Journal of Mathematical Inequalities
Volume 19, Number 1 (2025), 165–175
doi:10.7153/jmi-2025-19-11

A WEIGHTED WELCH INEQUALITY

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(*Communicated by S. Furuichi*)

Abstract. In this paper, we give a generalization of the Welch inequality, a weighted version in the presence of a normal matrix as a weight. Also, we obtain connections with tight frames and we give some examples.

1. Introduction

In 1974, L. R. Welch [14] proved the following inequality

$$\sum_{i,j=1}^{N} |\langle f_i, f_j\rangle|^2 \geqslant \frac{N^2}{n}, \tag{1}$$

for $f_1, f_2, \ldots, f_N$ unit vectors in $\mathbb{C}^n$, $N \geqslant n$. Here and in the following, $\langle \cdot,\cdot\rangle$ denotes the inner product in $\mathbb{C}^n$ i.e,

$$\langle z, w\rangle = \sum_{i=1}^{n} z_i \overline{w_i}, \quad \text{for } z, w \in \mathbb{C}^n.$$

The unit vectors for which we have equality in the above inequality are called Welch Bound Equality sequences (in short, WBE sequences). They were used for Code-Division Multiple-Acces Systems (CDMA systems) [10]. From the above inequality, L. R. Welch gave a lower bound on the maximal cross corelation:

$$\max_{i\neq j} |\langle f_i, f_j\rangle| \geqslant \sqrt{\frac{N-n}{n(N-1)}}. \tag{2}$$

The Welch's inequalities were used also in compressed sensing [5], in connection to potential energy [1] and informationally complete quantum measurements [11, 12].

A generalization of (1) to a finite number of vectors which need not to have unit norm was given in 2003 by S. Waldron [13]:

$$\sum_{i,j=1}^{N} |\langle f_i, f_j\rangle|^2 \geqslant \frac{1}{n}\left(\sum_{i=1}^{N} \|f_i\|^2\right)^2. \tag{3}$$

Mathematics subject classification (2020): 15Axx, 42C15.
Keywords and phrases: Welch inequality, frames.

It is proven that we have equality in (3) if and only if the family $\{f_1,\dots,f_N\}$ is a tight frame. In [7], P. Găvruţa gave an extension of Waldron's result to an infinite family of elements, which is a Bessel sequence, with a different proof.

Frames were introduced by R. J. Duffin and A. C. Schaffer [6] in 1952, while working on some problems concerning nonharmonic Fourier series. For many years frames were not paid attention to until the fundamental paper of I. Daubechies, A. Grossman and Y. Meyer [3] in 1986, where they were brought to life and their importance to signal processing was shown.

In the following we recall the definition of a frame. Let $\mathbb{F}^n$ $(\mathbb{F}=\mathbb{R},\ \mathbb{C})$ be an n-dimensional space, with inner product.

DEFINITION 1.1. A finite sequence of vectors $\{f_k\}_{k=1}^N$ in $\mathbb{F}^n$, with $N \geqslant n$, is a frame if there exists a,b strictly positive constants such that, for all $x \in \mathbb{F}$,

$$a\|x\|^2 \leqslant \sum_{k=1}^{N} |\langle x, f_k\rangle|^2 \leqslant b\|x\|^2.$$

a and b called the lower, respectively, upper frame bound and they are not unique. If $a=b$, we say that the frame is a-*tight* and a *Parseval frame* if $a=b=1$.

A useful characterization for tight frames is the following.

PROPOSITION 1.1. [8] *Let $M=[f_1 \quad f_2 \quad \dots \quad f_N]$ be an $n\times N$ matrix with f_i being the column vectors of M. Then $\{f_1,f_2,\dots,f_N\}$ is a tight frame for $\mathbb{C}^n$ if and only if the set of row vectors of M is a pairwise orthogonal collection of vectors all having the same norm.*

Tight frames can be obtained in $\mathbb{C}^n$ by projecting the discrete Fourier transform basis in any $\mathbb{C}^N$, $N>n$, onto $\mathbb{C}^n$:

PROPOSITION 1.2. [2] *Let $N>n$ and define the vectors $\{f_j\}_{j=1}^N$ in $\mathbb{C}^n$ by*

$$f_j = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ e^{2\pi i \frac{j-1}{N}} \\ \vdots \\ e^{2\pi i (n-1)\frac{j-1}{N}} \end{pmatrix}, \quad j=1,\dots N.$$

Then $\{f_j\}_{j=1}^N$ is a tight frame for $\mathbb{C}^n$ with frame bound equal to 1 and $\|f_j\| = \sqrt{\dfrac{n}{N}}$, for all j.

