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RATES OF CONVERGENCE FOR ITERATES OF POSITIVE LINEAR OPERATORS

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Abstract. We are concerned with positive linear operators defined on $C(X)$, where X is a simplex or a hypercube. We assume that the operators preserve the affine functions. After identifying an eigenvalue $a \in [0,1)$ of such an operator L, we show that the sequence $(L^k f)_{k\geqslant 1}$ has a limit Vf, $f \in C(X)$ and $|L^k f(x) - Vf(x)|$ is dominated by a^k multiplied by a factor depending on L, f and x. These general results are applied to several classical or recently introduced operators acting on simplices and hypercubes.

1. Introduction

Positive linear operators are important tools in Approximation Theory. Their properties are investigated from various points of view and with various methods, corresponding to the applications where they are required. A rich literature was devoted to the study of the iterates of such operators, see, e.g., [4], [5], [6], [7], [8], [11], [12], [13], [14], [17], [19], [21], [24], [25], and the references therein. Given a positive linear operator L, the sequences of its iterates $(L^k)_{k\geqslant 1}$ is an object of study in Approximation Theory, Ergodic Theory, Linear Algebra, Functional Analysis and other areas of research. In particular, the limit $\lim_{k\to\infty} L^k$ was intensively investigated. Qualitative results were obtained, as well as rates of convergence of the sequence $(L^k)_{k\geqslant 1}$ toward the limit. We mention here the important results involving rates of convergence obtained in [10], [13], [14], [24], [25]; see also the references therein. In these papers the degree of approximation is estimated in terms of moduli of continuity, K-functionals and other suitable tools.

In this paper we are concerned with positive linear operators defined on $C(X)$, where X is a simplex or a hypercube. We assume that the operators preserve the affine functions. After identifying an eigenvalue $a \in [0,1)$ of such an operator L, we show that the sequence $(L^k f)_{k\geqslant 1}$ has a limit Vf, $f \in C(X)$ and $|L^k f(x) - Vf(x)|$ is dominated by a^k multiplied by a factor depending on L, f and x. These general results are applied to several classical or recently introduced operators acting on simplices and hypercubes (Bernstein, Beta-type, genuine Bernstein-Durrmeyer, U_n^ρ operators).

Rates of convergence for operators which preserve only the constant functions, not all the affine functions, can be found in [3], [20] and the references therein.

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2. Main results

In this section we consider a positive linear operator $L: C[0,1] \to C[0,1]$ which preserves the functions 1, x and transforms x^2 into a polynomial of degree 2. We show that it has an eigenvalue $a \in [0,1)$ corresponding to the eigenpolynomial $x - x^2$. For each $f \in C[0,1]$ the sequence $(L^k f)_{k \geqslant 1}$ has a limit Vf. Theorem 2.2 shows that $|L^k f(x) - Vf(x)|$ is dominated by a^k multiplied by a factor depending on f and x. This general result will be illustrated by specific examples and will be extended to multivariate operators.

LEMMA 2.1. *Let* $x \in [0,1]$.

(i) If $f \in C^1[0,1]$, *then*

$$|f(x) - (1-x)f(0) - xf(1)| \leqslant 2x(1-x)\|f'\|_\infty. \tag{2.1}$$

(ii) If $f \in C^2[0,1]$, *then*

$$|f(x) - (1-x)f(0) - xf(1)| \leqslant \frac{1}{2}x(1-x)\|f''\|_\infty. \tag{2.2}$$

Proof. (i) Let $f \in C^1[0,1]$. We have

$$\begin{aligned} |f(x) - (1-x)f(0) - xf(1)| &= (1-x)(f(x) - f(0)) + x(f(x) - f(1)) \\ &\leqslant (1-x)x\|f'\|_\infty + x(1-x)\|f'\|_\infty, \end{aligned}$$

and this leads to (2.1).

(ii) For each $f \in C[0,1]$ the Lagrange interpolation formula gives us

$$f(x) - (1-x)f(0) - xf(1) = x(x-1)[0,x,1;f], \tag{2.3}$$

where $[0,x,1;f]$ is the divided difference of f on the nodes $0,x,1$. If $f \in C^2[0,1]$, then

$$[0,x,1;f] = \frac{1}{2}f''(t)$$

for a suitable $t \in [0,1]$. Therefore, $|[0,x,1;f]| \leqslant \frac{1}{2}\|f''\|_\infty$, and this, combined with (2.3), leads to (2.2). $\square$

We use the notation $e_k(x) = x^k$, $x \in [0,1]$, $k = 0,1,\dots$

Let I be the identity operator.

THEOREM 2.1. *Let* $L: C[0,1] \to C[0,1]$ *be a positive linear operator,* $L \neq I$, $Le_0 = e_0$, $Le_1 = e_1$, Le_2 *is a polynomial of degree 2. Then* $e_1 - e_2$ *is an eigenpolynomial of* L, *associated with an eigenvalue* $a \in [0,1)$.

Proof. It is well known that from $Le_0 = e_0$ and $Le_1 = e_1$ it follows that $Lf \geqslant f$ for each convex function $f \in C[0,1]$. In particular, $e_2 \leqslant Le_2$. We have $e_2 \leqslant e_1$, hence $e_2 \leqslant Le_2 \leqslant Le_1 = e_1$.

