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SOME YANG–WANG–REN TYPE INEQUALITIES

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Abstract. In this paper, we shall present some weighted power mean inequalities: Let $a\sharp_{p,v}b = \left((1-v)a^p + vb^p\right)^{\frac{1}{p}}$. If $p\in[\frac{1}{2},1]$, $0<v\leqslant\tau<1$ and $m\in\mathbb{N}^+$, then we have

$$\frac{(a\sharp_{p,v}b)^m-(a\sharp_v b)^m}{(a\sharp_{p,\tau}b)^m-(a\sharp_\tau b)^m}\geqslant\frac{v(1-v)}{\tau(1-\tau)}$$

and

$$\frac{(a\sharp_{p,v}b)^m-(a!_v b)^m}{(a\sharp_{p,\tau}b)^m-(a!_\tau b)^m}\geqslant\frac{v(1-v)}{\tau(1-\tau)}$$

for $0<b\leqslant a$; and the inequalities are reversed for $b\geqslant a>0$. As applications, we obtain some inequalities for operator and determinant.

1. Introduction

Let $(\mathbb{H},\langle\cdot,\cdot\rangle)$ be a complex Hilbert space and let $\mathbb{B}(\mathbb{H})$ denote the algebra of all bounded linear operators acting on $\mathbb{H}$. A self-adjoint operator A is said to be positive if $\langle Ax,x\rangle\geqslant 0$ for all $x\in\mathbb{H}$, while it is said to be strictly positive if A is positive and invertible, denoted by $A\geqslant 0$ and $A>0$ respectively. $A-B\geqslant 0$ means $A\geqslant B$. Moreover, let $M_n(\mathbb{C})$ be the space of $n\times n$ complex matrices and $M_n{}^{++}(\mathbb{C})$ be the set of positive definite matrices in $M_n(\mathbb{C})$. In addition, $s(A)=(s_1(A),\cdots,s_n(A))$ denotes the singular values of $A\in M_n(\mathbb{C})$, that is, the eigenvalues of the positive semi-definite matrix $|A|=(A^*A)^{\frac{1}{2}}$, arranged in non-increasing order.

In this paper, we define v-weighted arithmetic-geometric-harmonic means (AM-GM-HM) by

$$a\nabla_v b=(1-v)a+vb,\ \ a\sharp_v b=a^{1-v}b^v\ \text{ and }\ a!_v b=\left((1-v)a^{-1}+vb^{-1}\right)^{-1}$$

for $a,b>0$ and $v\in[0,1]$. The corresponding v-weighted operator AM-GM-HM are

$$A\nabla_v B=(1-v)A+vB,\ A\sharp_v B=A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^vA^{\frac{1}{2}}\ \text{ and }\ A!_v B=\left((1-v)A^{-1}+vB^{-1}\right)^{-1}$$

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for $A,B>0$ and $v\in[0,1]$, respectively. A more generalized v-weighted mean is the weighted power mean, defined as

$$a\sharp_{p,v}b=\left((1-v)a^p+vb^p\right)^{\frac{1}{p}}$$

for $a,b>0,\ p\neq 0$ and $v\in[0,1]$. As usual, we define the weighted operator power mean as follows: if $A,B>0,\ v\in[0,1]$ and $p\neq 0$, then

$$A\sharp_{p,v}B=A^{\frac{1}{2}}\left((1-v)I+v(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^p\right)^{\frac{1}{p}}A^{\frac{1}{2}}.$$

The famous Young's inequality reads

$$a^{1-v}b^v\leqslant(1-v)a+vb, \tag{1.1}$$

where $a,b\geqslant 0$, and $v\in[0,1]$. Hirzallah, Kittaneh and Manasrah [3, 7, 8] showed some refinements and reverses of (1.1) in the following forms

$$r_0(\sqrt{a}-\sqrt{b}\,)^2\leqslant\left((1-v)a+vb\right)-a^{1-v}b^v\leqslant R_0(\sqrt{a}-\sqrt{b}\,)^2 \tag{1.2}$$

and

$$r_0^2(a-b)^2\leqslant\left((1-v)a+vb\right)^2-(a^{1-v}b^v)^2\leqslant R_0^2(a-b)^2, \tag{1.3}$$