From this Proposition, we have the following Corollary.

COROLLARY 1.1. [2] *For any $N \geqslant n$, there exists a tight frame in $\mathbb{C}^n$ consisting of N normalized vectors.*

In the following, we give a pedagogical proof of the Welch inequality and, with the same technique, we give a weighted version of the Welch inequality in the presence of a normal matrix as a weight. Also, we give an operatorial proof for the weighted Welch inequality when the weight is a positive matrix. Some examples and connections with tight frames are also given.

2. A pedagogical proof of the Welch inequality

Let $\mathscr{F} = \{f_1, f_2, \ldots, f_N\}$ be $N \geqslant n$ vectors in $\mathbb{R}^n$(or $\mathbb{C}^n$). In the following, we give another proof of (3), different from the ones in [7] and [13].

Let be

$$f_i = \begin{pmatrix} f_{i1} \\ f_{i2} \\ \vdots \\ f_{in} \end{pmatrix}, \quad 1 \leqslant i \leqslant N.$$

Proof. We consider the associated matrix

$$\begin{pmatrix} f_{11} & f_{21} & \cdots & f_{N1} \\ f_{12} & f_{22} & \cdots & f_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ f_{1n} & f_{2n} & \cdots & f_{Nn} \end{pmatrix}$$

Let f^p be the row vector of this matrix, from the position p, i.e.

$$f^p = (f_{1p}, f_{2p}, \ldots, f_{Np}),\ 1 \leqslant p \leqslant n.$$

Next, we show that

$$\sum_{i,j=1}^{N} |\langle f_i, f_j \rangle|^2 = \sum_{p,q=1}^{n} |\langle f^p, f^q \rangle|^2. \tag{4}$$

For a fixed i, $1 \leqslant i \leqslant N$, we have

$$\begin{aligned} \sum_{j=1}^{N} |\langle f_i, f_j \rangle|^2 &= \sum_{j=1}^{N} \langle f_i, f_j \rangle \overline{\langle f_i, f_j \rangle} \\ &= \sum_{j=1}^{N} \left(\sum_{p=1}^{n} f_{ip} \overline{f_{jp}} \right) \left(\overline{\sum_{q=1}^{n} f_{iq} \overline{f_{jq}}} \right) \\ &= \sum_{p,q=1}^{n} f_{ip} \overline{f_{iq}} \sum_{j=1}^{N} f_{jq} \overline{f_{jp}} \\ &= \sum_{p,q=1}^{n} f_{ip} \overline{f_{iq}} \langle f^q, f^p \rangle \end{aligned}$$

and by summation after i we obtain (4).

We use the following elementary inequality for positive numbers

$$n\Big(\sum_{p=1}^{n}\alpha_p^2\Big)\geqslant\Big(\sum_{p=1}^{n}\alpha_p\Big)^2, \tag{5}$$

with equality if and only if $\alpha_1=\alpha_2=\ldots=\alpha_n$.

We also have

$$\sum_{j=1}^{N}\|f_j\|^2=\sum_{j=1}^{N}\sum_{p=1}^{n}|f_{jp}|^2 \quad\text{and}\quad \sum_{p=1}^{n}\|f^p\|^2=\sum_{p=1}^{n}\sum_{j=1}^{N}|f_{jp}|^2$$

hence

$$\sum_{j=1}^{N}\|f_j\|^2=\sum_{p=1}^{n}\|f^p\|^2.$$

Using the above equality and relations (4) and (5), we have

$$\begin{aligned} &n\sum_{i,j=1}^{N}|\langle f_i,f_j\rangle|^2-\Big(\sum_{j=1}^{N}\|f_j\|^2\Big)^2\\ &=n\sum_{p,q=1}^{n}|\langle f^p,f^q\rangle|^2-\Big(\sum_{p=1}^{n}\|f^p\|^2\Big)^2\\ &=n\sum_{p=1}^{n}\|f^p\|^4-\Big(\sum_{p=1}^{n}\|f^p\|^2\Big)^2+n\sum_{p\neq q}|\langle f^p,f^q\rangle|^2\geqslant 0\end{aligned}$$

With equality iff $\|f^p\|^2=\|f^q\|^2$, for all $p,q, 1\leqslant p,q\leqslant n$ and $f^p\perp f^q,\ p\neq q$. Next, we apply Proposition 1.1 to obtain the conclusion. □

3. A weighted Welch inequality with a normal matrix weight

Next, we prove the following Theorem, which is a generalization of the Welch inequality.