Let $Le_2(x) = ax^2 + bx + c$, $x \in [0,1]$. We deduce that

$$x^2 \leqslant ax^2 + bx + c \leqslant x, \ \ x \in [0,1].$$

For $x = 0$, this implies $c = 0$, so that

$$x^2 \leqslant ax^2 + bx \leqslant x, \ \ x \in [0,1].$$

For $x = 1$ we get $a + b = 1$, i.e.,

$$x^2 \leqslant ax^2 + (1-a)x \leqslant x.$$

Therefore, $x \leqslant ax + 1 - a \leqslant 1$. Setting $x = 0$, one sees that $a \geqslant 0$. Moreover, $(1-a)(1-x) \geqslant 0$ leads to $a \leqslant 1$.

So, $Le_2 = ae_2 + (1-a)e_1$. If $a = 1$, then $Le_2 = e_2$ and Korovkin's Theorem shows that $L = I$. It follows that $a \in [0,1)$ and $L(e_1 - e_2) = a(e_1 - e_2)$. This concludes the proof. □

Let L be an operator as in Theorem 2.1 and $a \in [0,1)$ the corresponding eigenvalue. Consider also the operator

$$V : C[0,1] \to C[0,1], \ \ Vf(x) := (1-x)f(0) + xf(1).$$

THEOREM 2.2. *(i) If $f \in C^1[0,1]$, then*

$$|L^k f(x) - Vf(x)| \leqslant 2x(1-x)a^k \|f'\|_\infty, \ \ x \in [0,1], \ \ k \in \mathbb{N}. \tag{2.4}$$

(ii) If $f \in C^2[0,1]$, then

$$|L^k f(x) - Vf(x)| \leqslant \frac{1}{2} x(1-x)a^k \|f''\|_\infty, \ \ x \in [0,1], \ \ k \in \mathbb{N}. \tag{2.5}$$

(iii) If $f \in C[0,1]$ and $Lf \in C^1[0,1]$, then

$$|L^{k+1} f(x) - Vf(x)| \leqslant 2x(1-x)a^k \|(Lf)'\|_\infty, \ \ x \in [0,1], \ \ k \in \mathbb{N}. \tag{2.6}$$

(iv) If $f \in C[0,1]$ and $Lf \in C^2[0,1]$, then

$$|L^{k+1} f(x) - Vf(x)| \leqslant \frac{1}{2} x(1-x)a^k \|(Lf)''\|_\infty, \ \ x \in [0,1], \ \ k \in \mathbb{N}. \tag{2.7}$$

Proof. (i) Since L satisfies the hypotheses of Theorem 2.1, we have $Lf(0) = f(0)$, $Lf(1) = f(1)$, and so

$$LV = V = VL, \tag{2.8}$$

$$L^k(e_1 - e_2) = a^k(e_1 - e_2). \tag{2.9}$$

Moreover, according to (2.1),

$$|f - Vf| \leqslant 2(e_1 - e_2)\|f'\|_\infty. \tag{2.10}$$

Combining (2.8)–(2.10) we get (2.4). The proof of (ii) is similar. Finally, (iii) follows from (i), and (iv) from (ii), observing that $VL = V$. □

3. Applications I

3.1. Bernstein operators

The Bernstein operators are given by

$$B_n : C[0,1] \to C[0,1], \quad B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \quad \text{where}$$

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \;\; x \in [0,1].$$

It is well known that

$$B_n e_0 = e_0, \;\; B_n e_1 = e_1, \;\; B_n(e_1 - e_2) = \frac{n-1}{n}(e_1 - e_2). \tag{3.1}$$

Moreover, for $f \in C[0,1]$,

$$(B_n f)'(x) = n \sum_{j=0}^{n-1} p_{n-1,j}(x) \left(f\left(\frac{j+1}{n}\right) - f\left(\frac{j}{n}\right) \right), \tag{3.2}$$

$$(B_n f)''(x) = n(n-1) \sum_{j=0}^{n-2} p_{n-2,j}(x) \left(f\left(\frac{j+2}{n}\right) - 2f\left(\frac{j+1}{n}\right) + f\left(\frac{j}{n}\right) \right). \tag{3.3}$$

Therefore, from (3.2) and (3.3),

$$\|(B_n f)'\|_\infty \leqslant n \max_{j=0,\cdots,n-1} \left| f\left(\frac{j+1}{n}\right) - f\left(\frac{j}{n}\right) \right|, \tag{3.4}$$

$$\|(B_n f)''\|_\infty \leqslant n(n-1) \max_{j=0,\cdots,n-2} \left| f\left(\frac{j+2}{n}\right) - 2f\left(\frac{j+1}{n}\right) + f\left(\frac{j}{n}\right) \right|. \tag{3.5}$$

Using (3.1), (3.4), (3.5) and Theorem 2.2 we get

COROLLARY 3.1. *(i) If $f \in C^1[0,1]$ and $x \in [0,1]$, then*

$$|B_n^k f(x) - Vf(x)| \leqslant 2x(1-x)\left(\frac{n-1}{n}\right)^k \|f'\|_\infty.$$

