(H)$ of all bounded linear operators $X : H \to H$, i.e.,

(i) $\omega(X) \geqslant 0$ for any $X \in \mathscr{B}(H)$ and $\omega(X) = 0$ if and only if $X = 0$;

(ii) $\omega(\lambda X) = |\lambda|\,\omega(X)$ for any $\lambda \in \mathbb{C}$ and $X \in \mathscr{B}(H)$;

(iii) $\omega(X+Y) \leqslant \omega(X) + \omega(Y)$ for any $X, Y \in \mathscr{B}(H)$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$\omega(X) \leqslant \|X\| \leqslant 2\omega(X) \tag{1.3}$$

for any $X \in \mathscr{B}(H)$.

In the recent paper [1], P. Bhunia obtained the following interesting inequalities for the numerical radius of a product of two operators in Hilbert spaces.

Let $B,\ C \in \mathscr{B}(H)$, then

$$\omega^2(BC) \leqslant \frac{1}{2}\left\| |B^*|^4 + |C|^4 \right\|,$$

$$\omega^2(BC) \leqslant \frac{1}{2}\left(\|B\|^2\|C\|^2 + \omega\left(|B^*|^2|C|^2\right)\right),$$

$$\omega^2(BC) \leqslant \frac{1}{2}\left(\frac{1}{2}\left\| |B^*|^4 + |C|^4 \right\| + \omega\left(|B^*|^2|C|^2\right)\right)$$

and

$$\omega^2(BC) \leqslant \left\| \alpha|B^*|^2 + (1-\alpha)|C|^2 \right\| \|B\|^{2(1-\alpha)}\|C\|^{2\alpha}$$

for all $\alpha \in [0,1]$.

In particular, for $\alpha = 1/2$, we get the symmetrical upper bound

$$\omega^2(BC) \leqslant \frac{1}{2}\left\| |B^*|^2 + |C|^2 \right\| \|B\|\|C\|.$$

For more related results for the numerical radius of a product, see for instance the recent books [3], [7] and research papers [2] and [13].

In 1988, F. Kittaneh obtained the following generalization of Schwarz inequality [10]:

THEOREM 1. *Assume that f and g are non-negative functions on $[0,\infty)$ which are continuous and satisfying the relation $f(t)g(t)=t$ for all $t\in[0,\infty)$. For any $X\in\mathscr{B}(H)$*

$$|\langle Xx,y\rangle|\leqslant\|f(|X|)x\|\,\|g(|X^*|)y\| \tag{1.4}$$

for all $x,\ y\in H$.

If we take $f(t)=t^{\lambda},\ g(t)=t^{1-\lambda}$ with $\lambda\in[0,1]$, then we obtain the celebrated *Kato's inequality* [9]

$$|\langle Xx,y\rangle|\leqslant\left\||X|^{\lambda}x\right\|\left\||X^*|^{1-\lambda}y\right\| \tag{1.5}$$

for all $x,\ y\in H$.

In [10] F. Kittaneh also obtained the following result for multiplication of n-operators:

LEMMA 1. *Let $T_1,\ldots,T_n$ be operators in $\mathscr{B}(H)$ with $n\geqslant 2$, then for $\alpha\in[0,1]$*

$$\begin{aligned}|\langle(T_1T_2\ldots T_{n-1}T_n)x,y\rangle|^2\leqslant{}&\|T_1\|^{2\alpha}\|T_2\|\ldots\|T_{n-1}\|\,\|T_n\|^{2(1-\alpha)}\\&\times\left\langle|T_n|^{2\alpha}x,x\right\rangle\left\langle|T_1^*|^{2(1-\alpha)}y,y\right\rangle\end{aligned} \tag{1.6}$$

for all $x,\ y\in H$.

Let $A,\ B,\ C,\ D$ be operators in $\mathscr{B}(H)$. For $T_1=D^*,\ T_2=A,\ T_3=B$ and $T_4=C$ in (1.6) we get

$$|\langle D^*ABCx,y\rangle|^2\leqslant\|D\|^{2\alpha}\|A\|\,\|B\|\,\|C\|^{2(1-\alpha)}\left\langle|C|^{2\alpha}x,x\right\rangle\left\langle|D|^{2(1-\alpha)}y,y\right\rangle \tag{1.7}$$

for all $x,\ y\in H$.

In particular, for $AB=I$ in (1.7) we obtain the Schwarz's type inequality

$$|\langle D^*Cx,y\rangle|^2\leqslant\|D\|^{2\alpha}\|A\|\,\|B\|\,\|C\|^{2(1-\alpha)}\left\langle|C|^{2\alpha}x,x\right\rangle\left\langle|D|^{2(1-\alpha)}y,y\right\rangle \tag{1.8}$$

for all $x,\ y\in H$ and $\alpha\in[0,1]$.

We consider the power series with complex coefficients $f(z):=\sum_{k=0}^{\infty}a_kz^k$ with $a_k\in\mathbb{C}$ for $k\in\mathbb{N}:=\{0,1,\ldots\}$. Suppose that this power series is convergent on the open disk $D(0,R):=\{z\in\mathbb{C}|\ z<R\}$. If $R=\infty$, then $D(0,R)=\mathbb{C}$. We define $f_a(z):=\sum_{k=0}^{\infty}|a_k|z^k$ which has the same radius of convergence R.

Assume that the power series $f(z)=\sum_{i=0}^{\infty}a_iz^i$ is convergent on the open disk $D(0,R_1)$ and $g(z)=\sum_{i=0}^{\infty}b_iz^i$ is convergent on $D(0,R_2)$ and $A,\ B,\ C,\ D$ are operators in $\mathscr{B}(H)$ with $\|A\|^{1/2},\|A\|<R_1$ and $\|B\|^{1/2},\ \|B\|<R_2$. We show among others that the following extension of Kittaneh result (1.6) in terms of the power series f and g can be stated

$$\begin{aligned}|\langle D^*Af(A)g(B)BCx,y\rangle|\leqslant{}&\|A\|^{\alpha}\|B\|^{1-\alpha}f_a\left(\|A\|^{1/2}\right)g_a\left(\|B\|^{1/2}\right)\\&\times\left\langle\left||B|^{\alpha}C\right|^2x,x\right\rangle^{1/2}\left\langle\left||A^*|^{1-\alpha}D\right|^2y,y\right\rangle^{1/2}\end{aligned}$$

for all $\alpha \in [0,1]$ and $x,\ y \in H$.

Motivated by Bunia's inequalities for the numerical radius of a product of two noncommutative operators mentioned above, we provide norm and numerical radius inequalities for the operators product $D^* A f(A) g(B) BC$ under certain natural assumptions for the operators involved. Some particular cases for fundamental power series in Complex Analysis like $f(z) = (1 \pm z)^{-1},\ |z| < 1$ and $f(z) = \exp(z)$, with $z \in \mathbb{C}$, are also given.

2. Some vector inequalities

The following result for powers of two noncommutative operators holds:

LEMMA 2. *Let $A,\ B$ be operators in $\mathscr{B}(H)$ and $i,\ j \geqslant 1$, then for $\alpha \in [0,1]$*

$$\left|\left\langle A^i B^j x, y\right\rangle\right|^2 \leqslant \|A\|^{2\alpha+i-1} \|B\|^{2(1-\alpha)+j-1} \left\langle |B|^{2\alpha} x, x\right\rangle \left\langle |A^*|^{2(1-\alpha)} y, y\right\rangle \tag{2.1}$$

for all $x, y \in H$.

