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OPERATOR INEQUALITIES FOR h–CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. In this paper, we generalize the operator version of Jensen's inequality and the converse one for the class of h-convex functions. We extend the Hermite-Hadamard's type inequality and a multiple operator version of Jensen's inequality for this class of functions. We also provide a refinement of Jensen's inequality for convex functions. In particular, the operator h-convexity can be reduced to usual h-convexity in some sense and some results for the other classes of functions can be deduced by choosing an appropriate function h. The superiority of our results is that our results can recover some known results.

1. Introduction

Throughout this paper, let $\mathscr{H}$ be a Hilbert space and $B(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$. Let $A \in B(\mathscr{H})$. The operator $U \in B(\mathscr{H})$ is the adjoint of the operator A if $\langle Ax, y\rangle = \langle x, Uy\rangle$ for every $x, y \in \mathscr{H}$. The operator U is denoted by A^* and we say that the operator A is self-adjoint if $A = A^*$. The subalgebra of all self-adjoint operators in $B(\mathscr{H})$ is denoted by $B_{sa}(\mathscr{H})$. An operator A in $B_{sa}(\mathscr{H})$ is positive whenever $\langle Ax, x\rangle \geqslant 0$ for all $x \in \mathscr{H}$ and we write $A \geqslant 0$. W denote by $Sp(A)$ the spectrum of an operator $A \in B(\mathscr{H})$.

The convexity of functions is an important issue in many fields of science, for instance in economy and optimization. A function $f : \mathbb{I} \to \mathbb{R}$, $\mathbb{I} \subseteq \mathbb{R}$ is convex whenever the following inequality

$$f(\lambda u + (1-\lambda)v) \leqslant \lambda f(u) + (1-\lambda) f(v)$$

holds for all $u, v \in \mathbb{I}$, for all $\lambda \in [0,1]$ and the function $f : \mathbb{I} \to \mathbb{R}$ is concave whenever $-f$ is convex.

In 1979, Breckner [2] introduced the class of s-convex functions in the second sense. A function $f : [0,\infty) \to \mathbb{R}$ is s-convex in the second sense whenever

$$f(\lambda u + (1-\lambda)v) \leqslant \lambda^s f(u) + (1-\lambda)^s f(v) \tag{1}$$

holds for all $u, v \in [0,\infty)$, for all $\lambda \in [0,1]$ and for some fixed $s \in (0,1]$. Note that all s-convex functions in the second sense are non-negative. Hudzik and Maligranda

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(1994) [13] remarked two senses of s-convexity of real-valued functions are known in the literature. A function $f:[0,\infty)\to\mathbb{R}$ is s-convex in the first sense if

$$f(\alpha u+\beta v)\leqslant \alpha^s f(u)+\beta^s f(v) \tag{2}$$

holds for all $u,v\in[0,\infty)$, for all $\alpha,\beta\geqslant 0$ with $\alpha^s+\beta^s=1$ and for some fixed $s\in(0,1]$. There is an identity between the class of 1-convex functions and the class of convex functions. Indeed, the s-convexity means just the convexity when $s=1$, no mater in the first sense or in the second sense. For more details and examples on s-convex functions we refer to see [4, 9, 11, 13, 17, 18, 22].

In 1985, Godunova and Levin (see [14]) introduced the class of Godunova-Levin functions. A function $f:\mathbb{I}\to\mathbb{R}$ is a Godunova-Levin function on $\mathbb{I}$ if

$$f(\lambda u+(1-\lambda)v)\leqslant \frac{f(u)}{\lambda}+\frac{f(v)}{1-\lambda}, \tag{3}$$

where $u,v\in\mathbb{I}$ and $\lambda\in(0,1)$. Note that all non-negative monotonic and non-negative convex functions belong to this class [7]. The function f is s-Godunova-Levin type if

$$f(\lambda u+(1-\lambda)v)\leqslant \frac{f(u)}{\lambda^s}+\frac{f(v)}{(1-\lambda)^s}, \tag{4}$$

where $u,v\in\mathbb{I}$ and $\lambda\in(0,1)$.

In 1999, Pearce and Rubinov [21] introduced a new class of convex functions which is called P-class functions. A function $f:\mathbb{I}\to\mathbb{R}$ is a P-class function on $\mathbb{I}$ if

$$f(\lambda u+(1-\lambda)v)\leqslant f(u)+f(v), \tag{5}$$

where $u,v\in\mathbb{I}$ and $\lambda\in[0,1]$. The inequalities (1) and (2) reduce to P-class functions when $s\to 0$. Some properties of P-class functions can be found in [7, 8, 16].

In 2007, in order to unify the above concepts for functions of real variable Varošanec [23] introduced a wide class of functions the so called h-convex functions which generalizes convex, s-convex, Godunova-Levin, and P-class functions. A non-negative function $f:\mathbb{I}\to\mathbb{R}$ is h-convex on $\mathbb{I}$ if

$$f(\lambda u+(1-\lambda)v)\leqslant h(\lambda)f(u)+h(1-\lambda)f(v), \tag{6}$$

where h is a non-negative function defined on the real interval $\mathbb{J}$, $u,v\in\mathbb{I}$ and $\lambda\in[0,1]\subseteq\mathbb{J}$. For more results and generalizations regarding h-convexity, we refer the readers to see [1, 5, 12, 20]. For other types of convexity, we refer the readers to see [3, 19].