(ii) If $f \in C^2[0,1]$ and $x \in [0,1]$, then

$$|B_n^k f(x) - Vf(x)| \leqslant \frac{1}{2}x(1-x)\left(\frac{n-1}{n}\right)^k \|f''\|_\infty.$$

(iii) If $f \in C[0,1]$ and $x \in [0,1]$, then

$$|B_n^{k+1} f(x) - Vf(x)| \leqslant 2x(1-x)\left(\frac{n-1}{n}\right)^k n \max_{j=0,\cdots,n-1} \left| f\left(\frac{j+1}{n}\right) - f\left(\frac{j}{n}\right) \right|,$$

$$|B_n^{k+1} f(x) - Vf(x)| \leqslant \frac{1}{2}x(1-x)\left(\frac{n-1}{n}\right)^k n(n-1) \cdot \max_{j=0,\cdots,n-2} \left| f\left(\frac{j+2}{n}\right) - 2f\left(\frac{j+1}{n}\right) + f\left(\frac{j}{n}\right) \right|.$$

3.2. Beta-type operators

The Beta-type operators $\overline{\mathbb{B}}_n$ were introduced by A. Lupaş in his German thesis [22]. For $n = 1,2,3,\ldots$ and $f \in C[0,1]$ they are given by

$$\overline{\mathbb{B}}_n(f;x) := \begin{cases} f(0), & x = 0, \\ \dfrac{1}{B(nx,n-nx)} \displaystyle\int_0^1 t^{nx-1}(1-t)^{n-1-nx} f(t)dt, & 0 < x < 1, \\ f(1), & x = 1, \end{cases}$$

where $B(\cdot,\cdot)$ is the Euler's Beta function.

It is well known that $\overline{\mathbb{B}}_n(e_0;x) = e_0$, $\overline{\mathbb{B}}_n(e_1;x) = e_1$, $\overline{\mathbb{B}}_n(e_2;x) = \dfrac{nx(nx+1)}{n(n+1)}$. Therefore,

$$\overline{\mathbb{B}}_n(e_1 - e_2;x) = x(1-x)\frac{n}{n+1},$$

and applying Theorem 2.2 with $a = \dfrac{n}{n+1}$, we get

COROLLARY 3.2. *(i) If $f \in C^1[0,1]$ and $x \in [0,1]$, then*

$$|\overline{\mathbb{B}}_n^k f(x) - Vf(x)| \leqslant 2x(1-x)\left(\frac{n}{n+1}\right)^k \|f'\|_\infty.$$

(ii) If $f \in C^2[0,1]$ and $x \in [0,1]$, then

$$|\overline{\mathbb{B}}_n^k f(x) - Vf(x)| \leqslant \frac{1}{2}x(1-x)\left(\frac{n}{n+1}\right)^k \|f''\|_\infty.$$

3.3. Genuine Bernstein-Durrmeyer operators

The genuine Bernstein-Durrmeyer operators are introduced as a composition of Bernstein operators and Beta operators, namely $U_n = B_n \circ \overline{\mathbb{B}}_n$ (see [9], [18]). These are given in explicit form by

$$\begin{aligned} U_n(f;x) &= (1-x)^n f(0) + x^n f(1) \\ &\quad + (n-1)\sum_{k=1}^{n-1}\left(\int_0^1 f(t)p_{n-2,k-1}(t)dt\right)p_{n,k}(x), \quad f \in C[0,1]. \end{aligned}$$

These operators are defined as Bernstein operators at the end points and have a Durrmeyer-like construction inside of $[0,1]$.

It is well known that

$$U_n(e_0;x) = e_0, \;\; U_n(e_1;x) = e_1, \;\; U_n(e_2;x) = x^2 + \frac{2x(1-x)}{n+1}.$$

Therefore,

$$U_n(e_1 - e_2; x) = x(1-x)\frac{n-1}{n+1},$$

and applying Theorem 2.2 with $a = \dfrac{n-1}{n+1}$, we get

COROLLARY 3.3. *(i) If $f \in C^1[0,1]$ and $x \in [0,1]$, then*

$$|U_n^k f(x) - Vf(x)| \leqslant 2x(1-x)\left(\frac{n-1}{n+1}\right)^k \|f'\|_\infty.$$

(ii) If $f \in C^2[0,1]$ and $x \in [0,1]$, then

$$|U_n^k f(x) - Vf(x)| \leqslant \frac{1}{2}x(1-x)\left(\frac{n-1}{n+1}\right)^k \|f''\|_\infty.$$

3.4. The operator U_n^ρ

Let us consider the class of operators U_n^ρ introduced in [23] by Păltănea and further investigated by Păltănea and Gonska in [16] and [15].

Let $\rho > 0$ and $n \in \mathbb{N}$. The operators $U_n^\rho : C[0,1] \to \prod_n$ are defined by

$$\begin{aligned} U_n^\rho(f;x) &:= \sum_{k=0}^{n} F_k^\rho(f) p_{n,k}(x) \\ &:= \sum_{k=1}^{n-1} \left(\int_0^1 \frac{t^{k\rho-1}(1-t)^{(n-k)\rho-1}}{B(k\rho,(n-k)\rho)} f(t)dt \right) p_{n,k}(x) + f(0)(1-x)^n + f(1)x^n, \end{aligned}$$

for $f \in C[0,1]$, $x \in [0,1]$.