Proof. We have by (1.6) for $n = i + j \geqslant 2$, and

$$T_1 = A, \ldots, T_i = A, \quad T_{i+1} = B, \ldots, T_{i+j} = B$$

that

$$\begin{aligned}\left|\left\langle A^i B^j x, y\right\rangle\right|^2 &\leqslant \|A\|^{2\alpha} \ldots \|A\| \|B\| \ldots \|B\|^{2(1-\alpha)} \left\langle |B|^{2\alpha} x, x\right\rangle \left\langle |A^*|^{2(1-\alpha)} y, y\right\rangle \\ &= \|A\|^{2\alpha+i-1} \|B\|^{2(1-\alpha)+j-1} \left\langle |B|^{2\alpha} x, x\right\rangle \left\langle |A^*|^{2(1-\alpha)} y, y\right\rangle\end{aligned}$$

for all $x, y \in H$, which proves (2.1). □

LEMMA 3. *Let $A,\ B,\ C,\ D$ be operators in $\mathscr{B}(H)$ and $i,\ j \geqslant 1$, then for $\alpha \in [0,1]$*

$$\begin{aligned}&\left|\left\langle D^* A^i B^j C x, y\right\rangle\right|^2 \\ &\leqslant \|A\|^{2\alpha+i-1} \|B\|^{2(1-\alpha)+j-1} \left\langle \left||B|^{\alpha} C\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} D\right|^2 y, y\right\rangle\end{aligned} \tag{2.2}$$

for all $x, y \in H$.

Proof. If we take Cx instead of x and Dy instead of y in Lemma 2, then we get

$$\begin{aligned}&\left|\left\langle D^* A^i B^j C x, y\right\rangle\right|^2 \\ &\leqslant \|A\|^{2\alpha+i-1} \|B\|^{2(1-\alpha)+j-1} \left\langle C^* |B|^{2\alpha} C x, x\right\rangle \left\langle D^* |A^*|^{2(1-\alpha)} D y, y\right\rangle\end{aligned} \tag{2.3}$$

for all $x, y \in H$.

Since

$$C^{*}\left|B\right|^{2\alpha}C=\left|\left|B\right|^{\alpha}C\right|^{2}\quad\text{and}\quad D^{*}\left|A^{*}\right|^{2(1-\alpha)}D=\left|\left|A^{*}\right|^{1-\alpha}D\right|^{2},$$

hence by (2.3) we obtain (2.2). □

As some natural examples for power series that are useful for applications, we can point out that, if

$$f(\lambda)=\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n}\lambda^{n}=\ln\frac{1}{1+\lambda},\ \lambda\in D(0,1); \tag{2.4}$$

$$g(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^{n}}{(2n)!}\lambda^{2n}=\cos\lambda,\ \lambda\in\mathbb{C};$$

$$h(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^{n}}{(2n+1)!}\lambda^{2n+1}=\sin\lambda,\ \lambda\in\mathbb{C};$$

$$l(\lambda)=\sum_{n=0}^{\infty}(-1)^{n}\lambda^{n}=\frac{1}{1+\lambda},\ \lambda\in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_{a}(\lambda)=\sum_{n=1}^{\infty}\frac{1}{n}\lambda^{n}=\ln\frac{1}{1-\lambda},\ \lambda\in D(0,1); \tag{2.5}$$

$$g_{a}(\lambda)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}\lambda^{2n}=\cosh\lambda,\ \lambda\in\mathbb{C};$$

$$h_{a}(\lambda)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\lambda^{2n+1}=\sinh\lambda,\ \lambda\in\mathbb{C};$$

$$l_{a}(\lambda)=\sum_{n=0}^{\infty}\lambda^{n}=\frac{1}{1-\lambda},\ \lambda\in D(0,1).$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\exp(\lambda)=\sum_{n=0}^{\infty}\frac{1}{n!}\lambda^{n},\quad \lambda\in\mathbb{C}; \tag{2.6}$$

$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1},\quad \lambda\in D(0,1);$$

$$\sin^{-1}(\lambda)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}\lambda^{2n+1},\quad \lambda\in D(0,1);$$

$$\tanh^{-1}(\lambda)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1},\quad \lambda\in D(0,1);$$

$${}_{2}F_{1}(\alpha,\beta,\gamma,\lambda)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^{n},\alpha,\beta,\gamma>0,\quad \lambda\in D(0,1);$$

where Γ is *Gamma function*.

The following result is an extension of Kittaneh result (1.6) in terms of the power series f and g:

THEOREM 2. *Consider the power series $f(z) = \sum_{i=0}^{\infty} a_i z^i$ convergent on the open disk $D(0,R_1)$ and $g(z) = \sum_{i=0}^{\infty} b_i z^i$ convergent on $D(0,R_2)$ and $A,\ B,\ C,\ D$ be operators in $\mathscr{B}(H)$ with $\|A\|^{1/2}, \|A\| < R_1$ and $\|B\|^{1/2},\ \|B\| < R_2$, then*

$$|\langle D^* A f(A) g(B) BCx, y\rangle| \leqslant \|A\|^{\alpha} \|B\|^{1-\alpha} f_a\left(\|A\|^{1/2}\right) g_a\left(\|B\|^{1/2}\right) \times \left\langle \left||B|^{\alpha} C\right|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} D\right|^2 y, y\right\rangle^{1/2} \tag{2.7}$$

for all $\alpha \in [0,1]$ and $x,\ y \in H$.

Proof. If we take in (2.2) instead of $i,\ i+1$ and instead of $j,\ j+1$, then we get

$$\left|\left\langle D^* A A^i B^j BCx, y\right\rangle\right|^2 \leqslant \|A\|^{2\alpha+i} \|B\|^{2(1-\alpha)+j} \left\langle \left||B|^{\alpha} C\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} D\right|^2 y, y\right\rangle$$

for all $x,\ y \in H$.

By taking the square root, we get

$$\left|\left\langle D^* A A^i B^j BCx, y\right\rangle\right| \leqslant \|A\|^{\alpha} \|B\|^{1-\alpha} \left\langle \left||B|^{\alpha} C\right|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} D\right|^2 y, y\right\rangle^{1/2} \|A\|^{i/2} \|B\|^{j/1}$$

for all $i,\ j = 0,1,\ldots; \alpha \in [0,1]$ and all $x, y \in H$.

If we multiply by $|a_i|,\ \left|b_j\right|$ and sum, then we get

$$\begin{aligned}
&\left|\left\langle D^* A \left(\sum_{i=0}^{n} a_i A^i\right)\left(\sum_{j=0}^{m} b_j B^j\right) BCx, y\right\rangle\right| \\
&\leqslant \sum_{i=0}^{n}\sum_{j=0}^{m} |a_i| \left|b_j\right| \left|\left\langle D^* A A^i B^j BCx, y\right\rangle\right| \\
&\leqslant \|A\|^{\alpha} \|B\|^{2(1-\alpha)} \left\langle \left||B|^{\alpha} C\right|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} D\right|^2 y, y\right\rangle^{1/2} \\
&\quad \times \sum_{i=0}^{n}\sum_{j=0}^{m} |a_i| \left|b_j\right| \|A\|^{i/2} \|B\|^{j/2} \\
&= \|A\|^{\alpha} \|B\|^{1-\alpha} \left\langle \left||B|^{\alpha} C\right|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} D\right|^2 y, y\right\rangle^{1/2} \\
&\quad \times \sum_{i=0}^{n} |a_i| \|A\|^{i/2} \sum_{j=0}^{m} \left|b_j\right| \|B\|^{j/2}
\end{aligned} \tag{2.8}$$

for all $\alpha \in [0,1]$ and $x,\ y \in H$.