In the following, we denote by $\Re\lambda$ the real part of λ, where λ is an arbitrary complex number.

THEOREM 3.1. *Let be $\mathscr{F}=\{f_1,f_2,\ldots,f_N\}\subset\mathbb{C}^n$, with $N\geqslant n$ a sequence of vectors and A be a normal $n\times n$ matrix, $A\neq 0$, with eigenvalues $\{\lambda_1,\ldots,\lambda_n\}$. We suppose that $\Re\overline{\lambda_k}\lambda_l\geqslant 0$, $(\forall)\ k,l=1,\ldots,n$. Then*

$$\sum_{i,j=1}^{N}|\langle f_i,Af_j\rangle|^2\geqslant\frac{\Big(\sum_{i=1}^{N}\|Af_i\|^2\Big)^2}{\sum_{k=1}^{n}|\lambda_k|^2}. \tag{6}$$

If $\Re\overline{\lambda_k}\lambda_l>0$, $(\forall)\ k,l=1,\ldots,n$, we have equality in relation (6) if and only if $\mathscr{F}$ is a tight frame.

Proof. From the Spectral Theorem, there exists

$$\mathscr{E} = \{e_1, \dots, e_n\}$$

an orthonormal basis of $\mathbb{C}^n$ which consists from the eigenvalues of A

$$Ae_k = \lambda_k e_k, \quad k = 1, \dots, n.$$

We can write

$$f_j = \sum_{k=1}^{n} f_j(k) e_k, \quad j = 1, 2, \dots, N.$$

Then

$$Af_j = \sum_{k=1}^{n} f_j(k) A e_k = \sum_{k=1}^{n} f_j(k) \lambda_k e_k.$$

Let $x = \sum_{k=1}^{n} x(k) e_k$ be an arbitrary vector from $\mathbb{C}^n$. We also have

$$\langle x, Af_j \rangle = \sum_{k=1}^{n} x(k) \overline{f_j(k) \lambda_k}.$$

So

$$\begin{aligned} |\langle x, Af_j \rangle|^2 &= \langle x, Af_j \rangle \overline{\langle x, Af_j \rangle} \\ &= \left(\sum_{k=1}^{n} x(k) \overline{f_j(k) \lambda_k} \right) \left(\sum_{l=1}^{n} \overline{x(l)} f_j(l) \lambda_l \right) \\ &= \sum_{k,l=1}^{n} x(k) \overline{x(l) f_j(k)} f_j(l) \overline{\lambda_k} \lambda_l \end{aligned}$$

Thus

$$\begin{aligned} \sum_{j=1}^{N} |\langle x, Af_j \rangle|^2 &= \sum_{j=1}^{N} \left(\sum_{k,l=1}^{n} x(k) \overline{x(l) f_j(k)} f_j(l) \overline{\lambda_k} \lambda_l \right) \\ &= \sum_{k,l=1}^{n} \sum_{j=1}^{N} x(k) \overline{x(l) f_j(k)} f_j(l) \overline{\lambda_k} \lambda_l \\ &= \sum_{k,l=1}^{n} x(k) \overline{x(l) \lambda_k} \lambda_l \sum_{j=1}^{N} \overline{f_j(k)} f_j(l) \end{aligned}$$

Let be the matrix

$$\begin{pmatrix} f_1(1) & f_2(1) & \cdots & f_N(1) \\ f_1(2) & f_2(2) & \cdots & f_N(2) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(n) & f_2(n) & \cdots & f_N(n) \end{pmatrix} \tag{7}$$

and

$$f^k = (f_1(k), f_2(k), \ldots, f_N(k)), \quad k = 1, \ldots, n$$

Then

$$\langle f^l, f^k \rangle = \sum_{j=1}^{N} f_j(l)\overline{f_j(k)}.$$

We obtain

$$\sum_{j=1}^{N} |\langle x, Af_j \rangle|^2 = \sum_{k,l=1}^{n} \overline{\lambda_k}\lambda_l x(k)\overline{x(l)}\langle f^l, f^k \rangle \tag{8}$$

In equation (8) we take $x = f_i$, $i \in \{1, 2, \ldots N\}$ and sum after i to get