Jensen's inequality for convex functions is one of the most important result in the theory of inequalities and many other famous inequalities are particular cases of this inequality. An operator version of the Jensen inequality for a convex function has been proved by Mond and Pečarić as follows ([15], [10]):

THEOREM 1. *Let $f:[m,M]\to\mathbb{R}$ be a convex function. Then,*

$$f(\langle Ax,x\rangle)\leqslant\langle f(A)x,x\rangle$$

for every $x\in\mathscr{H}$ with $\langle x,x\rangle=1$ and every self-adjoint operator A such that $mI\leqslant A\leqslant MI$.

In this paper, we prove some inequalities for self-adjoint operators on a Hilbert space including an operator version of Jensen's inequality and its converse for h-convex functions. Moreover, we refine Jensen's inequality for convex functions. We prove the Hermite-Hadamard's type inequality and a multiple operator version of Jensen's inequality for h-convex functions. In particular, we obtain Jensen's inequality for non-negative convex, P-class, s-convex, Godunova-Levin, and s-Godunova-Levin functions by choosing an appropriate function h. We show that the operator h-convexity can be reduced to usual h-convexity in some sense.

2. Mond-Pečarić inequality for h-convex functions

We indicate that an operator version of the Jensen inequality for h-convex functions still holds similar to that Mond-Pečarić considered for convex functions.

THEOREM 2. *Let A be a self-adjoint operator on a Hilbert space $\mathscr{H}$ and assume that $Sp(A)\subseteq[m,M]$ for some scalars m,M with $0<m<M$. Let $h:[0,1]\to\mathbb{R}$ be a non-negative and non-zero function. If f is a continuous h-convex function on $[m,M]$, then*

$$f(\langle Ax,x\rangle)\leqslant 2h\left(\frac{1}{2}\right)\langle f(A)x,x\rangle \tag{7}$$

for each $x\in\mathscr{H}$ with $||x||=1$.

Proof. It follows from h-convexity of f that

$$f\left(\frac{a+b}{2}\right)-h\left(\frac{1}{2}\right)f(b)\leqslant h\left(\frac{1}{2}\right)f(a) \tag{8}$$

for all $a,b\in[m,M]$. By dividing both sides of (8) with $\frac{1}{2}$, one can reach

$$\frac{f\left(\frac{a+b}{2}\right)-h\left(\frac{1}{2}\right)f(b)}{\frac{1}{2}}\leqslant\frac{h\left(\frac{1}{2}\right)}{\frac{1}{2}}f(a) \tag{9}$$

for all $a,b\in[m,M]$. Define

$$\alpha:=\min_{b\in[m,M]}\frac{f\left(\frac{a+b}{2}\right)-h\left(\frac{1}{2}\right)f(b)}{\frac{1}{2}(a-b)}. \tag{10}$$

The inequalities (9) and (10) entail that

$$\alpha(a-b)\leqslant\frac{h\left(\frac{1}{2}\right)}{\frac{1}{2}}f(a) \tag{11}$$

for all $a,b \in [m,M]$. Consider the linear function $l(t) := \alpha(t-b)$. The inequality (11) implies $l(a) \leqslant 2h\left(\frac{1}{2}\right)f(a)$ for all $a \in [m,M]$. Put $\bar{g} = \langle Ax,x\rangle$. So, it is clear that $m \leqslant \bar{g} \leqslant M$. Consider the straight line $l'(t) := \alpha(t-\bar{g}) + f(\bar{g})$ passing through the point $(\bar{g}, f(\bar{g}))$ and parallel to the line l. The continuity of the function f ensures that

$$l'(\bar{g}) \geqslant f(\bar{g}) - \varepsilon \tag{12}$$

for all $\varepsilon > 0$. We now consider two cases:

(i) Assume that $l'(t) \leqslant 2h\left(\frac{1}{2}\right)f(t)$ for every $t \in [m,M]$. By using the functional calculus, one has $l'(A) \leqslant 2h\left(\frac{1}{2}\right)f(A)$ and consequently

$$\langle l'(A)x,x\rangle \leqslant 2h\left(\frac{1}{2}\right)\langle f(A)x,x\rangle \tag{13}$$

for each $x \in \mathscr{H}$ with $||x|| = 1$. The linearity of the function l' and the inequalities (12) and (13) imply

$$f(\langle Ax,x\rangle) - \varepsilon \leqslant l'(\langle Ax,x\rangle) = \langle l'(A)x,x\rangle \leqslant 2h\left(\frac{1}{2}\right)\langle f(A)x,x\rangle.$$

Since ε is arbitrary, we observe that

$$f(\langle Ax,x\rangle) \leqslant 2h\left(\frac{1}{2}\right)\langle f(A)x,x\rangle. \tag{14}$$

(ii) Assume that there exist some points $t \in [m,M]$ such that $l'(t) > \frac{h(\lambda)}{\lambda} f(t)$. Define the sets T and S as follows:

$$T := \left\{t \in [m,\bar{g}] : l'(t) > 2h\left(\frac{1}{2}\right)f(t)\right\},$$

$$S := \left\{t \in [\bar{g},M] : l'(t) > 2h\left(\frac{1}{2}\right)f(t)\right\}.$$

Consider $t_T := \max\{t : t \in T\}$ and $t_S := \min\{t : t \in S\}$. We use two lines passing through the points $(t_T, 0)$, $(\bar{g}, f(\bar{g}))$ and $(t_S, 0)$ and $(\bar{g}, f(\bar{g}))$, respectively. Let l_T be the line passing through the points $(t_T, 0)$ and $(\bar{g}, f(\bar{g}))$ and l_S the line passing through the points $(t_S, 0)$ and $(\bar{g}, f(\bar{g}))$. Define the function L as follows:

$$L(t) := \begin{cases} l_T(t), & t \in [m,\bar{g}], \\ l_S(t), & t \in [\bar{g},M]. \end{cases}$$