Since the following series are convergent and

$$\sum_{i=0}^{\infty} a_i A^i = f(A), \ \sum_{j=0}^{\infty} b_j B^j = g(B),$$

while

$$\sum_{i=0}^{\infty} |a_i| \|A\|^{i/2} = f_a\left(\|A\|^{1/2}\right) \text{ and } \sum_{j=0}^{m} |b_j| \|B\|^{j/2} = g_a\left(\|B\|^{1/2}\right),$$

then by taking $n,\ m \to \infty$ in (2.8), we deduce (2.7). □

We observe that for $f = g \equiv 1$ in (2.7) we get for $A,B,C,D \in \mathscr{B}(H)$ the following "*dual version*" of (1.7)

$$|\langle D^* ABCx, y\rangle| \leqslant \|A\|^{\alpha} \|B\|^{1-\alpha} \left\langle \left||B|^{\alpha} C\right|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} D\right|^2 y, y\right\rangle^{1/2} \tag{2.9}$$

for all $\alpha \in [0,1]$ and $x,\ y \in H$.

Also, if we take $f(z) = g(z) = z$ in (2.7), then we get for $A,B,C,D \in \mathscr{B}(H)$ that

$$\begin{aligned} &\left|\left\langle D^* A^2 B^2 Cx, y\right\rangle\right| \\ &\leqslant \|A\|^{\alpha+1/2} \|B\|^{3/2-\alpha} \left\langle \left||B|^{\alpha} C\right|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} D\right|^2 y, y\right\rangle^{1/2} \end{aligned} \tag{2.10}$$

for all $\alpha \in [0,1]$ and $x,\ y \in H$.

REMARK 1. With the assumptions in Theorem 2, if we take $D = C = I$ in (2.7), then we get

$$\begin{aligned} |\langle A f(A) g(B) Bx, y\rangle| &\leqslant \|A\|^{\alpha} \|B\|^{1-\alpha} f_a\left(\|A\|^{1/2}\right) g_a\left(\|B\|^{1/2}\right) \\ &\times \left\langle |B|^{2\alpha} x, x\right\rangle^{1/2} \left\langle |A^*|^{2(1-\alpha)} y, y\right\rangle^{1/2} \end{aligned} \tag{2.11}$$

for all $\alpha \in [0,1]$ and $x,\ y \in H$.

If A and B are invertible and we take $D = (A^*)^{-1}$ and $C = B^{-1}$, then we get

$$\begin{aligned} |\langle f(A) g(B) x, y\rangle| &\leqslant \|A\|^{\alpha} \|B\|^{1-\alpha} f_a\left(\|A\|^{1/2}\right) g_a\left(\|B\|^{1/2}\right) \\ &\times \left\langle \left||B|^{\alpha} B^{-1}\right|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} (A^*)^{-1}\right|^2 y, y\right\rangle^{1/2} \end{aligned} \tag{2.12}$$

for all $\alpha \in [0,1]$ and $x, y \in H$.

EXAMPLE 1. Let $A,\ B,\ C,\ D$ be operators in $\mathscr{B}(H)$. Assume that $\|A\|,\|B\|<1$ and take $f(z):=(1\pm z)^{-1}$ and $g(w):=(1\pm w)^{-1}$ with $|z|,\ |w|<1$ in Theorem 2, then

$$\begin{aligned} &\left|\left\langle D^*A(1\pm A)^{-1}(1\pm B)^{-1}BCx,y\right\rangle\right| \\ &\leqslant \|A\|^{\alpha}\left(1-\|A\|^{1/2}\right)^{-1}\|B\|^{1-\alpha}\left(1-\|B\|^{1/2}\right)^{-1} \\ &\quad\times\left\langle\left||B|^{\alpha}C\right|^2x,x\right\rangle^{1/2}\left\langle\left||A^*|^{1-\alpha}D\right|^2y,y\right\rangle^{1/2} \end{aligned} \tag{2.13}$$

for all $\alpha\in[0,1]$ and $x,\ y\in H$.

For $C=D=I$, we obtain

$$\begin{aligned} &\left|\left\langle A(1\pm A)^{-1}(1\pm B)^{-1}Bx,y\right\rangle\right| \\ &\leqslant \|A\|^{\alpha}\left(1-\|A\|^{1/2}\right)^{-1}\|B\|^{1-\alpha}\left(1-\|B\|^{1/2}\right)^{-1} \\ &\quad\times\left\langle|B|^{2\alpha}x,x\right\rangle^{1/2}\left\langle|A^*|^{2(1-\alpha)}y,y\right\rangle^{1/2} \end{aligned} \tag{2.14}$$

for all $\alpha\in[0,1]$ and $x,\ y\in H$.

Let $A,\ B,\ C,\ D$ be operators in $\mathscr{B}(H)$. If we take $f(z):=\exp(\gamma z)$ and $g(w):=\exp(\beta z)$ with $\alpha,\ \beta\in\mathbb{C}$ then by Theorem 2 we get

$$\begin{aligned} &|\langle D^*A\exp(\gamma A)\exp(\beta B)BCx,y\rangle| \\ &\leqslant \|A\|^{\alpha}\|B\|^{1-\alpha}\exp\left(|\gamma|\,\|A\|^{1/2}+|\beta|\,\|B\|^{1/2}\right) \\ &\quad\times\left\langle\left||B|^{\alpha}C\right|^2x,x\right\rangle^{1/2}\left\langle\left||A^*|^{1-\alpha}D\right|^2y,y\right\rangle^{1/2} \end{aligned} \tag{2.15}$$

for all $\alpha\in[0,1]$ and $x,\ y\in H$.

If A and B are commutative, then

$$\begin{aligned} &|\langle D^*AB\exp(\gamma A+\beta B)Cx,y\rangle| \\ &\leqslant \|A\|^{\alpha}\|B\|^{1-\alpha}\exp\left(|\gamma|\,\|A\|^{1/2}+|\beta|\,\|B\|^{1/2}\right) \\ &\quad\times\left\langle\left||B|^{\alpha}C\right|^2x,x\right\rangle^{1/2}\left\langle\left||A^*|^{1-\alpha}D\right|^2y,y\right\rangle^{1/2} \end{aligned} \tag{2.16}$$

for all $\alpha\in[0,1]$ and $x,\ y\in H$.

COROLLARY 1. *Consider the power series* $u(z)=\sum_{i=0}^{\infty}c_iz^i$ *convergent on the open disk* $D(0,R_1)$ *and* $v(z)=\sum_{i=0}^{\infty}d_iz^i$ *convergent on* $D(0,R_2)$ *and* $A,\ B,\ C,\ D$ *be* *operators in* $\mathscr{B}(H)$, $A,B \neq 0$ *with* $\|A\|^{1/2}, \|A\| < R_1$ *and* $\|B\|^{1/2}$, $\|B\| < R_2$, *then*

$$\begin{aligned} &\left|\left\langle D^*\left[u(A)-u(0)\right]\left[v(B)-v(0)\right]Cx,y\right\rangle\right| \\ &\leqslant \|A\|^{\alpha-1/2}\|B\|^{1/2-\alpha}\left[u_a\left(\|A\|^{1/2}\right)-u_a(0)\right]\left[v_a\left(\|B\|^{1/2}\right)-v_a(0)\right] \\ &\quad\times\left\langle\left||B|^{\alpha}C\right|^2x,x\right\rangle^{1/2}\left\langle\left||A^*|^{1-\alpha}D\right|^2y,y\right\rangle^{1/2} \end{aligned} \tag{2.17}$$

for all $\alpha \in [0,1]$ *and* x, $y \in H$.