where $r_0=\min\{v,1-v\}$, $R_0=\max\{v,1-v\}$ and $v\in[0,1]$. Choi [2] gave the generalizations of (1.2) and (1.3) as

$$\begin{aligned} r_0^m\left((a+b)^m-2^m(ab)^{\frac{m}{2}}\right)&\leqslant\left((1-v)a+vb\right)^m-\left(a^{1-v}b^v\right)^m\\ &\leqslant R_0^m\left((a+b)^m-2^m(ab)^{\frac{m}{2}}\right), \end{aligned} \tag{1.4}$$

where $a,b\geqslant 0,\ m\in\mathbb{N}^+,\ r_0=\min\{v,1-v\},\ R_0=\max\{v,1-v\}$ and $v\in[0,1]$. In 2015, Alzer, Fonseca and Kovačec [1] presented the following AM-GM inequalities

$$\left(\frac{v}{\tau}\right)^m\leqslant\frac{(a\nabla_v b)^m-(a\sharp_v b)^m}{(a\nabla_\tau b)^m-(a\sharp_\tau b)^m}\leqslant\left(\frac{1-v}{1-\tau}\right)^m \tag{1.5}$$

for $a,b\geqslant 0$, $0<v\leqslant\tau<1$ and $m\geqslant 1$. In fact, we can obtain (1.4) by (1.5) when $v=\frac{1}{2}$ and $\tau=\frac{1}{2}$, respectively. Later, Liao and Wu [9] replicated (1.5) as

$$\left(\frac{v}{\tau}\right)^m\leqslant\frac{(a\nabla_v b)^m-(a!_v b)^m}{(a\nabla_\tau b)^m-(a!_\tau b)^m}\leqslant\left(\frac{1-v}{1-\tau}\right)^m \tag{1.6}$$

for $a,b\geqslant 0,\ 0<v\leqslant\tau<1$ and $m\geqslant 1$. The following remark explain (1.6) is a new Alzer-Fonseca-Kovačec's type inequalities:

REMARK 1.1. When $a=1,\ b=2,\ v=\frac{1}{4},\ \tau=\frac{2}{3},\ m=1$, then

$$\frac{(a\nabla_v b)^m-(a\sharp_v b)^m}{(a\nabla_\tau b)^m-(a\sharp_\tau b)^m}\approx 0.767>0.643\approx\frac{(a\nabla_v b)^m-(a!_v b)^m}{(a\nabla_\tau b)^m-(a!_\tau b)^m}.$$

When $a=2,\ b=1,\ v=\frac{1}{4},\ \tau=\frac{2}{3},\ m=1$, then

$$\frac{(a\nabla_v b)^m-(a\sharp_v b)^m}{(a\nabla_\tau b)^m-(a\sharp_\tau b)^m}\approx 0.929<1.125=\frac{(a\nabla_v b)^m-(a!_v b)^m}{(a\nabla_\tau b)^m-(a!_\tau b)^m}.$$

In 2023, Yang and Wang [15] improved (1.5) under some conditions: Let $a,b>0$, $0<v\leqslant\tau<1$ and $m\in\mathbb{N}^+$. Then

$$\frac{(a\nabla_v b)^m-(a\sharp_v b)^m}{(a\nabla_\tau b)^m-(a\sharp_\tau b)^m}\geqslant\frac{v(1-v)}{\tau(1-\tau)} \tag{1.7}$$

for $b\leqslant a$; and the inequalities is reversed for $b\geqslant a$.

Very recently, Ren [10] improved (1.6) under the same conditions as in (1.7), then

$$\frac{(a\nabla_v b)^m-(a!_v b)^m}{(a\nabla_\tau b)^m-(a!_\tau b)^m}\geqslant\frac{v(1-v)}{\tau(1-\tau)} \tag{1.8}$$

for $b\leqslant a$; and the inequalities is reversed for $b\geqslant a$.

We refer the readers to [5, 6, 11, 12, 13, 14] and references therein for some other results about Alzer-Fonseca-Kovačec's type inequalities.

In this paper, we shall present some weighted power mean inequalities, which extend the inequalities (1.7) and (1.8). As applications, we obtain some inequalities for operator and determinant.