We prove that the inequality $L(t) \leqslant 2h\left(\frac{1}{2}\right)f(t)$ holds for all $t \in [m,M]$. We consider the partition $\{m, t_T, \bar{g}, t_S, M\}$ for the closed interval $[m,M]$ and we notice that $l_T(t) \leqslant 0$ for every $t \in [m, t_T]$. Since $f(t) \geqslant 0$, we clearly observe that $l_T(t) \leqslant 2h\left(\frac{1}{2}\right)f(t)$ for every $t \in [m, t_T]$. On the other hand, we see that

$$l'(t) \leqslant 2h\left(\frac{1}{2}\right)f(t) \tag{15}$$

for every $t \in (t_T, \bar{g}]$; otherwise, there exists $t_0 \in (t_T, \bar{g}]$ such that $l'(t_0) > 2h\left(\frac{1}{2}\right) f(t_0)$ and so $t_0 \in T$ and $t_0 < t_T$, which is a contradiction. So, it follows from (15) by letting t tends to t_T from right that

$$l'(t_T) \leqslant 2h\left(\frac{1}{2}\right) f(t_T). \tag{16}$$

Moreover, since t_T is in the closure of the set T, the reversed inequality holds in (16) and hence $l'(t_T) = 2h\left(\frac{1}{2}\right) f(t_T)$. It follows that l' is the line passing through the points $\left(t_T, 2h\left(\frac{1}{2}\right) f(t_T)\right)$ and $(\bar{g}, f(\bar{g}))$ and its slope is $\alpha = \frac{f(\bar{g}) - 2h\left(\frac{1}{2}\right) f(t_T)}{\bar{g} - t_T}$, where the slope of the line l_T is $\alpha' = \frac{f(\bar{g})}{\bar{g} - t_T}$. By the inequality (15) we observe that

$$l_T(t) = \alpha'(t - \bar{g}) + f(\bar{g}) \leqslant \alpha(t - \bar{g}) + f(\bar{g}) = l'(t) \leqslant 2h\left(\frac{1}{2}\right) f(t)$$

for every $t \in (t_T, \bar{g}]$. So, $L(t) = l_T(t) \leqslant 2h\left(\frac{1}{2}\right) f(t)$ for every $t \in [m, \bar{g}]$.

By the similar methods one can show that $L(t) = l_S(t) \leqslant 2h\left(\frac{1}{2}\right) f(t)$ for every $t \in [\bar{g}, M]$. Note that the lines l_T and l_S are joining at the point along the length of $\bar{g}$ and so $l_T(\bar{g}) = l_S(\bar{g})$ and since f is continuous,

$$l_T(\bar{g}) = f(\bar{g}) \geqslant f(\bar{g}) - \varepsilon \tag{17}$$

for arbitrary $\varepsilon > 0$. For the case $Sp(A) \subseteq [m, \bar{g}]$, we have

$$f(\langle Ax, x\rangle) - \varepsilon \leqslant l_T(\langle Ax, x\rangle) = \langle l_T(A)x, x\rangle \leqslant 2h\left(\frac{1}{2}\right) \langle f(A)x, x\rangle.$$

Moreover, for the case $Sp(A) \subseteq [\bar{g}, M]$, we have

$$\begin{aligned} f(\langle Ax, x\rangle) - \varepsilon \leqslant l_T(\langle Ax, x\rangle) = l_S(\langle Ax, x\rangle) = \langle l_S(A)x, x\rangle \\ \leqslant 2h\left(\frac{1}{2}\right) \langle f(A)x, x\rangle \end{aligned}$$

and consequently one can deduce (7). $\square$

We demonstrate that the constant $2h\left(\frac{1}{2}\right)$ is the best possible in (7) such one in the following example.

EXAMPLE 1. Let $h(t) = \sqrt{t}$ and $t > 0$. Define $g : [0, \infty) \to \mathbb{R}$ by $g(t) = \sqrt{t}$. Note that the function g is h-convex, since

$$\begin{aligned} g(\alpha x + (1-\alpha)y) &= (\alpha x + (1-\alpha)y)^{\frac{1}{2}} \\ &\leqslant (\alpha x)^{\frac{1}{2}} + ((1-\alpha)y)^{\frac{1}{2}} \\ &= \alpha^{\frac{1}{2}} x^{\frac{1}{2}} + (1-\alpha)^{\frac{1}{2}} y^{\frac{1}{2}} \\ &= h(\alpha) x^{\frac{1}{2}} + h(1-\alpha) y^{\frac{1}{2}} \end{aligned}$$

for every $x,y \geqslant 0$ and $\alpha \in [0,1]$. Consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $x = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. A simple calculation shows that $g(\langle Ax,x\rangle) = g(\frac{1}{2}) = \sqrt{\frac{1}{2}}$ and $\langle g(A)x,x\rangle = \frac{1}{2}$. Therefore,

$$g(\langle Ax,x\rangle) = 2h\left(\frac{1}{2}\right)\langle g(A)x,x\rangle.$$

REMARK 1. Applying Theorem 2 and considering the unital positive linear map $\Phi(A) = \langle Ax,x\rangle$ and $p = 1$ in [6, Corollary 3.7], we see that the operator h-convexity of f can reduce to the usual h-convexity without any condition on the function h. Note that the operator h-convex functions are h-convex, but the converse is not true in general.