Proof. We consider the function

$$f(z) := \frac{u(z)-u(0)}{z} = \sum_{i=1}^{\infty} c_i z^i$$

that is convergent on the open disk $D(0,R_1)$ and

$$g(z) := \frac{v(z)-v(0)}{z} = \sum_{i=1}^{\infty} d_i z^i,$$

which is convergent on $D(0,R_2)$.

Then

$$Af(A) = u(A)-u(0) \quad \text{and} \quad f_a\left(\|A\|^{1/2}\right) = \frac{u_a\left(\|A\|^{1/2}\right)-u_a(0)}{\|A\|^{1/2}}$$

and

$$g(B)B = v(B)-v(0) \quad \text{and} \quad g_a\left(\|B\|^{1/2}\right) = \frac{v_a\left(\|B\|^{1/2}\right)-v_a(0)}{\|B\|^{1/2}}.$$

By making use of the inequality (2.7), we derive

$$\begin{aligned} &\left|\left\langle D^*\left[u(A)-u(0)\right]\left[v(B)-v(0)\right]Cx,y\right\rangle\right| \\ &\leqslant \|A\|^{\alpha}\|B\|^{1-\alpha}\frac{u_a\left(\|A\|^{1/2}\right)-u_a(0)}{\|A\|^{1/2}}\frac{v_a\left(\|B\|^{1/2}\right)-v_a(0)}{\|B\|^{1/2}} \\ &\quad\times\left\langle\left||B|^{\alpha}C\right|^2x,x\right\rangle^{1/2}\left\langle\left||A^*|^{1-\alpha}D\right|^2y,y\right\rangle^{1/2} \end{aligned}$$

for all $\alpha \in [0,1]$ and x, $y \in H$ and the inequality (2.17) is thus obtained. □

REMARK 2. If we take, for instance, $u(z) = v(z) = \sin z$, then $u_a(z) = v_a(z) = \sinh z$, $z \in \mathbb{C}$ and by (2.17) we get

$$\begin{aligned} &\left|\left\langle D^*(\sin A \sin B)Cx,y\right\rangle\right| \\ &\leqslant \|A\|^{\alpha-1/2}\|B\|^{1/2-\alpha}\sinh\left(\|A\|^{1/2}\right)\sinh\left(\|B\|^{1/2}\right) \\ &\quad\times\left\langle\left||B|^{\alpha}C\right|^2x,x\right\rangle^{1/2}\left\langle\left||A^*|^{1-\alpha}D\right|^2y,y\right\rangle^{1/2} \end{aligned} \tag{2.18}$$

for $A,\ B,\ C,\ D$ operators in $\mathscr{B}(H),\ A,B\neq 0,\ \alpha\in[0,1]$ and $x,\ y\in H$.

One can obtain other similar inequalities by taking some specific examples for the operators and the functions involved, however the details are omitted.

3. Norm and numerical radius inequalities

Our first result concerning the operator norm and numerical radius is as follows:

THEOREM 3. *Consider the power series* $f(z)=\sum_{i=0}^{\infty}a_i z^i$ *convergent on the open disk* $D(0,R_1)$ *and* $g(z)=\sum_{i=0}^{\infty}b_i z^i$ *convergent on* $D(0,R_2)$ *and* $A,\ B,\ C,\ D$ *be operators in* $\mathscr{B}(H)$ *with* $\|A\|^{1/2},\|A\|<R_1$ *and* $\|B\|^{1/2},\ \|B\|<R_2$*, then*

$$\begin{aligned}
&\|D^*Af(A)g(B)BC\| \\
&\leqslant f_a\left(\|A\|^{1/2}\right)g_a\left(\|B\|^{1/2}\right)\|A\|^{\alpha}\|B\|^{1-\alpha}\left\||B|^{\alpha}C\right\|\left\||A^*|^{1-\alpha}D\right\| \\
&\leqslant f_a\left(\|A\|^{1/2}\right)g_a\left(\|B\|^{1/2}\right)\|A\|\,\|B\|\,\|C\|\,\|D\|
\end{aligned}\tag{3.1}$$

for all $\alpha\in[0,1]$.

Proof. We take the supremum over $\|x\|=\|y\|=1$ in (2.7), then we get

$$\begin{aligned}
\|D^*Af(A)g(B)BC\| &= \sup_{\|x\|=\|y\|=1}|\langle D^*Af(A)g(B)BCx,y\rangle| \\
&\leqslant \|A\|^{\alpha}\|B\|^{1-\alpha}f_a\left(\|A\|^{1/2}\right)g_a\left(\|B\|^{1/2}\right) \\
&\quad\times\sup_{\|x\|=\|y\|=1}\left[\left\langle\left||B|^{\alpha}C\right|^2x,x\right\rangle^{1/2}\left\langle\left||A^*|^{1-\alpha}D\right|^2y,y\right\rangle^{1/2}\right] \\
&= \|A\|^{\alpha}\|B\|^{1-\alpha}f_a\left(\|A\|^{1/2}\right)g_a\left(\|B\|^{1/2}\right) \\
&\quad\times\sup_{\|x\|=1}\left\langle\left||B|^{\alpha}C\right|^2x,x\right\rangle^{1/2}\sup_{\|y\|=1}\left\langle\left||A^*|^{1-\alpha}D\right|^2y,y\right\rangle^{1/2} \\
&= \|A\|^{\alpha}\|B\|^{1-\alpha}f_a\left(\|A\|^{1/2}\right)g_a\left(\|B\|^{1/2}\right) \\
&\quad\times\left\|\left||B|^{\alpha}C\right|^2\right\|^{1/2}\left\|\left||A^*|^{1-\alpha}D\right|^2\right\|^{1/2} \\
&= \|A\|^{\alpha}\|B\|^{1-\alpha}f_a\left(\|A\|^{1/2}\right)g_a\left(\|B\|^{1/2}\right)\left\||B|^{\alpha}C\right\|\left\||A^*|^{1-\alpha}D\right\|,
\end{aligned}$$

which proves the first part of (3.1).