It is well known that

$$U_n^\rho(e_0;x) = e_0,\ \ U_n^\rho(e_1;x) = e_1,\ \ U_n^\rho(e_2;x) = x^2 + \frac{x(1-x)(\rho+1)}{n\rho+1}.$$

Therefore,

$$U_n^\rho(e_1 - e_2; x) = x(1-x)\frac{\rho(n-1)}{n\rho+1},$$

and applying Theorem 2.2 with $a = \dfrac{\rho(n-1)}{n\rho+1}$, we get

COROLLARY 3.4. *(i) If $f \in C^1[0,1]$ and $x \in [0,1]$, then*

$$|(U_n^\rho)^k f(x) - Vf(x)| \leqslant 2x(1-x)\left(\frac{\rho(n-1)}{n\rho+1}\right)^k \|f'\|_\infty.$$

(ii) If $f \in C^2[0,1]$ *and* $x \in [0,1]$, *then*

$$|(U_n^\rho)^k f(x) - Vf(x)| \leqslant \frac{1}{2}x(1-x)\left(\frac{\rho(n-1)}{n\rho+1}\right)^k \|f''\|_\infty.$$

(iii) If $f \in C[0,1]$, *then*

$$|(U_n^\rho)^{k+1} f(x) - Vf(x)| \leqslant 2x(1-x)\left(\frac{\rho(n-1)}{n\rho+1}\right)^k \|(U_n^\rho f)'\|_\infty, \ \ x \in [0,1], \ \ k \in \mathbb{N},$$

$$|(U_n^\rho)^{k+1} f(x) - Vf(x)| \leqslant \frac{1}{2}x(1-x)\left(\frac{\rho(n-1)}{n\rho+1}\right)^k \|(U_n^\rho f)''\|_\infty, \ \ x \in [0,1], \ \ k \in \mathbb{N}.$$

The derivatives of the polynomial $U_n^\rho f$ are investigated in [16].

4. Operators on simplices

For the sake simplicity we consider only the bidimensional simplex

$$S := \left\{(x,y) \in \mathbb{R}^2 | \ x \geqslant 0, \ \ y \geqslant 0, \ \ x+y \leqslant 1\right\},$$

but the results can be easily extended to arbitrary dimensional simplices.

Let $C^1(S)$ be the space of all functions $f \in C(S)$ having continuous partial derivatives of first order on the interior of S, which can be continuously extended to S. For $f \in C^1(S)$ denote

$$M(f) := \max\left\{\left\|\frac{\partial f}{\partial x}\right\|_\infty, \left\|\frac{\partial f}{\partial y}\right\|_\infty\right\}.$$

Consider the operator $U : C(S) \to C(S)$,

$$Uf(x,y) = (1-x-y)f(0,0) + xf(1,0) + yf(0,1).$$

Let $(x,y) \in S$ and $f \in C^1(S)$. Then

$$\begin{aligned}
&|f(x,y) - Uf(x,y)| \\
&= |(1-x-y)(f(x,y) - f(0,0)) + x(f(x,y) - f(1,0) + y(f(x,y) - f(0,1))| \\
&\leqslant ((1-x-y)(x+y) + x(1-x+y) + y(x+1-y))M(f).
\end{aligned}$$

Therefore,

$$|f(x,y) - Uf(x,y)| \leqslant 2(x - x^2 + y - y^2)M(f). \tag{4.1}$$

Let $L : C(S) \to C(S)$ be a positive linear operator, $L \neq I$. Suppose that L preserves the affine functions. Consider the functions $p,q \in C(S)$, $p(x,y) = x^2$, $q(x,y) = y^2$, $(x,y) \in S$. Suppose that there exist real numbers a,b,c such that

$$Lp(x,y) = ax^2 + bx + c, \ \ Lq(x,y) = ay^2 + by + c, \ \ (x,y) \in S.$$

Denote $u(x,y) = x - x^2$, $v(x,y) = y - y^2$, $(x,y) \in S$.

THEOREM 4.1. *With the above notation,* $a \in [0,1)$ *and* $Lu = au$, $Lv = av$.

Proof. Let $f \in C[0,1]$. Consider the function $\tilde{f} \in C(S)$, $\tilde{f}(x,y) = f(x)$, $(x,y) \in S$, and the operator $\tilde{L} : C[0,1] \to C[0,1]$ defined as

$$\tilde{L}f(x) := L\tilde{f}(x,0), \;\; f \in C[0,1], \;\; x \in [0,1].$$

Then $\tilde{L}$ is a positive linear operator which preserves the affine functions. We have $\tilde{L}e_2(x) = L\tilde{e}_2(x,0) = Lp(x,0) = ax^2 + bx + c$. As in the proof of Theorem 2.1, we see that $c = 0$, $b = 1-a$, $a \in [0,1]$. Now $Lp(x,y) = ax^2 + (1-a)x$, hence $Lu(x,y) = x - ax^2 - (1-a)x = a(x-x^2) = au(x,y)$. Similarly, $Lv(x,y) = av(x,y)$. If $a = 1$, then L preserves the functions p and q; according to Korovkin's theory, $L = I$, a contradiction. So, $a \in [0,1)$ and the proof is finished. □