We now compare the results of this article with the work done by others. We generally cover all the work done for some specific functions. It is remarkable that the superiority of our results is that our results can recover the other works.

COROLLARY 1. *Let A be a self-adjoint operator on a Hilbert space $\mathscr{H}$ and assume that $Sp(A) \subseteq [m,M]$ for some scalars m,M with $0 < m < M$ and $x \in \mathscr{H}$ with* $||x|| = 1$.

(1) ([15, Theorem 1]) *If f is a non-negative convex function on $[m,M]$, then*

$$f(\langle Ax,x\rangle) \leqslant \langle f(A)x,x\rangle. \tag{18}$$

(2) ([16, Theorem 2.1]) *If f is a continuous P-class function on $[m,M]$, then*

$$f(\langle Ax,x\rangle) \leqslant 2\langle f(A)x,x\rangle. \tag{19}$$

(3) ([17, Theorem 2]) *If f is a continuous s-convex function on $[m,M]$ in the second sense, then*

$$f(\langle Ax,x\rangle) \leqslant 2^{1-s}\langle f(A)x,x\rangle. \tag{20}$$

(4) *If f is a continuous Godunova-Levin function on $[m,M]$, then*

$$f(\langle Ax,x\rangle) \leqslant 4\langle f(A)x,x\rangle. \tag{21}$$

(5) *If f is a continuous s-Godunova-Levin function on $[m,M]$, then*

$$f(\langle Ax,x\rangle) \leqslant 2^{1+s}\langle f(A)x,x\rangle. \tag{22}$$

Proof. Consider $h(t) = t$, $h(t) = 1$, $h(t) = t^s$, $h(t) = \frac{1}{t}$, and $h(t) = \frac{1}{t^s}$ in parts (1)–(5), respectively. Then, the coefficient $2h(\frac{1}{2})$ can be calculated in each case and a simple calculation gets the desired result in each case. □

We provide a refinement of the Mond and Pečarić inequality for convex functions.

COROLLARY 2. *Let the conditions of Theorem 2 be satisfied and* $h\left(\frac{1}{2}\right) \leqslant \frac{1}{2}$*. Then,*

$$f(\langle Ax,x\rangle) \leqslant 2h\left(\frac{1}{2}\right)\langle f(A)x,x\rangle \leqslant \langle f(A)x,x\rangle \tag{23}$$

for each $x \in \mathscr{H}$ *with* $||x|| = 1$.

Proof. Note that the function f is mid convex, since $h\left(\frac{1}{2}\right) \leqslant \frac{1}{2}$ and so f is convex. The first inequality follows from Theorem 2 and the second one follows from the fact that $2h\left(\frac{1}{2}\right) \leqslant 1$. □

THEOREM 3. *Let the conditions of Theorem 2 be satisfied. Then,*

$$\langle f(A)x,x\rangle \leqslant 2h\left(\frac{1}{2}\right)\left(\frac{M-\langle Ax,x\rangle}{M-m}f(m) + \frac{\langle Ax,x\rangle - m}{M-m}f(M)\right). \tag{24}$$

Proof. Consider $D = \begin{pmatrix} m & 0 \\ 0 & M \end{pmatrix}$ and $x = \begin{pmatrix} \sqrt{\frac{M-t}{M-m}} \\ \sqrt{\frac{t-m}{M-m}} \end{pmatrix}$. By applying Theorem 2, we have

$$\begin{aligned} f(t) &= f(\langle Dx,x\rangle) \\ &\leqslant 2h\left(\frac{1}{2}\right)\langle f(D)x,x\rangle \\ &= 2h\left(\frac{1}{2}\right)\left(\frac{M-t}{M-m}f(m) + \frac{t-m}{M-m}f(M)\right). \end{aligned}$$

Since the operator $2h\left(\frac{1}{2}\right)\left(\frac{M-A}{M-m}f(m) + \frac{A-m}{M-m}f(M)\right) - f(A)$ is positive, we get (24). □

THEOREM 4. *Let the conditions of Theorem 2 be satisfied. Let* J *be an interval such that* $f([m,M]) \subset J$*. If* $F(u,v)$ *is a real function defined on* $J \times J$ *and non–decreasing in* u*, then*

$$\begin{aligned} &F(\langle f(A)x,x\rangle, f(\langle Ax,x\rangle)) \\ &\leqslant \max_{t\in[m,M]} F\left(2h\left(\frac{1}{2}\right)\left(\frac{M-t}{M-m}f(m) + \frac{t-m}{M-m}f(M)\right), f(t)\right) \\ &= \max_{\theta\in[0,1]} F\left(2h\left(\frac{1}{2}\right)(\theta f(m) + (1-\theta)f(M)), f(\theta m + (1-\theta)M)\right). \end{aligned} \tag{25}$$

Proof. Since $\bar{g} = \langle Ax,x\rangle \in [m,M]$, by the non-decreasing character of F and Theorem 3, one has

$$\begin{aligned} &F(\langle f(A)x,x\rangle, f(\langle Ax,x\rangle)) \\ &\leqslant F\left(2h\left(\frac{1}{2}\right)\left(\frac{M-\bar{g}}{M-m}f(m) + \frac{\bar{g}-m}{M-m}f(M)\right), f(\bar{g})\right) \\ &\leqslant \max_{t\in[m,M]} F\left(2h\left(\frac{1}{2}\right)\left(\frac{M-t}{M-m}f(m) + \frac{t-m}{M-m}f(M)\right), f(t)\right). \end{aligned}$$