2. Main results

We firstly show a generalization of (1.7) using the following lemma.

LEMMA 2.1. *Let $a,b>0$, $p\in[\frac{1}{2},1]$ and $0<v\leqslant\tau<1$.*
(i) *If $b\leqslant a$, then*

$$\frac{a\sharp_{p,v}b-a\sharp_v b}{a\sharp_{p,\tau}b-a\sharp_\tau b}\geqslant\frac{v(1-v)}{\tau(1-\tau)}. \tag{2.1}$$

(ii) *If $b\geqslant a$, then*

$$\frac{a\sharp_{p,v}b-a\sharp_v b}{a\sharp_{p,\tau}b-a\sharp_\tau b}\leqslant\frac{v(1-v)}{\tau(1-\tau)}. \tag{2.2}$$

Proof. To prove our results, we set $f(v)=\frac{(1-v+vx^p)^{\frac{1}{p}}-x^v}{v(1-v)}$. Then $f'(v)=\frac{1}{(v-v^2)^2}g(x)$, where

$$g(x)=(1-v+vx^p)^{\frac{1}{p}-1}\left(\frac{1}{p}\left(v-v^2\right)(x^p-1)-(1-2v)(1-v+vx^p)\right)$$
$$+(1-2v)x^v-\left(v-v^2\right)x^v\ln x.$$

So we have

$$g'(x) = \left(\frac{1}{p}-1\right)(1-v+vx^p)^{\frac{1}{p}-2} vpx^{p-1}\left(\frac{1}{p}\left(v-v^2\right)(x^p-1)-(1-2v)(1-v+vx^p)\right)$$
$$+(1-v+vx^p)^{\frac{1}{p}-1}\left(\frac{1}{p}\left(v-v^2\right)px^{p-1}-(1-2v)vpx^{p-1}\right)+v(1-2v)x^{v-1}$$
$$+\left(v^2-v\right)vx^{v-1}\ln x+\left(v^2-v\right)x^{v-1}.$$

Thus,

$$g''(x) = v^2(v-1)^2x^{v-2}\ln x + v^2(1-v+vx^p)^{\frac{1}{p}-3}x^{p-2}\frac{1}{p}(1-p)(1-v)^2(x^p-1)(2p-1).$$

When $0<x\leqslant 1$, then $g''(x)\leqslant 0 \Rightarrow g'(x)\geqslant g'(1)=0$, so $g(x)\leqslant g(1)=0$, that is $f'(v)\leqslant 0$, which means $f(v)\geqslant f(\tau)$. Therefore,

$$\frac{(1-v+vx^p)^{\frac{1}{p}}-x^v}{v(1-v)} \geqslant \frac{(1-\tau+\tau x^p)^{\frac{1}{p}}-x^\tau}{\tau(1-\tau)};$$

When $x\geqslant 1$, then $g''(x)\geqslant 0 \Rightarrow g'(x)\geqslant g'(1)=0$, so $g(x)\geqslant g(1)=0$, that is $f'(v)\geqslant 0$, which means $f(v)\leqslant f(\tau)$. Therefore,

$$\frac{(1-v+vx^p)^{\frac{1}{p}}-x^v}{v(1-v)} \leqslant \frac{(1-\tau+\tau x^p)^{\frac{1}{p}}-x^\tau}{\tau(1-\tau)}.$$

Taking $x=\frac{b}{a}$, we can get the desired inequalities. □

A generalization of (1.7) is as follows.

THEOREM 2.2. *Let* $a,b>0$, $p\in[\frac{1}{2},1]$, $0<v\leqslant\tau<1$ *and* $m\in\mathbb{N}^+$.
(i) *If* $b\leqslant a$, *then*

$$\frac{(a\sharp_{p,v}b)^m-(a\sharp_v b)^m}{(a\sharp_{p,\tau}b)^m-(a\sharp_\tau b)^m} \geqslant \frac{v(1-v)}{\tau(1-\tau)}. \tag{2.3}$$

(ii) *If* $b\geqslant a$, *then*

$$\frac{(a\sharp_{p,v}b)^m-(a\sharp_v b)^m}{(a\sharp_{p,\tau}b)^m-(a\sharp_\tau b)^m} \leqslant \frac{v(1-v)}{\tau(1-\tau)}. \tag{2.4}$$