THEOREM 4.2. *Let* $L : C(S) \to C(S)$ *be an operator as in Theorem 4.1.*
(i) If $f \in C^1(S)$, *then*

$$|L^k f(x,y) - Uf(x,y)| \leqslant 2a^k(x+y-x^2-y^2)M(f), \;\; k \in \mathbb{N}.$$

(ii) If $f \in C(S)$ *and* $Lf \in C^1(S)$, *then*

$$|L^{k+1} f(x,y) - Uf(x,y)| \leqslant 2a^k(x+y-x^2-y^2)M(Lf), \;\; k \in \mathbb{N}.$$

Proof. It suffices to apply Theorem 4.1 in conjunction with (4.1), observing that $UL = U = LU$. □

5. Operators on hypercubes

Again we consider, for the sake of simplicity, only the bidimensional case, i.e., $H = [0,1] \times [0,1]$. As before, for $f \in C^1(H)$ let

$$M(f) := \max\left\{ \left\| \frac{\partial f}{\partial x} \right\|_\infty, \left\| \frac{\partial f}{\partial y} \right\|_\infty \right\}.$$

Consider the operator $W : C(H) \to C(H)$,

$$Wf(x,y) = (1-x)(1-y)f(0,0) + x(1-y)f(1,0) + y(1-x)f(0,1) + xyf(1,1).$$

We have

$$\begin{aligned} |f(x,y) - Wf(x,y)| &= |(1-x)(1-y)(f(x,y)-f(0,0)) + x(1-y)(f(x,y)-f(1,0)) \\ &\quad + y(1-x)(f(x,y)-f(0,1)) + xy(f(x,y)-f(1,1))| \\ &\leqslant M(f)\,((1-x)(1-y)(x+y) + x(1-y)(1-x+y) \\ &\quad + y(1-x)(x+1-y) + xy(2-x-y)), \end{aligned}$$

for all $f \in C^1(H)$ and $(x,y) \in H$. Consequently,

$$|f(x,y) - Wf(x,y)| \leqslant 2(x - x^2 + y - y^2)M(f). \tag{5.1}$$

Let now $L : C(H) \to C$ be a positive linear operator, $L \neq I$, such that L preserves the affine functions. Let $p(x,y) = x^2$, $q(x,y) = y^2$, $(x,y) \in H$. Suppose that there exist $a,b,c \in \mathbb{R}$ such that

$$Lp(x,y) = ax^2 + bx + c, \ \ Lq(x,y) = ax^2 + bx + c, \ \ (x,y) \in H.$$

Set $u(x,y) = x - x^2$, $v(x,y) = y - y^2$, $(x,y) \in H$.

THEOREM 5.1. *For the above operator L one has $a \in [0,1)$ and $Lu = au$, $Lv = av$.*

Proof. The proof is similar to that of Theorem 4.1 and we omit the details. □

THEOREM 5.2. *Let $L : C(H) \to C(H)$ satisfying the hypotheses of Theorem 5.1. Suppose that, in addition, L preserves the function $(x,y) \in H \to xy$.*

(i) If $f \in C^1(H)$, then

$$|L^k f(x,y) - Wf(x,y)| \leqslant 2a^k(x - x^2 + y - y^2)M(f), \ \ k \in \mathbb{N}.$$

(ii) If $f \in C(H)$ and $Lf \in C^1(H)$, then

$$|L^{k+1} f(x,y) - Wf(x,y)| \leqslant 2a^k(x - x^2 + y - y^2)M(Lf), \ \ k \in \mathbb{N}.$$

Proof. To prove (i) it suffices to combine Theorem 5.1 with (5.1), observing that $LW = W$. Now (i) implies (ii) since $WL = W$. □

6. Applications II

6.1. Bernstein operators on the simplex S

Let $(x,y) \in S$, i.e., $x \geqslant 0$, $y \geqslant 0$, $x + y \leqslant 1$. For $n \in \mathbb{N}$ and $i, j \in \mathbb{N}_0$, $i + j \leqslant n$, let

$$p_{n,i,j}(x,y) = \frac{n!}{i!j!(n-i-j)!} x^i y^j (1 - x - y)^{n-i-j}.$$

The Bernstein operators $B_n : C(S) \to C(S)$ are defined by

$$B_n f(x,y) := \sum_{i+j \leqslant n} p_{n,i,j}(x,y) f\left(\frac{i}{n}, \frac{j}{n}\right).$$