The second form of the right side of (25) follows at once from the change of variable $\theta = \frac{M-t}{M-m}$, so that $t = \theta m + (1-\theta)M$, with $0 \leqslant \theta \leqslant 1$. □

DEFINITION 1. The function f is piecewise continuously twice differentiable on $[m,M]$ whenever the following conditions fulfil:

(1) f is continuous on $[m,M]$,

(2) there exists a finite subdivision $\{x_0,\ldots,x_n\}$ of $[m,M]$, $x_0 = a$, $x_n = b$ such that

(2.1) f is continuously twice differentiable on (x_{i-1},x_i) for every $i \in \{1,\ldots,n\}$,

(2.2) the one-sided limits $\lim_{x\to x_{i-1}^+} f'(x)$ and $\lim_{x\to x_i^-} f'(x)$ exist for every $i \in \{1,\ldots,n\}$,

(2.3) the one-sided limits $\lim_{x\to x_{i-1}^+} f''(x)$ and $\lim_{x\to x_i^-} f''(x)$ exist for every $i \in \{1,\ldots,n\}$.

We provide a converse inequality in Theorem 2.

THEOREM 5. *Let the conditions of Theorem 2 be satisfied. Moreover, let f be piecewise continuously twice differentiable on $[m,M]$.*

(i) There exists $\alpha \geqslant 1$ such that

$$\frac{1}{2h\left(\frac{1}{2}\right)\alpha}\langle f(A)x,x\rangle \leqslant f(\langle Ax,x\rangle).$$

(ii) There exists $\beta \geqslant 0$ such that

$$\frac{1}{2h\left(\frac{1}{2}\right)}\langle f(A)x,x\rangle - \beta \leqslant f(\langle Ax,x\rangle).$$

Proof. (i) Suppose $R = \{x_0,x_1,\ldots,x_n\}$, $x_0 = m$, $x_n = M$ is a finite subdivision of $[m,M]$ such that the conditions of Definition 1 fulfils. Consider $F(u,v) = \frac{u}{v}$, $J = (0,\infty)$, $\varphi_h(t) = 2h\left(\frac{1}{2}\right)\varphi_i(t)$ for every $t \in [x_{i-1},x_i]$, where $\varphi_i(t) = \frac{L_i(t)}{f(t)}$, $L_i(t) = f(x_{i-1}) + \mu_i(t-x_{i-1})$ and $\mu_i = \frac{f(x_i)-f(x_{i-1})}{x_i-x_{i-1}}$. According Theorem 4 we have

$$\frac{\langle f(A)x,x\rangle}{f(\langle Ax,x\rangle)} \leqslant \max_{t\in[m,M]} \varphi_h(t) = 2h\left(\frac{1}{2}\right)\max_{1\leqslant i\leqslant n}\max_{t\in[x_{i-1},x_i]} \varphi_i(t). \tag{26}$$

Now $\varphi_i'(t) = \frac{G_i(t)}{f(t)^2}$, where $G_i(t) = \mu_i f(t) - L_i(t)f'(t)$ for every $t \in [x_{i-1},x_i]$. If $\mu_i \neq 0$, then

$$G_i'(t) = -L_i(t)f''(t).$$

If $\mu_i = 0$ and $\bar{t_i} \in (x_{i-1},x_i)$ is a unique solution of the equation $f'(t) = 0$, then we consider

$$\lambda_i = \max_{t\in[x_{i-1},x_i]} \varphi_h(t) = 2h\left(\frac{1}{2}\right)\frac{f(x_{i-1})}{f(\bar{t_i})}.$$

Define

$$A=\{i:\lim_{t\to x_{i-1}^{+}} f''(t)>0,\lim_{t\to x_i^{-}} f''(t)>0, f''(t)>0, t\in(x_{i-1},x_i)\},$$

$$B=\{i:\lim_{t\to x_{i-1}^{+}} f''(t)<0,\lim_{t\to x_i^{-}} f''(t)<0, f''(t)<0, t\in(x_{i-1},x_i)\}.$$

Suppose $t\in[m,M]$. Then, there exists $i\in\{1,\ldots,n\}$ such that $t\in[x_{i-1},x_i]$.

(1) If $i\in A$, then $G_i'(t)<0$ and so G_i is decreasing on $[x_{i-1},x_i]$. So,

$$G_i(x_{i-1})G_i(x_i)=-f(x_{i-1})f(x_i)(\mu_i-f'(x_{i-1}))(f'(x_i)-\mu_i)<0.$$

This indicates the equation $G_i(t)=0$ has a unique solution at $\bar{t_i}\in(x_{i-1},x_i)$ and so the equation $\varphi_i'(t)=0$ has a unique solution at $\bar{t_i}\in(x_{i-1},x_i)$. Let $D_i=\begin{pmatrix} x_{i-1} & 0\\ 0 & x_i\end{pmatrix}$ and $x=\begin{pmatrix}\sqrt{\frac{x_i-t}{x_i-x_{i-1}}}\\ \sqrt{\frac{t-x_{i-1}}{x_i-x_{i-1}}}\end{pmatrix}$. Since $i\in A$, the function f is convex on $[x_{i-1},x_i]$. So, by Theorem 1, for the convex function f on $[x_{i-1},x_i]$, one can reach

$$\begin{aligned} f(t)&=f(\langle D_i x,x\rangle)\\ &\leqslant \langle f(D_i)x,x\rangle=\frac{x_i-t}{x_i-x_{i-1}}f(x_{i-1})+\frac{t-x_{i-1}}{x_i-x_{i-1}}f(x_i)=L_i(t). \end{aligned}$$