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{j=1}^{N} |\langle f_i, Af_j \rangle|^2 &= \sum_{i=1}^{N}\sum_{k,l=1}^{n} \overline{\lambda_k}\lambda_l f_i(k)\overline{f_i(l)}\langle f^l, f^k \rangle \\
&= \sum_{k,l=1}^{n} \overline{\lambda_k}\lambda_l \langle f^l, f^k \rangle \sum_{i=1}^{N} f_i(k)\overline{f_i(l)} \\
&= \sum_{k,l=1}^{n} \overline{\lambda_k}\lambda_l \langle f^l, f^k \rangle \langle f^k, f^l \rangle.
\end{aligned}$$

Thus, we obtain

$$\sum_{i,j=1}^{N} |\langle f_i, Af_j \rangle|^2 = \sum_{k,l=1}^{n} \overline{\lambda_k}\lambda_l \langle f^l, f^k \rangle \langle f^k, f^l \rangle. \tag{9}$$

We use the following Lagrange identity for complex numbers

$$\left(\sum_{k=1}^{n} |z_k|^2\right)\left(\sum_{k=1}^{n} |w_k|^2\right) - \left|\sum_{k=1}^{n} z_k w_k\right|^2 = \sum_{1 \leqslant k < l \leqslant n} |z_l\overline{w_k} - z_k\overline{w_l}|^2.$$

In the above equality, we take

$$z_k = \lambda_k, \quad w_k = \overline{\lambda_k}\langle f^k, f^k \rangle, \quad k = 1, \ldots, n$$

and we obtain

$$\begin{aligned}
&\left(\sum_{k=1}^{n} |\lambda_k|^2\right)\left(\sum_{k=1}^{n} |\lambda_k|^2 |\langle f^k, f^k \rangle|^2\right) - \left|\sum_{k=1}^{n} |\lambda_k|^2 \langle f^k, f^k \rangle\right|^2 \\
&= \sum_{1 \leqslant k < l \leqslant n} |\lambda_k|^2 |\lambda_l|^2 |\langle f^k, f^k \rangle - \langle f^l, f^l \rangle|^2
\end{aligned} \tag{10}$$

Above we saw that

$$Af_i = \sum_{k=1}^{n} f_i(k)\lambda_k e_k.$$

This implies that

$$\|Af_i\|^2 = \sum_{k=1}^{n} |f_i(k)|^2 |\lambda_k|^2.$$

On the other hand,

$$\|f^k\|^2 = \sum_{i=1}^{N} |f_i(k)|^2$$

and

$$\sum_{i=1}^{N} \|Af_i\|^2 = \sum_{i=1}^{N}\sum_{k=1}^{n} |f_i(k)|^2|\lambda_k|^2 = \sum_{k=1}^{n}\sum_{i=1}^{N} |f_i(k)|^2|\lambda_k|^2$$
$$= \sum_{k=1}^{n} |\lambda_k|^2\|f^k\|^2 = \sum_{k=1}^{n} |\lambda_k|^2\langle f^k, f^k\rangle.$$

Also, from equation (9), we have that

$$\sum_{i,j=1}^{N} |\langle f_i, Af_j\rangle|^2 = \sum_{k,l=1}^{n} \overline{\lambda_k}\lambda_l\langle f^l, f^k\rangle\langle f^k, f^l\rangle$$
$$= \sum_{k=1}^{n} |\lambda_k|^2|\langle f^k, f^k\rangle|^2 + \sum_{k,l,k\neq l} \overline{\lambda_k}\lambda_l|\langle f^k, f^l\rangle|^2$$

Thus, from (10), we obtain

$$\left(\sum_{k=1}^{n} |\lambda_k|^2\right)\left(\sum_{i,j=1}^{N} |\langle f_i, Af_j\rangle|^2\right) - \left(\sum_{i=1}^{N} \|Af_i\|^2\right)^2$$
$$= \left(\sum_{k=1}^{n} |\lambda_k|^2\right)\left(\sum_{k=1}^{n} |\lambda_k|^2|\langle f^k, f^k\rangle|^2 + \sum_{k,l=1,k\neq l}^{n} \overline{\lambda_k}\lambda_l|\langle f^k, f^l\rangle|^2\right) - \left(\sum_{k=1}^{n} |\lambda_k|^2\|f^k\|^2\right)^2$$
$$= \sum_{1\leqslant k<l\leqslant n} |\lambda_k|^2|\lambda_l|^2|\langle f^k, f^k\rangle - \langle f^l, f^l\rangle|^2 + \left(\sum_{k=1}^{n} |\lambda_k|^2\right)\left(\sum_{k,l=1,k\neq l}^{n} \overline{\lambda_k}\lambda_l|\langle f^k, f^l\rangle|^2\right).$$