Since,

$$\left\||B|^{\alpha}C\right\|\leqslant\left\||B|^{\alpha}\right\|\|C\|=\|B\|^{\alpha}\|C\|$$

and

$$\left\||A^*|^{1-\alpha}D\right\|\leqslant\left\||A^*|^{1-\alpha}\right\|\|D\|=\|A\|^{1-\alpha}\|D\|,$$

then

$$\begin{aligned}
&\|A\|^{\alpha}\|B\|^{1-\alpha} f_a\left(\|A\|^{1/2}\right) g_a\left(\|B\|^{1/2}\right) \left\||B|^{\alpha} C\right\| \left\||A^*|^{1-\alpha} D\right\| \\
&\leqslant \|A\|^{\alpha}\|B\|^{1-\alpha} f_a\left(\|A\|^{1/2}\right) g_a\left(\|B\|^{1/2}\right) \|B\|^{\alpha}\|C\| \|A\|^{1-\alpha}\|D\| \\
&= \|A\| \|B\| f_a\left(\|A\|^{1/2}\right) g_a\left(\|B\|^{1/2}\right) \|C\| \|D\|
\end{aligned}$$

and the last part of (3.1) is also proved. □

THEOREM 4. *With the assumptions of Theorem 3 we have the numerical radius inequalities*

$$\omega\left(D^* A f(A) g(B) B C\right) \leqslant f_a\left(\|A\|^{1/2}\right) g_a\left(\|B\|^{1/2}\right) \|A\|^{\alpha}\|B\|^{1-\alpha} \times \left\| \frac{\left||B|^{\alpha} C\right|^2 + \left||A^*|^{1-\alpha} D\right|^2}{2} \right\| \tag{3.2}$$

and

$$\begin{aligned}
&\omega^2\left(D^* A f(A) g(B) B C\right) \\
&\leqslant \frac{1}{2} f_a^2\left(\|A\|^{1/2}\right) g_a^2\left(\|B\|^{1/2}\right) \|A\|^{2\alpha}\|B\|^{2(1-\alpha)} \\
&\quad \times \left[\left\||B|^{\alpha} C\right\|^2 \left\||A^*|^{1-\alpha} D\right\|^2 + \omega\left(\left||A^*|^{1-\alpha} D\right|^2 \left||B|^{\alpha} C\right|^2\right)\right]
\end{aligned} \tag{3.3}$$

for all $\alpha \in [0,1]$.

Proof. From (2.7) we get for all $x \in H$ that

$$\left|\left\langle D^* A f(A) g(B) B C x, x\right\rangle\right| \leqslant \|A\|^{\alpha}\|B\|^{1-\alpha} f_a\left(\|A\|^{1/2}\right) g_a\left(\|B\|^{1/2}\right) \times \left\langle \left||B|^{\alpha} C\right|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} D\right|^2 x, x\right\rangle^{1/2}. \tag{3.4}$$

By the *A-G-inequality* we have

$$\begin{aligned}
&\left\langle \left||B|^{\alpha} C\right|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} D\right|^2 x, x\right\rangle^{1/2} \\
&\leqslant \frac{1}{2}\left[\left\langle \left||B|^{\alpha} C\right|^2 x, x\right\rangle + \left\langle \left||A^*|^{1-\alpha} D\right|^2 x, x\right\rangle\right] = \left\langle \frac{\left||B|^{\alpha} C\right|^2 + \left||A^*|^{1-\alpha} D\right|^2}{2} x, x\right\rangle
\end{aligned}$$

and by (3.4) we obtain

$$\begin{aligned}
&\omega\left(D^{*}Af\left(A\right)g\left(B\right)BC\right)\\
&=\sup_{\|x\|=1}\left|\left\langle D^{*}Af\left(A\right)g\left(B\right)BCx,x\right\rangle\right|\\
&\leqslant\|A\|^{\alpha}\|B\|^{1-\alpha}f_{a}\left(\|A\|^{1/2}\right)g_{a}\left(\|B\|^{1/2}\right)\sup_{\|x\|=1}\left\langle\frac{\left||B|^{\alpha}C\right|^{2}+\left||A^{*}|^{1-\alpha}D\right|^{2}}{2}x,x\right\rangle\\
&=\|A\|^{\alpha}\|B\|^{1-\alpha}f_{a}\left(\|A\|^{1/2}\right)g_{a}\left(\|B\|^{1/2}\right)\left\|\frac{\left||B|^{\alpha}C\right|^{2}+\left||A^{*}|^{1-\alpha}D\right|^{2}}{2}\right\|,
\end{aligned}$$

which proves (3.2).

Let $x\in H$, $\|x\|=1$, then by Buzano's inequality, we recall that [5]

$$\left|\langle u,e\rangle\langle e,v\rangle\right|\leqslant\frac{1}{2}\left[\|u\|\|v\|+\left|\langle u,v\rangle\right|\right]$$

holds for any $u,\ v,\ e\in H$ with $\|e\|=1,$ we derive

$$\begin{aligned}
&\left\langle\left||B|^{\alpha}C\right|^{2}x,x\right\rangle\left\langle x,\left||A^{*}|^{1-\alpha}D\right|^{2}x\right\rangle\\
&\leqslant\frac{1}{2}\left[\left\|\left||B|^{\alpha}C\right|^{2}x\right\|\left\|\left||A^{*}|^{1-\alpha}D\right|^{2}x\right\|+\left|\left\langle\left||B|^{\alpha}C\right|^{2}x,\left||A^{*}|^{1-\alpha}D\right|^{2}x\right\rangle\right|\right]\\
&=\frac{1}{2}\left[\left\|\left||B|^{\alpha}C\right|^{2}x\right\|\left\|\left||A^{*}|^{1-\alpha}D\right|^{2}x\right\|+\left|\left\langle\left||A^{*}|^{1-\alpha}D\right|^{2}\left||B|^{\alpha}C\right|^{2}x,x\right\rangle\right|\right].
\end{aligned}$$

By (3.4) we then obtain

$$\begin{aligned}
&\left|\left\langle D^{*}Af\left(A\right)g\left(B\right)BCx,x\right\rangle\right|^{2}\\
&\leqslant\frac{1}{2}\|A\|^{2\alpha}\|B\|^{2(1-\alpha)}f_{a}^{2}\left(\|A\|^{1/2}\right)g_{a}^{2}\left(\|B\|^{1/2}\right)\\
&\quad\times\left[\left\|\left||B|^{\alpha}C\right|^{2}x\right\|\left\|\left||A^{*}|^{1-\alpha}D\right|^{2}x\right\|+\left|\left\langle\left||A^{*}|^{1-\alpha}D\right|^{2}\left||B|^{\alpha}C\right|^{2}x,x\right\rangle\right|\right].
\end{aligned}\tag{3.5a}$$

If we take the supremum over $\|x\|=1,$ then we get

$$\begin{aligned}
&\omega^{2}\left(D^{*}Af\left(A\right)g\left(B\right)BC\right)\\
&=\sup_{\|x\|=1}\left|\left\langle D^{*}Af\left(A\right)g\left(B\right)BCx,x\right\rangle\right|^{2}\\
&\leqslant\frac{1}{2}\|A\|^{2\alpha}\|B\|^{2(1-\alpha)}f_{a}^{2}\left(\|A\|^{1/2}\right)g_{a}^{2}\left(\|B\|^{1/2}\right)\\
&\quad\times\sup_{\|x\|=1}\left[\left\|\left||B|^{\alpha}C\right|^{2}x\right\|\left\|\left||A^{*}|^{1-\alpha}D\right|^{2}x\right\|+\left|\left\langle\left||A^{*}|^{1-\alpha}D\right|^{2}\left||B|^{\alpha}C\right|^{2}x,x\right\rangle\right|\right]
\end{aligned}$$

$$\leqslant \frac{1}{2}\|A\|^{2\alpha}\|B\|^{2(1-\alpha)} f_a^2\left(\|A\|^{1/2}\right) g_a^2\left(\|B\|^{1/2}\right)$$
$$\times\left[\left\||B|^{\alpha} C\right\|^2 \left\||A^*|^{1-\alpha} D\right\|^2+\omega\left(\left|A^*|^{1-\alpha} D\right|^2 \left||B|^{\alpha} C\right|^2\right)\right],$$