They are positive linear operators preserving the affine functions. Let

$$u(x,y) = x - x^2, \ \ v(x,y) = y - y^2, \ \ (x,y) \in S.$$

Then

$$B_n u(x,y) = \frac{n-1}{n}(x-x^2), \;\; B_n v(x,y) = \frac{n-1}{n}(y-y^2).$$

Moreover,

$$\frac{\partial}{\partial x} B_n f(x,y) = n \sum_{i+j \leqslant n-1} p_{n-1,i,j}(x,y) \left[f\left(\frac{i+1}{n}, \frac{j}{n}\right) - f\left(\frac{i}{n}, \frac{j}{n}\right) \right]$$

and a similar formula for $\frac{\partial}{\partial y} B_n f(x,y)$. Consequently,

$$M(B_n f) = n \max_{i+j \leqslant n-1} \left\{ \left| f\left(\frac{i+1}{n}, \frac{j}{n}\right) - f\left(\frac{i}{n}, \frac{j}{n}\right) \right|, \left| f\left(\frac{i}{n}, \frac{j+1}{n}\right) - f\left(\frac{i}{n}, \frac{j}{n}\right) \right| \right\}.$$

So, we can apply Theorem 4.2 with $a = \frac{n-1}{n}$ and the above expression of $M(B_n f)$. Consequently, we get

COROLLARY 6.1. *(i) If $f \in C^1(S)$, then*

$$|B_n^k f(x,y) - U f(x,y)| \leqslant 2 \left(\frac{n-1}{n}\right)^k (x+y-x^2-y^2) M(f), \;\; k \in \mathbb{N}.$$

(ii) If $f \in C(S)$, then

$$|B_n^{k+1} f(x,y) - U f(x,y)| \leqslant 2 \left(\frac{n-1}{n}\right)^k (x+y-x^2-y^2) M(B_n f), \;\; k \in \mathbb{N}.$$

6.2. Genuine Bernstein-Durrmeyer operators on the simplex S

For $f \in C(S)$, the bivariate form of genuine Bernstein-Durrmeyer operators were considered in [1] as follows,

$$\begin{aligned} U_n(f)(x,y) &= f(0,0)(1-x-y)^n + f(1,0)x^n + f(0,1)y^n \\ &+ \sum_{l=1}^{n-1} p_{n,0,l}(x,y)(n-1) \int_0^1 p_{n-2,l-1}(t) f(0,t)\,dt \\ &+ \sum_{k=1}^{n-1} p_{n,k,0}(x,y)(n-1) \int_0^1 p_{n-2,k-1}(s) f(s,0)\,ds \\ &+ \sum_{k=1}^{n-1} p_{n,k,n-k}(x,y)(n-1) \int_0^1 p_{n-2,k-1}(t) f(t,1-t)\,dt \\ &+ \sum_{\substack{k+l \leqslant n-1 \\ k \geqslant 1, l \geqslant 1}} p_{n,k,l}(x,y)(n-1)(n-2) \int_0^1 \int_0^{1-t} p_{n-3,k-1,l-1}(s,t) f(s,t)\,ds\,dt \end{aligned}$$

where

$$p_{n,k,l}(x,y) := \frac{n!}{k!l!(n-k-l)!} x^k y^l (1-x-y)^{n-k-l}$$

with $k,l = 0,\dots,n,\ k+l \leqslant n,\ (x,y) \in S$.

These operators satisfy $U_n(f)(x,y) = f(x,y)$ at the vertices of S.

For $u(x,y) = x - x^2,\ v(x,y) = y - y^2$ we get

$$U_n u(x,y) = x(1-x)\frac{n-1}{n+1}, \quad U_n v(x,y) = y(1-y)\frac{n-1}{n+1}.$$

From Theorem 4.2 with $a = \dfrac{n-1}{n+1}$ we obtain

COROLLARY 6.2. *(i) If $f \in C^1(S)$, then*

$$|U_n^k f(x,y) - Uf(x,y)| \leqslant 2\left(\frac{n-1}{n+1}\right)^k (x+y-x^2-y^2) M(f), \quad k \in \mathbb{N}.$$

(ii) If $f \in C(S)$, then

$$|U_n^{k+1} f(x,y) - Uf(x,y)| \leqslant 2\left(\frac{n-1}{n+1}\right)^k (x+y-x^2-y^2) M(U_n f), \quad k \in \mathbb{N}.$$

6.3. The operators U_n^ρ on the simplex S

For $f \in C(S)$, $\rho > 0$, the bivariate form of the operators U_n^ρ were considered in [1] as follows,

$$\begin{aligned} U_n^\rho f(x,y) = {} & f(0,0)(1-x-y)^n + f(1,0)x^n + f(0,1)y^n \\ & + \sum_{l=1}^{n-1} F_{n,0,l}^\rho(f) p_{n,0,l}(x,y) + \sum_{k=1}^{n-1} F_{n,k,0}^\rho(f) p_{n,k,0}(x,y) \\ & + \sum_{k=1}^{n-1} F_{n,k,n-k}^\rho(f) p_{n,k,n-k}(x,y) + \sum_{\substack{k\geqslant 1,\ l\geqslant 1 \\ k+l\leqslant n-1,}} F_{n,k,l}^\rho(f) p_{n,k,l}(x,y), \end{aligned}$$

where

$$\begin{aligned} F_{n,0,l}^\rho(f) &:= \frac{\int_0^1 t^{l\rho-1}(1-t)^{(n-l)\rho-1} f(0,t)\,dt}{B(l\rho,(n-l)\rho)}, \\ F_{n,k,0}^\rho(f) &:= \frac{\int_0^1 s^{k\rho-1}(1-s)^{(n-k)\rho-1} f(s,0)\,ds}{B(k\rho,(n-k)\rho)}, \\ F_{n,k,n-k}^\rho(f) &:= \frac{\int_0^1 t^{k\rho-1}(1-t)^{(n-k)\rho-1} f(t,1-t)\,dt}{B(k\rho,(n-k)\rho)}, \\ F_{n,k,l}^\rho(f) &:= \frac{\iint_S s^{k\rho-1} t^{l\rho-1}(1-s-t)^{(n-k-l)\rho-1} f(s,t)\,ds\,dt}{\iint_S s^{k\rho-1} t^{l\rho-1}(1-s-t)^{(n-k-l)\rho-1}\,ds\,dt}. \end{aligned}$$