So

$$\left(\sum_{k=1}^{n} |\lambda_k|^2\right)\left(\sum_{i,j=1}^{N} |\langle f_i, Af_j\rangle|^2\right) - \left(\sum_{i=1}^{N} \|Af_i\|^2\right)^2$$
$$= \sum_{1\leqslant k<l\leqslant n} |\lambda_k|^2|\lambda_l|^2(\|f^k\|^2 - \|f^l\|^2)^2$$
$$+ 2\left(\sum_{k=1}^{n} |\lambda_k|^2\right)\left(\sum_{1\leqslant k<l\leqslant n} \Re\overline{\lambda_k}\lambda_l|\langle f^k, f^l\rangle|^2\right).$$

The conclusion follows from Proposition 1.1. □

COROLLARY 3.1. *Let A be a unitary matrix with $\Re\overline{\lambda_k}\lambda_l > 0$, $(\forall)\ k,l = 1,2,\ldots,n$. Then it takes place the following inequality*

$$\sum_{i,j=1}^{N} |\langle f_i, Af_j\rangle|^2 \geqslant \frac{1}{n}\left(\sum_{i=1}^{N} \|f_i\|^2\right)^2$$

and the equality occurs if and only if $\mathscr{F}$ is a tight frame.

In the case when A is identity matrix, we have exactly Welch's result.

REMARK 3.1. The condition $\Re\overline{\lambda_k}\lambda_l \geqslant 0$ in Theorem 3.1 is essential for (6) to take place.

Indeed, for the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$

the eigenvalues are $\lambda_1 = -1,\ \lambda_2 = 3$ so $\Re\lambda_1\lambda_2 < 0$.

In this case, the inequality (6), for a single vector is

$$|\langle x, Ax\rangle|^2 \geqslant \frac{\|Ax\|^4}{|\lambda_1|^2 + |\lambda_2|^2}, \quad \text{where } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

which is equivalent with

$$(x_1^2 + 4x_1x_2 + x_2^2)^2 \geqslant \frac{(5x_1^2 + 8x_1x_2 + 5x_2^2)^2}{10}.$$

This can not take place for $x_2 = 0$ and x_1 arbitrary.

In the previous example, the matrix A is self-adjoint. We can give an example of an unitary matrix. Indeed, the matrix

$$A = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

is unitary and its eigenvalues are $\lambda_1 = -1,\ \lambda_2 = 1$ so $\Re\overline{\lambda_1}\lambda_2 = -1 < 0$.

In this case the inequality (6) for a single vector is equivalent with

$$|\langle x, Ax\rangle|^2 \geqslant \frac{\|Ax\|^4}{2}$$

i.e. with

$$\sqrt{2}|-x_1^2 + 2x_1x_2 - x_2^2| \geqslant x_1^2x_2^2, \quad \text{for } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

This it can not take place if $x_2 = (\sqrt{2} - 1)x_1$.

4. An operatorial proof for the weighted Welch inequality when the weight is a positive definite matrix

An operatorial proof of the Welch inequality (3) was given in [9]. In the following we give an operatorial proof for the weighted Welch inequality when the weight is a positive definite matrix A. This result follows immediately from Theorem 3.1.

THEOREM 4.1. *Let $\{f_i\}_{i=1}^N$ be a collection of N vectors in $\mathbb{C}^n$ and A be a positive definite matrix. Then it takes place the following inequality*

$$\sum_{i,j=1}^{N} |\langle f_i, Af_j\rangle|^2 \geqslant \frac{\left(\sum_{i=1}^{N} \|Af_i\|^2\right)^2}{\sum_{k=1}^{n} \lambda_k^2}.$$

The equality holds if and only if $\mathscr{F}$ is a tight frame.