Consequently, $\frac{L_i(t)}{f(t)}\geqslant 1$ for every $t\in[x_{i-1},x_i]$ and

$$\varphi_h(t)=2h\left(\frac{1}{2}\right)\frac{L_i(t)}{f(t)}\geqslant 2h\left(\frac{1}{2}\right)$$

for every $t\in[x_{i-1},x_i]$ where the equality occurs at x_{i-1} and x_i. Note that the maximum value of φ_i is attained in $\bar{t_i}\in[x_{i-1},x_i]$, since $\varphi_i''(\bar{t_i})<0$. We consider

$$\begin{aligned} \lambda_i&=\max_{t\in[x_{i-1},x_i]}\varphi_h(t)\\ &=2h\left(\frac{1}{2}\right)\max_{t\in[x_{i-1},x_i]}\varphi_i(t)\\ &=2h\left(\frac{1}{2}\right)\varphi_i(\bar{t_i})\\ &=2h\left(\frac{1}{2}\right)\frac{L_i(\bar{t_i})}{f(\bar{t_i})}\\ &=2h\left(\frac{1}{2}\right)\frac{\mu_i}{f'(\bar{t_i})}. \end{aligned}$$

The last equality comes from the fact that $G_i(\bar{t_i})=0$.

(2) If $i \in B$, then define D_i and x as the part (1) and apply Theorem 1 for the concave function f on $[x_{i-1}, x_i]$. So, $\frac{L_i(t)}{f(t)} \leqslant 1$ and this inequality yields

$$0 \leqslant \varphi_h(t) = 2h\left(\frac{1}{2}\right)\frac{L_i(t)}{f(t)} \leqslant 2h\left(\frac{1}{2}\right)$$

for every $t \in [x_{i-1}, x_i]$ where equality occurs at x_{i-1} and x_i. We consider

$$\lambda_i = \max_{t\in[x_{i-1},x_i]} \varphi_h(t) = 2h\left(\frac{1}{2}\right).$$

It follows from the cases (1) and (2) in the part (i) that

$$\lambda_i = \begin{cases} 2h\left(\frac{1}{2}\right)\frac{f(x_{i-1})}{f(\bar{t_i})}, & \mu_i = 0,\ i \in \{1,\dots,n\}\setminus A\cup B, \\ 2h\left(\frac{1}{2}\right)\frac{\mu_i}{f'(\bar{t_i})}, & \mu_i \neq 0,\ i \in A, \\ 2h\left(\frac{1}{2}\right), & \mu_i \neq 0,\ i \in B. \end{cases}$$

Define $\lambda = \max_{1\leqslant i\leqslant n} \lambda_i$. Then, $\lambda = 2h\left(\frac{1}{2}\right)\alpha$, where

$$\alpha = \max\left\{\max_{i\in(A\cup B)^c}\frac{f(x_{i-1})}{f(\bar{t_i})}, \max_{i\in A}\frac{\mu_i}{f'(\bar{t_i})}, 1\right\}.$$

By virtue of (26), we deduce

$$\frac{\langle f(A)x,x\rangle}{f(\langle Ax,x\rangle)} \leqslant \max_{t\in[m,M]} \varphi_h(t) = 2h\left(\frac{1}{2}\right)\alpha.$$

(ii) Consider the sets R, P, A, and B as the part (i) and define $F(u,v) = u - 2h\left(\frac{1}{2}\right)v$, $J = \mathbb{R}$ and $\varphi_h(t) = 2h\left(\frac{1}{2}\right)\varphi_i(t)$ for every $t \in [x_{i-1}, x_i]$, where $\varphi_i(t) = L_i(t) - f(t)$ and $L_i(t)$ defined in the part (i). By virtue of Theorem 4 we yield

$$\begin{aligned} \langle f(A)x,x\rangle - 2h\left(\frac{1}{2}\right) f(\langle Ax,x\rangle) &\leqslant \max_{t\in[m,M]} \varphi_h(t) \\ &= 2h\left(\frac{1}{2}\right)\max_{1\leqslant i\leqslant n}\max_{t\in[x_{i-1},x_i]} \varphi_i(t). \end{aligned} \tag{27}$$

If $\mu_i \neq 0$, then $\varphi_i''(t) = -f''(t)$ and if $\mu_i = 0$ and $\bar{t_i} \in (x_{i-1}, x_i)$ is the unique solution of the equation $f'(t) = 0$, then we define

$$\lambda_i = \max_{t\in[x_{i-1},x_i]} \varphi_h(t) = 2h\left(\frac{1}{2}\right)(f(x_{i-1}) - f(\bar{t_i})).$$

Suppose $t \in [m, M]$. Then, there exists $i \in \{1,\dots,n\}$ such that $t \in [x_{i-1}, x_i]$.