Proof. By computations, we have

$$(1-v+vx^p)^{\frac{m}{p}}-x^{mv} = \left((1-v+vx^p)^{\frac{1}{p}}-x^v\right)\left(\sum_{i=1}^{m}(1-v+vx^p)^{\frac{m-i}{p}}x^{(i-1)v}\right).$$

Let

$$f(v)=\sum_{i=1}^{m}(1-v+vx^p)^{\frac{m-i}{p}}x^{(i-1)v}.$$

Then

$$f'(v) = \frac{(x^p-1)}{p}\left(\sum_{i=1}^{m}(m-i)(1-v+vx^p)^{\frac{m-i}{p}-1}x^{(i-1)v}\right) + \left(\sum_{i=1}^{m}(i-1)(1-v+vx^p)^{\frac{m-i}{p}}x^{(i-1)v}\right)\ln x.$$

(i) When $0<x\leqslant 1$, we have $\frac{(x^p-1)}{p}\leqslant 0$ and $\ln x\leqslant 0$, so $f'(v)\leqslant 0$, which means $\frac{f(v)}{f(\tau)}\geqslant 1$. Therefore,

$$\begin{aligned}\frac{(1-v+vx^p)^{\frac{m}{p}}-x^{mv}}{(1-\tau+\tau x^p)^{\frac{m}{p}}-x^{m\tau}} &= \frac{\left((1-v+vx^p)^{\frac{1}{p}}-x^v\right)f(v)}{\left((1-\tau+\tau x^p)^{\frac{1}{p}}-x^\tau\right)f(\tau)}\\ &\geqslant \frac{(1-v+vx^p)^{\frac{1}{p}}-x^v}{(1-\tau+\tau x^p)^{\frac{1}{p}}-x^\tau}\\ &\geqslant \frac{v(1-v)}{\tau(1-\tau)}\ \ (\text{by } (2.1)).\end{aligned}$$

(ii) When $x\geqslant 1$, we have $\frac{(x^p-1)}{p}\geqslant 0$ and $\ln x\geqslant 0$, so $f'(v)\geqslant 0$, which means $\frac{f(v)}{f(\tau)}\leqslant 1$. Therefore,

$$\begin{aligned}\frac{(1-v+vx^p)^{\frac{m}{p}}-x^{mv}}{(1-\tau+\tau x^p)^{\frac{m}{p}}-x^{m\tau}} &= \frac{\left((1-v+vx^p)^{\frac{1}{p}}-x^v\right)f(v)}{\left((1-\tau+\tau x^p)^{\frac{1}{p}}-x^\tau\right)f(\tau)}\\ &\leqslant \frac{(1-v+vx^p)^{\frac{1}{p}}-x^v}{(1-\tau+\tau x^p)^{\frac{1}{p}}-x^\tau}\\ &\leqslant \frac{v(1-v)}{\tau(1-\tau)}\ \ (\text{by } (2.2)).\end{aligned}$$

Taking $x=\frac{b}{a}$, we can get the desired results. □

Following the ideas as above, we now show a generalization of (1.8).

LEMMA 2.3. *Let* $a,b>0$, $p\in[\frac{1}{2},1]$ *and* $0<v\leqslant\tau<1$.
(i) *If* $b\leqslant a$, *then*

$$\frac{a\sharp_{p,v}b-a!_v b}{a\sharp_{p,\tau}b-a!_\tau b}\geqslant\frac{v(1-v)}{\tau(1-\tau)}. \tag{2.5}$$

(ii) *If* $b\geqslant a$, *then*

$$\frac{a\sharp_{p,v}b-a!_v b}{a\sharp_{p,\tau}b-a!_\tau b}\leqslant\frac{v(1-v)}{\tau(1-\tau)}. \tag{2.6}$$