An operatorial proof of Theorem 4.1. We consider the Hilbert space of all $n \times n$ complex matrices equipped with the inner product

$$\langle B, C\rangle := \mathrm{Tr}(B^*C)$$

Here, B^* is the standard adjoint (conjugate transpose) of B and for an arbitrary $n \times n$ matrix D, $\mathrm{Tr}(D)$ is the trace of the matrix. Let F be the $n \times N$ matrix that has f_i as its i^{th} column (i.e., the matrix (7))

For a $n \times n$ positive definite matrix A, the cycle property of the trace gives

$$\begin{aligned}\|AF\|^2 &= \langle AF, AF\rangle = \mathrm{Tr}[(AF)^*(AF)] \\ &= \mathrm{Tr}(F^*AAF) = \mathrm{Tr}[A(A^{\frac{1}{2}}FF^*A^{\frac{1}{2}})],\end{aligned}$$

i.e.,

$$\|AF\|^2 = \langle A, A^{\frac{1}{2}}FF^*A^{\frac{1}{2}}\rangle.$$

By the Cauchy-Schwarz inequality, we have

$$\|AF\|^2 = \langle A, A^{\frac{1}{2}}FF^*A^{\frac{1}{2}}\rangle \leqslant \|A\|\|A^{\frac{1}{2}}FF^*A^{\frac{1}{2}}\|.$$

We have equality if and only if A is a scalar multiple of $A^{\frac{1}{2}}FF^*A^{\frac{1}{2}}$, namely iff $\{f_i\}_{i=1}^N$ is a tight frame. This inequality is a restatement of the result of Theorem 3.1. To elaborate, squaring both sides gives

$$\|AF\|^4 \leqslant \|A\|^2\|A^{\frac{1}{2}}FF^*A^{\frac{1}{2}}\|^2$$

Here, AF is the $n \times N$ matrix whose i^{th} column is Af_i, meaning

$$\|AF\|^2 = \sum_{i=1}^{N} \|Af_i\|^2.$$

Meanwhile, letting $\{\lambda_k\}_{k=1}^n$ be the eigenvalues of A gives

$$\|A\|^2 = \mathrm{Tr}(A^2) = \sum_{k=1}^{n} \lambda_k^2$$

Finally, since F^*AF is $N \times N$ matrix whose $(i,j)^{\text{th}}$ entry is $\langle f_i, Af_j \rangle$,

$$\begin{aligned}\|A^{\frac{1}{2}}FF^*A^{\frac{1}{2}}\|^2 &= \mathrm{Tr} A^{\frac{1}{2}}FF^*A^{\frac{1}{2}}A^{\frac{1}{2}}FF^*A^{\frac{1}{2}} \\ &= \mathrm{Tr}[(F^*AF)^2] = \|F^*AF\|^2 \\ &= \sum_{i,j=1}^{N} |\langle f_i, Af_j \rangle|^2\end{aligned}$$

Putting all of this together, the inequality $\|AF\|^4 \leqslant \|A\|^2 \|A^{\frac{1}{2}}FF^*A^{\frac{1}{2}}\|^2$ becomes

$$\left(\sum_{i=1}^{N} \|Af_i\|^2\right)^2 \leqslant \left(\sum_{k=1}^{n} \lambda_k^2\right)\left(\sum_{i,j=1}^{N} |\langle f_i, Af_j \rangle|^2\right) \quad \square$$

COROLLARY 4.1. *Let be $\{f_j\}_{j=1}^N \subset \mathbb{C}$ so that $\{Af_j\}_{j=1}^N$ is a $a-$tight frame for A positive definite matrix. Then*

$$a = \frac{\sum_{j=1}^{N} \|f_j\|^2}{\sum_{k=1}^{n} \frac{1}{\lambda_k^2}}.$$

Proof. If $\{Af_j\}_{j=1}^N$ is a $a-$tight frame and A is positive definite, then

$$AFF^*A = (AF)(AF)^* = aI$$

and so $FF^* = aA^{-2}$ implying

$$\sum_{j=1}^{N} \|f_j\|^2 = \mathrm{Tr}(F^*f) = \mathrm{Tr}(FF^*) = \mathrm{Tr}(aA^{-2}) = a\sum_{k=1}^{n} \lambda_k^{-2}. \quad \square$$

OPEN PROBLEM. The inequality (1) was generalized in the following form

$$\sum_{i=1}^{N} |\langle f_i, f_j \rangle|^{2k} \geqslant \frac{N^2}{\binom{n+N-1}{k}} \tag{11}$$

for the unit vectors $\{f_i\}_{i=1}^N$ in $\mathbb{C}^n$ and $k \geqslant 1$ integer. The problem is to generalize (6) for all $k \geqslant 1$ to incorporate (11).

Acknowledgements. After this paper was seen as a preprint, we received useful suggestions and observations. We would like to thank colleagues and references who contributed to the improvement of this work.

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(Received June 12, 2024)

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