It can be easily seen that, for $\rho = 1$, we obtain the genuine Bernstein-Durrmeyer operators U_n.

For $u(x,y) = x - x^2$, $v(x,y) = y - y^2$ we get

$$U_n^{\rho} u(x,y) = x(1-x)\frac{\rho(n-1)}{n\rho+1}, \quad U_n^{\rho} v(x,y) = y(1-y)\frac{\rho(n-1)}{n\rho+1}.$$

From Theorem 4.2 with $a = \dfrac{\rho(n-1)}{n\rho+1}$ we obtain

COROLLARY 6.3. *(i) If $f \in C^1(S)$, then*

$$|(U_n^{\rho})^k f(x,y) - Uf(x,y)| \leqslant 2\left(\frac{\rho(n-1)}{n\rho+1}\right)^k (x+y-x^2-y^2)M(f), \quad k \in \mathbb{N}.$$

(ii) If $f \in C(S)$, then

$$|(U_n^{\rho})^{k+1} f(x,y) - Uf(x,y)| \leqslant 2\left(\frac{\rho(n-1)}{n\rho+1}\right)^k (x+y-x^2-y^2)M(U_n^{\rho} f), \quad k \in \mathbb{N}.$$

6.4. Beta operators on the simplex S

For $\rho \in (0,\infty)$, $f \in C(S)$ and $(x,y) \in S$, the Beta operators on the simplex S were introduced in [1] as follows,

$$\mathscr{B}_{\rho}(f)(x,y) = \begin{cases} f(x,y), \quad (x,y) \in \{(0,0),(1,0),(0,1)\}, \\ \dfrac{\int_0^1 s^{\rho x-1}(1-s)^{\rho(1-x)-1} f(s,0)ds}{B(\rho x, \rho(1-x))}, \quad x \in (0,1), \quad y=0, \\ \dfrac{\int_0^1 t^{\rho y-1}(1-t)^{\rho(1-y)-1} f(0,t)dt}{B(\rho y, \rho(1-y))}, \quad x=0, \quad y \in (0,1), \\ \dfrac{\int_0^1 u^{\rho x-1}(1-u)^{\rho(1-x)-1} f(u,1-u)du}{B(\rho x, \rho(1-x))}, \quad y = 1-x, \quad x \in (0,1), \\ \dfrac{\iint_S s^{\rho x-1} t^{\rho y-1}(1-s-t)^{\rho-\rho x-\rho y-1} f(s,t)dsdt}{\iint_S s^{\rho x-1} t^{\rho y-1}(1-s-t)^{\rho-\rho x-\rho y-1} dsdt}, \quad (x,y) \in int(S). \end{cases}$$

For $u(x,y) = x - x^2$, $v(x,y) = y - y^2$ we get

$$\mathscr{B}_{\rho} u(x,y) = x(1-x)\frac{\rho}{\rho+1}, \quad \mathscr{B}_{\rho} v(x,y) = y(1-y)\frac{\rho}{\rho+1}.$$

From Theorem 4.2 with $a = \dfrac{\rho}{\rho+1}$ we obtain

COROLLARY 6.4. *(i) If $f \in C^1(S)$, then*

$$|(\mathscr{B}_\rho)^k f(x,y) - Uf(x,y)| \leqslant 2\left(\frac{\rho}{\rho+1}\right)^k (x+y-x^2-y^2)M(f), \quad k \in \mathbb{N}.$$

(ii) If $f \in C(S)$ and $\mathscr{B}_\rho f \in C^1(S)$ then

$$|(\mathscr{B}_\rho)^{k+1} f(x,y) - Uf(x,y)| \leqslant 2\left(\frac{\rho}{\rho+1}\right)^k (x+y-x^2-y^2)M(\mathscr{B}_\rho f), \quad k \in \mathbb{N}.$$

REMARK 6.1. We the above notations it can be verified that $U_n^\rho = B_n \circ \mathscr{B}_{n\rho}$ on $C(S)$.