Proof. Let $f(v)=\frac{(1-v+vx^p)^{\frac{1}{p}}-\left(1-v+vx^{-1}\right)^{-1}}{v(1-v)}$. Then we have $f'(v)=\frac{1}{\left(v-v^2\right)^2}g(x)$, where

$$g(x)=v(1-v)\left(\frac{1}{p}(1-v+vx^p)^{\frac{1}{p}-1}(x^p-1)+\left(1-v+vx^{-1}\right)^{-2}\left(x^{-1}-1\right)\right)\\ -(1-2v)\left((1-v+vx^p)^{\frac{1}{p}}-\left(1-v+vx^{-1}\right)^{-1}\right).$$

So we have

$$g'(x)=\frac{v^2}{p}x^{p-1}(1-v+vx^p)^{\frac{1}{p}-2}\left((1-v)(1-p)(x^p-1)+p(1-v+vx^p)\right)\\ +v^2x^{-2}\left(1-v+vx^{-1}\right)^{-3}\left(2(1-v)\left(x^{-1}-1\right)-\left(1-v+vx^{-1}\right)\right).$$

Thus,

$$g''(x)=(1-v+vx^p)^{\frac{1}{p}-3}\frac{v^2}{p}x^{p-2}(2p-1)(x^p-1)(1-p)(1-v)^2\\ +6\left(1-v+vx^{-1}\right)^{-4}v^2x^{-4}(x-1)(1-v)^2.$$

When $0<x\leqslant 1$, then $g''(x)\leqslant 0\Rightarrow g'(x)\geqslant g'(1)=0$, so $g(x)\leqslant g(1)=0$, that is $f'(v)\leqslant 0$, which means $f(v)\geqslant f(\tau)$. Therefore,

$$\frac{(1-v+vx^p)^{\frac{1}{p}}-\left(1-v+vx^{-1}\right)^{-1}}{v(1-v)}\geqslant\frac{(1-\tau+\tau x^p)^{\frac{1}{p}}-\left(1-\tau+\tau x^{-1}\right)^{-1}}{\tau(1-\tau)}.$$

When $x\geqslant 1$, then $g''(x)\geqslant 0\Rightarrow g'(x)\geqslant g'(1)=0$, so $g(x)\geqslant g(1)=0$, that is $f'(v)\geqslant 0$, which means $f(v)\leqslant f(\tau)$. Therefore,

$$\frac{(1-v+vx^p)^{\frac{1}{p}}-\left(1-v+vx^{-1}\right)^{-1}}{v(1-v)}\leqslant\frac{(1-\tau+\tau x^p)^{\frac{1}{p}}-\left(1-\tau+\tau x^{-1}\right)^{-1}}{\tau(1-\tau)}.$$

Taking $x=\frac{b}{a}$, we can get the desired inequalities. □

THEOREM 2.4. *Let* $a,b>0$, $p\in[\frac{1}{2},1]$, $0<v\leqslant\tau<1$ *and* $m\in\mathbb{N}^+$.
(i) *If* $b\leqslant a$, *then*

$$\frac{(a\sharp_{p,v}b)^m-(a!_vb)^m}{(a\sharp_{p,\tau}b)^m-(a!_\tau b)^m}\geqslant\frac{v(1-v)}{\tau(1-\tau)}. \tag{2.7}$$

(ii) *If* $b\geqslant a$, *then*

$$\frac{(a\sharp_{p,v}b)^m-(a!_vb)^m}{(a\sharp_{p,\tau}b)^m-(a!_\tau b)^m}\leqslant\frac{v(1-v)}{\tau(1-\tau)}. \tag{2.8}$$

Proof. By computations, we have

$$\begin{aligned}&(1-v+vx^p)^{\frac{m}{p}}-\left(1-v+vx^{-1}\right)^{-m}\\&=\left((1-v+vx^p)^{\frac{1}{p}}-\left(1-v+vx^{-1}\right)^{-1}\right)\left(\sum_{i=1}^{m}(1-v+vx^p)^{\frac{m-i}{p}}\left(1-v+vx^{-1}\right)^{-(i-1)}\right).\end{aligned}$$

Let

$$f(v)=\sum_{i=1}^{m}(1-v+vx^p)^{\frac{m-i}{p}}\left(1-v+vx^{-1}\right)^{-(i-1)}.$$