which proves (3.3). $\square$

REMARK 3. If we take in Theorem 4 $D=C=I$, then we derive

$$\omega(Af(A)g(B)B) \tag{3.6}$$
$$\leqslant f_a\left(\|A\|^{1/2}\right) g_a\left(\|B\|^{1/2}\right)\|A\|^{\alpha}\|B\|^{1-\alpha}\left\|\frac{|B|^{2\alpha}+|A^*|^{2(1-\alpha)}}{2}\right\|$$

and

$$\omega^2(Af(A)g(B)B) \tag{3.7}$$
$$\leqslant \frac{1}{2} f_a^2\left(\|A\|^{1/2}\right) g_a^2\left(\|B\|^{1/2}\right)\|A\|^{2\alpha}\|B\|^{2(1-\alpha)}$$
$$\times\left[\|B\|^{2\alpha}\|A\|^{2(1-\alpha)}+\omega\left(|A^*|^{2(1-\alpha)}|B|^{2\alpha}\right)\right]$$

for all $\alpha\in[0,1]$.

We recall now *Young's inequality* for $a,b\geqslant 0$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$ab\leqslant \frac{1}{p}a^p+\frac{1}{q}b^q$$

and *McCarthy inequality* for operator $P\geqslant 0$,

$$\langle Px,x\rangle^s\leqslant\langle P^s x,x\rangle$$

where $s\geqslant 1$ and $x\in H$ with $\|x\|=1$.

THEOREM 5. *Assume that the conditions of Theorem 2 are satisfied and* $\alpha\in[0,1]$. *If* $r>0$, $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $pr,\ qr\geqslant 1$, *then*

$$\omega^{2r}(D^*Af(A)g(B)BC) \tag{3.8}$$
$$\leqslant f_a^{2r}\left(\|A\|^{1/2}\right) g_a^{2r}\left(\|B\|^{1/2}\right)\|A\|^{2r\alpha}\|B\|^{2r(1-\alpha)}$$
$$\times\left\|\frac{1}{p}\left||B|^{\alpha} C\right|^{2pr}+\frac{1}{q}\left||A^*|^{1-\alpha} D\right|^{2qr}\right\|$$

and, in particular

$$\omega^{2r}(D^*Af(A)g(B)BC) \tag{3.9}$$
$$\leqslant f_a^{2r}\left(\|A\|^{1/2}\right) g_a^{2r}\left(\|B\|^{1/2}\right)\|A\|^{r}\|B\|^{r}$$
$$\times\left\|\frac{1}{p}\left||B|^{1/2} C\right|^{2pr}+\frac{1}{q}\left||A^*|^{1/2} D\right|^{2qr}\right\|.$$

If $r \geqslant 1$, then

$$\omega^{2r}\left(D^{*}Af\left(A\right)g\left(B\right)BC\right) \leqslant f_a^{2r}\left(\|A\|^{1/2}\right)g_a^{2r}\left(\|B\|^{1/2}\right)\|A\|^{2r\alpha}\|B\|^{2r(1-\alpha)} \times \frac{\left\||B|^{\alpha}C\right\|^{2r}\left\||A^{*}|^{1-\alpha}D\right\|^{2r}+\omega^{r}\left(\left|A^{*}|^{1-\alpha}D\right|^{2}\left||B|^{\alpha}C\right|^{2}\right)}{2} \tag{3.10}$$

and, in particular

$$\omega^{2r}\left(D^{*}Af\left(A\right)g\left(B\right)BC\right) \leqslant f_a^{2r}\left(\|A\|^{1/2}\right)g_a^{2r}\left(\|B\|^{1/2}\right)\|A\|^{r}\|B\|^{r} \times \frac{\left\||B|^{1/2}C\right\|^{2r}\left\||A^{*}|^{1/2}D\right\|^{2r}+\omega^{r}\left(\left|A^{*}|^{1/2}D\right|^{2}\left||B|^{1/2}C\right|^{2}\right)}{2}. \tag{3.11}$$

If $r \geqslant 1$, p, $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ and pr, $qr \geqslant 2$, then also

$$\omega^{2r}\left(D^{*}Af\left(A\right)g\left(B\right)BC\right) \leqslant f_a^{2r}\left(\|A\|^{1/2}\right)g_a^{2r}\left(\|B\|^{1/2}\right)\|A\|^{2r\alpha}\|B\|^{2r(1-\alpha)} \times \frac{\left\|\frac{1}{p}\left||B|^{\alpha}C\right|^{2rp}+\frac{1}{q}\left||A^{*}|^{1-\alpha}D\right|^{2rq}\right\|+\omega^{r}\left(\left|A^{*}|^{1-\alpha}D\right|^{2}\left||B|^{\alpha}C\right|^{2}\right)}{2} \tag{3.12}$$

and, in particular

$$\omega^{2r}\left(D^{*}Af\left(A\right)g\left(B\right)BC\right) \leqslant f_a^{2r}\left(\|A\|^{1/2}\right)g_a^{2r}\left(\|B\|^{1/2}\right)\|A\|^{r}\|B\|^{r} \times \frac{\left\|\frac{1}{p}\left||B|^{1/2}C\right|^{2rp}+\frac{1}{q}\left||A^{*}|^{1/2}D\right|^{2rq}\right\|+\omega^{r}\left(\left|A^{*}|^{1/2}D\right|^{2}\left||B|^{1/2}C\right|^{2}\right)}{2}. \tag{3.13}$$

Proof. From (2.7) we get by taking the power $2r > 0$ for all $x \in H$ that

$$\left|\left\langle D^{*}Af\left(A\right)g\left(B\right)BCx,x\right\rangle\right|^{2r} \leqslant \|A\|^{2r\alpha}\|B\|^{2r(1-\alpha)}f_a^{2r}\left(\|A\|^{1/2}\right)g_a^{2r}\left(\|B\|^{1/2}\right) \times\left\langle\left||B|^{\alpha}C\right|^{2}x,x\right\rangle^{r}\left\langle\left||A^{*}|^{1-\alpha}D\right|^{2}x,x\right\rangle^{r}. \tag{3.14}$$

By Young and McCarthy inequalities [6] we have,

$$\begin{aligned}
&\left\langle \left||B|^{\alpha} C\right|^{2} x,x\right\rangle^{r} \left\langle \left||A^{*}|^{1-\alpha} D\right|^{2} x,x\right\rangle^{r} \\
&\leqslant \frac{1}{p}\left\langle \left||B|^{\alpha} C\right|^{2} x,x\right\rangle^{pr} + \frac{1}{q}\left\langle \left||A^{*}|^{1-\alpha} D\right|^{2} x,x\right\rangle^{qr} \\
&\leqslant \frac{1}{p}\left\langle \left||B|^{\alpha} C\right|^{2pr} x,x\right\rangle + \frac{1}{q}\left\langle \left||A^{*}|^{1-\alpha} D\right|^{2qr} x,x\right\rangle \\
&= \left\langle \left(\frac{1}{p}\left||B|^{\alpha} C\right|^{2pr} + \frac{1}{q}\left||A^{*}|^{1-\alpha} D\right|^{2qr}\right) x,x\right\rangle
\end{aligned} \tag{3.15}$$

for $x \in H$ with $\|x\| = 1$.