(1) If $i \in A$, then $\varphi_i''(t) < 0$ for every $t \in [x_{i-1}, x_i]$ and so φ_i' is decreasing on $[x_{i-1}, x_i]$. On the other hand, the equation $\varphi_i'(t) = 0$ has a unique solution at $t = \bar{t_i} \in$

$[x_{i-1},x_i]$, since $\varphi_i'(x_{i-1})\varphi_i'(x_i)<0$. Clearly, $\varphi_i''(\bar{t_i})<0$ and so the maximum value of φ_i is attained in $\bar{t_i}$. We define

$$\begin{aligned}\lambda_i &= \max_{t\in[x_{i-1},x_i]} \varphi_h(t)\\ &= 2h\left(\frac{1}{2}\right)\max_{t\in[x_{i-1},x_i]}\varphi_i(t)\\ &= 2h\left(\frac{1}{2}\right)\varphi_i(\bar{t_i})\\ &= 2h\left(\frac{1}{2}\right)(L_i(\bar{t_i})-f(\bar{t_i}))\\ &= 2h\left(\frac{1}{2}\right)(f(x_{i-1})+\mu_i(\bar{t_i}-x_{i-1})-f(\bar{t_i})).\end{aligned}$$

(2) If $i\in B$, then $f''(t)<0$ for every $t\in[x_{i-1},x_i]$. This means that f is concave on $[x_{i-1},x_i]$ and so $f(t)\geqslant L_i(t)$ for every $t\in[x_{i-1},x_i]$. This ensures $\varphi_i(t)=L_i(t)-f(t)\leqslant 0$ and this inequality entails

$$\max_{t\in[x_{i-1},x_i]}\varphi_i(t)\leqslant 0.$$

Since $\varphi_i(x_i)=0=\varphi_i(x_{i-1})$, φ_i attains its maximum value and the maximum value is 0. So that

$$\lambda_i=\max_{t\in[x_{i-1},x_i]}\varphi_h(t)=2h\left(\frac{1}{2}\right)\max_{t\in[x_{i-1},x_i]}\varphi_i(t)=\varphi_i(x_i)=0.$$

Consequently, it follows from the cases (1) and (2) in the part (ii) that

$$\lambda_i=\begin{cases}2h\left(\frac{1}{2}\right)(f(x_{i-1})-f(\bar{t_i})), & \mu_i=0,\ \ i\in\{1,\dots,n\}\setminus A\cup B,\\ 2h\left(\frac{1}{2}\right)(f(x_{i-1})+\mu_i(\bar{t_i}-x_{i-1})-f(\bar{t_i})), & \mu_i\neq 0,\ \ i\in A,\\ 0, & \mu_i\neq 0,\ \ i\in B.\end{cases}$$

Define $\lambda=\max_{1\leqslant i\leqslant n}\lambda_i$. Then, $\lambda=2h\left(\frac{1}{2}\right)\beta$, where

$$\beta=\max\{\max_{i\in(A\cup B)^c}(f(x_{i-1})-f(\bar{t_i})),\max_{i\in A}(f(x_{i-1})+\mu_i(\bar{t_i}-x_{i-1})-f(\bar{t_i})),0\}.$$

In view of (27), we deduce

$$\langle f(A)x,x\rangle-2h\left(\frac{1}{2}\right)f(\langle Ax,x\rangle)\leqslant\max_{t\in[m,M]}\varphi_h(t)=\lambda=2h\left(\frac{1}{2}\right)\beta.\quad\square$$

COROLLARY 3. *Let the function f be a piecewise continuously twice differentiable on $[m,M]$ and A a self-adjoint operator on a Hilbert space $\mathscr{H}$. Assume that $Sp(A)\subseteq[m,M]$ for some scalars m,M with $0<m<M$ and $x\in\mathscr{H}$ with $||x||=1$.*

(1) *If f is non-negative convex on $[m,M]$, then*

(i) there exists $\alpha \geqslant 1$ such that

$$\frac{1}{\alpha}\langle f(A)x,x\rangle \leqslant f(\langle Ax,x\rangle),$$

(ii) there exists $\beta \geqslant 0$ such that

$$\langle f(A)x,x\rangle - \beta \leqslant f(\langle Ax,x\rangle).$$

(2) *If f is P-class on $[m,M]$, then*

(i) there exists $\alpha \geqslant 1$ such that

$$\frac{1}{2\alpha}\langle f(A)x,x\rangle \leqslant f(\langle Ax,x\rangle),$$

(ii) there exists $\beta \geqslant 0$ such that

$$\frac{1}{2}\langle f(A)x,x\rangle - \beta \leqslant f(\langle Ax,x\rangle).$$

(3) *If f is s-convex on $[m,M]$ in the second sense, then*

(i) there exists $\alpha \geqslant 1$ such that

$$\frac{1}{2^{1-s}\alpha}\langle f(A)x,x\rangle \leqslant f(\langle Ax,x\rangle),$$

(ii) there exists $\beta \geqslant 0$ such that

$$\frac{1}{2^{1-s}}\langle f(A)x,x\rangle - \beta \leqslant f(\langle Ax,x\rangle).$$

(4) *If f is Godunova-Levin on $[m,M]$, then*

(i) there exists $\alpha \geqslant 1$ such that

$$\frac{1}{4\alpha}\langle f(A)x,x\rangle \leqslant f(\langle Ax,x\rangle),$$

(ii) there exists $\beta \geqslant 0$ such that

$$\frac{1}{4}\langle f(A)x,x\rangle - \beta \leqslant f(\langle Ax,x\rangle).$$

(5) *If f is s-Godunova-Levin on $[m,M]$, then*

(i) there exists $\alpha \geqslant 1$ such that

$$\frac{1}{2^{1+s}\alpha}\langle f(A)x,x\rangle \leqslant f(\langle Ax,x\rangle),$$

(ii) there exists $\beta \geqslant 0$ *such that*

$$\frac{1}{2^{1+s}}\langle f(A)x,x\rangle - \beta \leqslant f(\langle Ax,x\rangle).$$

Proof. Consider $h(t)=t$, $h(t)=1$, $h(t)=t^s$, $h(t)=\frac{1}{t}$, and $h(t)=\frac{1}{t^s}$ in parts (1)-(5), respectively and calculate the coefficient $2h(\frac{1}{2})$. According Theorem 5, we get the desired result in each part. □

3. Applications

In this section, we obtain the Hermite-Hadamard's type inequality for h-convex functions. Moreover, we obtain a multiple operator version of Theorem 2 for h-convex functions. In particular, one may reach a result for the convex, P-class, s-convex, Godunova-Levin, and s-Godunova-Levin functions.