6.5. Bernstein operators on $H = [0,1]^2$

Let $(x,y) \in H$, $f \in C(H)$. The Bernstein operators $B_n : C(H) \to C(H)$ are defined by

$$B_n f(x,y) := \sum_{i=0}^{n}\sum_{j=0}^{n} b_{n,i}(x)b_{n,j}(y) f\left(\frac{i}{n},\frac{j}{n}\right).$$

They are positive linear operators preserving the affine functions. Then,

$$B_n u(x,y) = \frac{n-1}{n}(x-x^2), \quad B_n v(x,y) = \frac{n-1}{n}(y-y^2).$$

Moreover,

$$\frac{\partial}{\partial x} B_n f(x,y) = n\sum_{i=0}^{n-1}\sum_{j=0}^{n} b_{n-1,i}(x)b_{n,j}(y)\left[f\left(\frac{i+1}{n},\frac{j}{n}\right) - f\left(\frac{i}{n},\frac{j}{n}\right)\right],$$

and a similar formula for $\frac{\partial}{\partial y} B_n f(x,y)$. It follows that

$$M(B_n f) = n \max_{i,j=0,1,\dots,n}\left\{\left|f\left(\frac{i+1}{n},\frac{j}{n}\right) - f\left(\frac{i}{n},\frac{j}{n}\right)\right|, \left|f\left(\frac{i}{n},\frac{j+1}{n}\right) - f\left(\frac{i}{n},\frac{j}{n}\right)\right|\right\}.$$

Since B_n preserves the function $(x,y) \in H \to xy$, we can apply Theorem 5.2 with $a = \frac{n-1}{n}$ and the above $M(B_n f)$.

6.6. Genuine Bernstein-Durrmeyer operators on $H = [0,1]^2$

Let $(x,y) \in H$, $f \in C(H)$. The genuine Bernstein-Durrmeyer operators $U_n : C(H) \to C(H)$ are defined by

$$\begin{aligned} U_n f(x,y) &= f(0,0)b_{n,0}(x)b_{n,0}(y) + f(1,0)b_{n,n}(x)b_{n,0}(y) + f(0,1)b_{n,0}(x)b_{n,n}(y) \\ &\quad + f(1,1)b_{n,n}(x)b_{n,n}(y) + (n-1)^2\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} b_{n,i}(x)b_{n,j}(y) \\ &\quad \times \int_0^1\int_0^1 b_{n-2,i-1}(s)b_{n-2,j-1}(t)f(s,t)\,dsdt. \end{aligned}$$

They are positive linear operators preserving the affine functions. Then

$$U_n u(x,y) = \frac{n-1}{n+1}(x-x^2), \;\; U_n v(x,y) = \frac{n-1}{n+1}(y-y^2).$$

Since U_n preserves the function $(x,y) \in H \to xy$, we can apply Theorem 5.2 with $a = \dfrac{n-1}{n+1}$.

6.7. The operators U_n^ρ on $H = [0,1]^2$

Let $(x,y) \in H$, $f \in C(H)$, $\rho > 0$. The operators $U_n^\rho : C(H) \to C(H)$ are defined by

$$\begin{aligned} U_n^\rho f(x,y) = {} & f(0,0)b_{n,0}(x)b_{n,0}(y) + f(1,0)b_{n,n}(x)b_{n,0}(y) + f(0,1)b_{n,0}(x)b_{n,n}(y) \\ & + f(1,1)b_{n,n}(x)b_{n,n}(y) + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} b_{n,i}(x)b_{n,j}(y) \\ & \times \frac{\int_0^1\int_0^1 s^{i\rho-1}(1-s)^{(n-i)\rho-1}t^{j\rho-1}(1-t)^{(n-j)\rho-1}f(s,t)dsdt}{\int_0^1\int_0^1 s^{i\rho-1}(1-s)^{(n-i)\rho-1}t^{j\rho-1}(1-t)^{(n-j)\rho-1}dsdt}. \end{aligned}$$

They are positive linear operators preserving the affine functions. Then

$$U_n^\rho u(x,y) = \frac{\rho(n-1)}{n\rho+1}(x-x^2), \;\; U_n^\rho v(x,y) = \frac{\rho(n-1)}{n\rho+1}(y-y^2).$$

Since U_n^ρ preserves the function $(x,y) \in H \to xy$, we can apply Theorem 5.2 with $a = \dfrac{\rho(n-1)}{n\rho+1}$.

6.8. Beta operators on $H = [0,1]^2$

Let $(x,y) \in H$, $f \in C(H)$, $\rho > 0$. The Beta operators $\mathscr{B}_\rho : C(H) \to C(H)$ are defined by

$$\mathscr{B}_\rho f(x,y) = \frac{\int_0^1\int_0^1 s^{\rho x-1}(1-s)^{\rho(1-x)-1}t^{\rho y-1}(1-t)^{\rho(1-y)-1}f(s,t)dsdt}{\int_0^1\int_0^1 s^{\rho x-1}(1-s)^{\rho(1-x)-1}t^{\rho y-1}(1-t)^{\rho(1-y)-1}dsdt}.$$

They are positive linear operators preserving the affine functions. Then

$$\mathscr{B}_\rho u(x,y) = \frac{\rho}{\rho+1}(x-x^2), \;\; \mathscr{B}_\rho v(x,y) = \frac{\rho}{\rho+1}(y-y^2).$$

Since $\mathscr{B}_\rho$ preserves the function $(x,y) \in H \to xy$, we can apply Theorem 5.2 with $a = \dfrac{\rho}{\rho+1}$.

REMARK 6.2. With the above notations it can be verified that $U_n^\rho = B_n \circ \mathscr{B}_{n\rho}$ on $C(H)$.

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