By (3.14) and (3.15) we obtain

$$\begin{aligned}
&\left|\langle D^{*}Af(A)\,g(B)\,BCx,x\rangle\right|^{2r} \\
&\leqslant \|A\|^{2r\alpha}\,\|B\|^{2r(1-\alpha)}\, f_{a}^{2r}\left(\|A\|^{1/2}\right) g_{a}^{2r}\left(\|B\|^{1/2}\right) \\
&\quad\times \left\langle \left(\frac{1}{p}\left||B|^{\alpha} C\right|^{2pr} + \frac{1}{q}\left||A^{*}|^{1-\alpha} D\right|^{2qr}\right) x,x\right\rangle
\end{aligned} \tag{3.16}$$

for $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$, we get (3.8).

By taking the power $r \geqslant 1$ in (3.5a) and by using the convexity of the power function, we obtain

$$\begin{aligned}
&\left|\langle D^{*}Af(A)\,g(B)\,BCx,x\rangle\right|^{2r} \\
&\leqslant \|A\|^{2r\alpha}\,\|B\|^{2r(1-\alpha)}\, f_{a}^{2r}\left(\|A\|^{1/2}\right) g_{a}^{2r}\left(\|B\|^{1/2}\right) \\
&\times \left[\frac{\left\|\left||B|^{\alpha} C\right|^{2} x\right\| \left\|\left||A^{*}|^{1-\alpha} D\right|^{2} x\right\| + \left|\left\langle \left||A^{*}|^{1-\alpha} D\right|^{2} \left||B|^{\alpha} C\right|^{2} x,x\right\rangle\right|}{2}\right]^{r} \\
&\leqslant \|A\|^{2r\alpha}\,\|B\|^{2r(1-\alpha)}\, f_{a}^{2r}\left(\|A\|^{1/2}\right) g_{a}^{2r}\left(\|B\|^{1/2}\right) \\
&\times \frac{\left\|\left||B|^{\alpha} C\right|^{2} x\right\|^{r} \left\|\left||A^{*}|^{1-\alpha} D\right|^{2} x\right\|^{r} + \left|\left\langle \left||A^{*}|^{1-\alpha} D\right|^{2} \left||B|^{\alpha} C\right|^{2} x,x\right\rangle\right|^{r}}{2}
\end{aligned} \tag{3.17}$$

for $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$, we get (3.10).

We also have by Young's inequality that

$$\begin{aligned}
&\left\| \left| |B|^{\alpha} C \right|^2 x \right\|^r \left\| \left| |A^*|^{1-\alpha} D \right|^2 x \right\|^r \\
&\leqslant \frac{1}{p} \left\| \left| |B|^{\alpha} C \right|^2 x \right\|^{rp} + \frac{1}{q} \left\| \left| |A^*|^{1-\alpha} D \right|^2 x \right\|^{rq} \\
&= \frac{1}{p} \left\| \left| |B|^{\alpha} C \right|^2 x \right\|^{2\frac{rp}{2}} + \frac{1}{q} \left\| \left| |A^*|^{1-\alpha} D \right|^2 x \right\|^{2\frac{rq}{2}} \\
&= \frac{1}{p} \left\langle \left| |B|^{\alpha} C \right|^4 x, x \right\rangle^{\frac{rp}{2}} + \frac{1}{q} \left\langle \left| |A^*|^{1-\alpha} D \right|^4 x, x \right\rangle^{\frac{rq}{2}}
\end{aligned}$$

for $x \in H$.

By McCarthy's inequality for $\frac{rp}{2},\ \frac{rq}{2} \geqslant 1$ we also have

$$\begin{aligned}
&\frac{1}{p} \left\langle \left| |B|^{\alpha} C \right|^4 x, x \right\rangle^{\frac{rp}{2}} + \frac{1}{q} \left\langle \left| |A^*|^{1-\alpha} D \right|^4 x, x \right\rangle^{\frac{rq}{2}} \\
&\leqslant \frac{1}{p} \left\langle \left| |B|^{\alpha} C \right|^{2rp} x, x \right\rangle + \frac{1}{q} \left\langle \left| |A^*|^{1-\alpha} D \right|^{2rq} x, x \right\rangle \\
&= \left\langle \left(\frac{1}{p} \left| |B|^{\alpha} C \right|^{2rp} + \frac{1}{q} \left| |A^*|^{1-\alpha} D \right|^{2rq} \right) x, x \right\rangle
\end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

Therefore,

$$\left\| \left| |B|^{\alpha} C \right|^2 x \right\|^r \left\| \left| |A^*|^{1-\alpha} D \right|^2 x \right\|^r \leqslant \left\langle \left(\frac{1}{p} \left| |B|^{\alpha} C \right|^{2rp} + \frac{1}{q} \left| |A^*|^{1-\alpha} D \right|^{2rq} \right) x, x \right\rangle$$

for $x \in H$ with $\|x\| = 1$.

By (3.17) we derive

$$\begin{aligned}
&|\langle D^* A f(A) g(B) BCx, x \rangle|^{2r} \\
&\leqslant \|A\|^{2r\alpha} \|B\|^{2r(1-\alpha)} f_a^{2r} \left(\|A\|^{1/2} \right) g_a^{2r} \left(\|B\|^{1/2} \right) \\
&\quad \times \frac{\left\langle \left(\frac{1}{p} \left| |B|^{\alpha} C \right|^{2rp} + \frac{1}{q} \left| |A^*|^{1-\alpha} D \right|^{2rq} \right) x, x \right\rangle + \left| \left\langle \left| |A^*|^{1-\alpha} D \right|^2 \left| |B|^{\alpha} C \right|^2 x, x \right\rangle \right|^r}{2}
\end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$, we obtain (3.12). □

REMARK 4. From Theorem 5 we get for $D = C = I$ that

$$\begin{aligned}
\omega^{2r} (A f(A) g(B) B) &\leqslant f_a^{2r} \left(\|A\|^{1/2} \right) g_a^{2r} \left(\|B\|^{1/2} \right) \\
&\quad \times \|A\|^{2r\alpha} \|B\|^{2r(1-\alpha)} \left\| \frac{1}{p} |B|^{2\alpha pr} + \frac{1}{q} |A^*|^{2(1-\alpha)qr} \right\|
\end{aligned} \tag{3.18}$$

and, in particular

$$\omega^{2r}\left(Af\left(A\right)g\left(B\right)B\right) \tag{3.19}$$
$$\leqslant f_a^{2r}\left(\|A\|^{1/2}\right)g_a^{2r}\left(\|B\|^{1/2}\right)\|A\|^r\|B\|^r\left\|\frac{1}{p}\left|B\right|^{pr}+\frac{1}{q}\left|A^*\right|^{qr}\right\|$$

for $r>0,\ p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $pr,\ qr\geqslant 1$.