COROLLARY 4. *Let the conditions of Theorem 2 be satisfied and let* p *and* q *be non-negative numbers, with* $p+q>0$, *for which*

$$\langle Ax,x\rangle = \frac{pm+qM}{p+q}.$$

Then,

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{pm+qM}{p+q}\right) \leqslant \langle f(A)x,x\rangle \leqslant 2h\left(\frac{1}{2}\right)\frac{pf(m)+qf(M)}{p+q}.$$

Proof. By virtue of Theorems 2 and 3 we reach

$$\begin{aligned} f\left(\frac{pm+qM}{p+q}\right) &= f(\langle Ax,x\rangle) \leqslant 2h\left(\frac{1}{2}\right)\langle f(A)x,x\rangle) \\ &\leqslant \left(2h\left(\frac{1}{2}\right)\right)^2 \left(\frac{M-\langle Ax,x\rangle}{M-m}f(m) + \frac{\langle Ax,x\rangle - m}{M-m}f(M)\right) \\ &= \left(2h\left(\frac{1}{2}\right)\right)^2 \frac{pf(m)+qf(M)}{p+q}. \quad \square \end{aligned}$$

We may consider a multiple operator version of Theorem 2 as follows and obtain some interesting corollaries.

COROLLARY 5. *Let* A_i *be self-adjoint operators with* $Sp(A_i) \subseteq [m,M]$ *for some scalars* $m<M$ *and* $x_i \in \mathscr{H}$, $i \in \{1,\dots,n\}$ *with* $\sum_{i=1}^n ||x_i||^2 = 1$. *If* f *is* h*-convex on* $[m,M]$, *then*

$$f\left(\sum_{i=1}^n \langle A_i x_i, x_i\rangle\right) \leqslant 2h\left(\frac{1}{2}\right)\sum_{i=1}^n \langle f(A_i)x_i,x_i\rangle.$$

Proof. We define

$$\tilde{A} = \begin{pmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_n \end{pmatrix} \quad \text{and} \quad \tilde{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

So, $Sp(\tilde{A}) \subseteq [m,M]$, $||\tilde{x}|| = 1$, and

$$f(\langle \tilde{A}\tilde{x}, \tilde{x}\rangle) = f\Big(\sum_{i=1}^{n} \langle A_i x_i, x_i\rangle\Big),$$

$$\langle f(\tilde{A})\tilde{x}, \tilde{x}\rangle = \sum_{i=1}^{n} \langle f(A_i)x_i, x_i\rangle.$$

In view of Theorem 2 the result follows. □

We obtain a complementary inequality in Corollary 5 as follows.

COROLLARY 6. *Let the conditions of Corollary 5 be satisfied.*

(i) There exists $\alpha \geqslant 1$ *such that*

$$\frac{1}{2h\left(\frac{1}{2}\right)\alpha} \sum_{i=1}^{n} \langle f(A_i)x_i, x_i\rangle \leqslant f(\langle \sum_{i=1}^{n} A_i x_i, x_i\rangle). \tag{28}$$

(ii) There exists $\beta \geqslant 0$ *such that*

$$\frac{1}{2h\left(\frac{1}{2}\right)} \sum_{i=1}^{n} \langle f(A_i)x_i, x_i\rangle - \beta \leqslant f(\langle \sum_{i=1}^{n} A_i x_i, x_i\rangle). \tag{29}$$

Proof. Consider $\tilde{A}$ and $\tilde{x}$ as in the proof of Corollary 5 and apply Theorem 5. □

COROLLARY 7. *Let* $A_1, \ldots, A_n$ *be self-adjoint operators with* $Sp(A_i) \subseteq [m,M]$, $i \in \{1, \ldots, n\}$ *for some scalars* $m < M$. *If* f *is* h*-convex on* $[m,M]$ *and* $p_i \geqslant 0$ *with* $\sum_{i=1}^{n} p_i = 1$, *then*

$$f\left(\sum_{i=1}^{n} p_i \langle A_i x, x\rangle\right) \leqslant 2h\left(\frac{1}{2}\right) \sum_{i=1}^{n} p_i \langle f(A_i)x, x\rangle$$

for every $x \in \mathscr{H}$ *with* $||x|| = 1$.

Proof. By using Corollary 5 and setting $x_i = \sqrt{p_i}x$, $i \in \{1, \ldots, n\}$ we can reach the result. □

4. Conclusions

We obtained a Mond-Pečarić and Hermite-Hadamard type inequalities for the class of h-convex functions and refined Jensen's inequality for convex functions. We proved some multiple operator versions for this class of functions. In particular, we discovered that the operator h-convexity can be reduced to the usual h-convexity in some sense. Moreover, we showed that some results for the other classes of functions such as the class of convex, P-class, s-convex, Godunova-Levin, and s-Godunova-Levin functions can be deduced by choosing an appropriate function h.

Declarations. We remark that the potential conflicts of interest and data sharing do not apply to this article; no data sets were generated during the current study.

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