If $r\geqslant 1$, then

$$\omega^{2r}\left(Af\left(A\right)g\left(B\right)B\right) \tag{3.20}$$
$$\leqslant f_a^{2r}\left(\|A\|^{1/2}\right)g_a^{2r}\left(\|B\|^{1/2}\right)\|A\|^{2r\alpha}\|B\|^{2r(1-\alpha)}$$
$$\times\frac{\|B\|^{2\alpha r}\|A\|^{2(1-\alpha)r}+\omega^r\left(\left|A^*\right|^{2(1-\alpha)}\left|B\right|^{2\alpha}\right)}{2}$$

and, in particular

$$\omega^{2r}\left(Af\left(A\right)g\left(B\right)B\right) \tag{3.21}$$
$$\leqslant f_a^{2r}\left(\|A\|^{1/2}\right)g_a^{2r}\left(\|B\|^{1/2}\right)\|A\|^r\|B\|^r\frac{\|B\|^r\|A\|^r+\omega^r\left(\left|A^*\right|\left|B\right|\right)}{2}.$$

Also, if $r\geqslant 1,\ p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $pr,\ qr\geqslant 2$, then also

$$\omega^{2r}\left(Af\left(A\right)g\left(B\right)B\right) \tag{3.22}$$
$$\leqslant f_a^{2r}\left(\|A\|^{1/2}\right)g_a^{2r}\left(\|B\|^{1/2}\right)\|A\|^{2r\alpha}\|B\|^{2r(1-\alpha)}$$
$$\times\frac{\left\|\frac{1}{p}\left|B\right|^{2\alpha rp}+\frac{1}{q}\left|A^*\right|^{2(1-\alpha)rq}\right\|+\omega^r\left(\left|A^*\right|^{2(1-\alpha)}\left|B\right|^{2\alpha}\right)}{2}$$

and, in particular

$$\omega^{2r}\left(Af\left(A\right)g\left(B\right)B\right)\leqslant f_a^{2r}\left(\|A\|^{1/2}\right)g_a^{2r}\left(\|B\|^{1/2}\right)\|A\|^r\|B\|^r \tag{3.23}$$
$$\times\frac{\left\|\frac{1}{p}\left|B\right|^{rp}+\frac{1}{q}\left|A^*\right|^{rq}\right\|+\omega^r\left(\left|A^*\right|\left|B\right|\right)}{2}.$$

4. Some examples

Consider the fundamental complex functions $f(z)=(1\pm z)^{-1}$ and $g(z)=\ln(1\pm z)^{-1}$ for $|z|<1$. From (3.6) and (3.7) we get

$$\omega\left(A\left(1\pm A\right)^{-1}B\ln\left(1\pm B\right)^{-1}\right) \tag{4.1}$$
$$\leqslant\|A\|^{\alpha}\left(1-\|A\|^{1/2}\right)^{-1}\|B\|^{1-\alpha}\ln\left(1-\|B\|^{1/2}\right)^{-1}$$
$$\times\left\|\frac{\left|B\right|^{2\alpha}+\left|A^*\right|^{2(1-\alpha)}}{2}\right\|$$

and

$$\omega\left(A(1\pm A)^{-1}B\ln(1\pm B)^{-1}\right) \tag{4.2}$$
$$\leqslant \|A\|^{\alpha}\left(1-\|A\|^{1/2}\right)^{-1}\|B\|^{1-\alpha}\ln\left(1-\|B\|^{1/2}\right)^{-1}$$
$$\times\left[\frac{\|B\|^{2\alpha}\|A\|^{2(1-\alpha)}+\omega\left(|A^*|^{2(1-\alpha)}|B|^{2\alpha}\right)}{2}\right]^{1/2}$$

for all $\alpha\in[0,1]$.

Also, from (3.18) we derive

$$\omega\left(A(1\pm A)^{-1}B\ln(1\pm B)^{-1}\right) \tag{4.3}$$
$$\leqslant \|A\|^{\alpha}\left(1-\|A\|^{1/2}\right)^{-1}\|B\|^{1-\alpha}\ln\left(1-\|B\|^{1/2}\right)^{-1}$$
$$\times\left\|\frac{1}{p}|B|^{2\alpha pr}+\frac{1}{q}|A^*|^{2(1-\alpha)qr}\right\|^{\frac{1}{2r}}$$

for $r>0,\ p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $pr,\ qr\geqslant 1$.

Moreover, if $r\geqslant 1,\ p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $pr,\ qr\geqslant 2,$ then by (3.22) we also obtain

$$\omega\left(A(1\pm A)^{-1}B\ln(1\pm B)^{-1}\right) \tag{4.4}$$
$$\leqslant \|A\|^{\alpha}\left(1-\|A\|^{1/2}\right)^{-1}\|B\|^{1-\alpha}\ln\left(1-\|B\|^{1/2}\right)^{-1}$$
$$\times\left(\frac{\left\|\frac{1}{p}|B|^{2\alpha rp}+\frac{1}{q}|A^*|^{2(1-\alpha)rq}\right\|+\omega^r\left(|A^*|^{2(1-\alpha)}|B|^{2\alpha}\right)}{2}\right)^{\frac{1}{2r}}$$

where $A,\ B$ are operators in $\mathscr{B}(H)$ with $\|A\|,\ \|B\|<1$ and $\alpha\in[0,1]$.

Now, consider the power series $f(z)=\sin z$ and $g(z)=\cos z,\ z\in\mathbb{C}$. From (3.6) and (3.7) we get

$$\omega(A\sin(A)\cos(B)B) \tag{4.5}$$
$$\leqslant \sinh\left(\|A\|^{1/2}\right)\cosh\left(\|B\|^{1/2}\right)\|A\|^{\alpha}\|B\|^{1-\alpha}\left\|\frac{|B|^{2\alpha}+|A^*|^{2(1-\alpha)}}{2}\right\|$$

and

$$\omega(A\sin(A)\cos(B)B) \tag{4.6}$$
$$\leqslant \sinh\left(\|A\|^{1/2}\right)\cosh\left(\|B\|^{1/2}\right)\|A\|^{\alpha}\|B\|^{1-\alpha}$$
$$\times\left[\frac{\|B\|^{2\alpha}\|A\|^{2(1-\alpha)}+\omega\left(|A^*|^{2(1-\alpha)}|B|^{2\alpha}\right)}{2}\right]^{1/2}$$

for all $\alpha \in [0,1]$.

Also, from (3.18) we obtain

$$\omega(A\sin(A)\cos(B)B) \leqslant \sinh\left(\|A\|^{1/2}\right)\cosh\left(\|B\|^{1/2}\right) \times \|A\|^{\alpha}\|B\|^{1-\alpha}\left\|\frac{1}{p}|B|^{2\alpha pr}+\frac{1}{q}|A^*|^{2(1-\alpha)qr}\right\|^{\frac{1}{2r}} \tag{4.7}$$

for $r > 0$, p, $q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$ and pr, $qr \geqslant 1$.

Moreover, if $r \geqslant 1$, p, $q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$ and pr, $qr \geqslant 2$, then by (3.22) we also derive

$$\begin{aligned}&\omega(A\sin(A)\cos(B)B) \\ &\leqslant \sinh\left(\|A\|^{1/2}\right)\cosh\left(\|B\|^{1/2}\right)\|A\|^{\alpha}\|B\|^{1-\alpha} \\ &\times\left(\frac{\left\|\frac{1}{p}|B|^{2\alpha rp}+\frac{1}{q}|A^*|^{2(1-\alpha)rq}\right\|+\omega^{r}\left(|A^*|^{2(1-\alpha)}|B|^{2\alpha}\right)}{2}\right)^{\frac{1}{2r}},\end{aligned} \tag{4.8}$$

where A, B are operators in $\mathscr{B}(H)$ and $\alpha \in [0,1]$.

The interested reader may state other similar results by employing the power series listed in (2.4)–(2.6). We omit the details.

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