Then

$$\begin{aligned}f'(v)=&\frac{(x^p-1)}{p}\left(\sum_{i=1}^{m}(m-i)(1-v+vx^p)^{\frac{m-i}{p}-1}\left(1-v+vx^{-1}\right)^{-(i-1)}\right)\\&-\left(x^{-1}-1\right)\left(\sum_{i=1}^{m}(i-1)(1-v+vx^p)^{\frac{m-i}{p}}\left(1-v+vx^{-1}\right)^{-(i-1)-1}\right).\end{aligned}$$

(i) When $0<x\leqslant 1$, we have $\frac{(x^p-1)}{p}\leqslant 0$ and $\left(x^{-1}-1\right)\geqslant 0$, so $f'(v)\leqslant 0$, which means $\frac{f(v)}{f(\tau)}\geqslant 1$. Therefore,

$$\begin{aligned}\frac{(1-v+vx^p)^{\frac{m}{p}}-\left(1-v+vx^{-1}\right)^{-m}}{(1-\tau+\tau x^p)^{\frac{m}{p}}-\left(1-\tau+\tau x^{-1}\right)^{-m}}&=\frac{\left((1-v+vx^p)^{\frac{1}{p}}-\left(1-v+vx^{-1}\right)^{-1}\right)f(v)}{\left((1-\tau+\tau x^p)^{\frac{1}{p}}-\left(1-\tau+\tau x^{-1}\right)^{-1}\right)f(\tau)}\\&\geqslant\frac{(1-v+vx^p)^{\frac{1}{p}}-\left(1-v+vx^{-1}\right)^{-1}}{(1-\tau+\tau x^p)^{\frac{1}{p}}-\left(1-\tau+\tau x^{-1}\right)^{-1}}\\&\geqslant\frac{v(1-v)}{\tau(1-\tau)}\quad\text{(by (2.5))}.\end{aligned}$$

(ii) When $x\geqslant 1$, we have $\frac{(x^p-1)}{p}\geqslant 0$ and $\left(x^{-1}-1\right)\leqslant 0$, so $f'(v)\geqslant 0$, which means $\frac{f(v)}{f(\tau)}\leqslant 1$. Therefore,

$$\begin{aligned}\frac{(1-v+vx^p)^{\frac{m}{p}}-\left(1-v+vx^{-1}\right)^{-m}}{(1-\tau+\tau x^p)^{\frac{m}{p}}-\left(1-\tau+\tau x^{-1}\right)^{-m}}&=\frac{\left((1-v+vx^p)^{\frac{1}{p}}-\left(1-v+vx^{-1}\right)^{-1}\right)f(v)}{\left((1-\tau+\tau x^p)^{\frac{1}{p}}-\left(1-\tau+\tau x^{-1}\right)^{-1}\right)f(\tau)}\\&\leqslant\frac{(1-v+vx^p)^{\frac{1}{p}}-\left(1-v+vx^{-1}\right)^{-1}}{(1-\tau+\tau x^p)^{\frac{1}{p}}-\left(1-\tau+\tau x^{-1}\right)^{-1}}\\&\leqslant\frac{v(1-v)}{\tau(1-\tau)}\quad\text{(by (2.6))}.\end{aligned}$$

Taking $x=\frac{b}{a}$, we can get the desired results directly. □

REMARK 2.5. It should be reminded readers that replacing v,τ,a,b with $1-\tau,1-v,b,a$, respectively, in the first inequalities of Lemma 2.1, Theorem 2.2, Lemma 2.3 and Theorem 2.4, then we can also obtained the second inequalities of them.

REMARK 2.6. When $p=1$, we can get inequalities (1.7) and (1.8) by Theorems 2.2 and 2.4, respectively.

Next, we give some operator inequalities as promised.

THEOREM 2.7. *Let* $A,B\in\mathbb{B}(\mathbb{H})$ *be strictly positive,* $0<v\leqslant\tau<1$ *and* $p\in[\frac{1}{2},1]$.
(i) *If* $B\leqslant A$*, then*

$$A\sharp_{p,v}B-A\sharp_vB\geqslant\frac{v(1-v)}{\tau(1-\tau)}(A\sharp_{p,\tau}B-A\sharp_\tau B) \tag{2.9}$$

and

$$A\sharp_{p,v}B-A!_vB\geqslant\frac{v(1-v)}{\tau(1-\tau)}(A\sharp_{p,\tau}B-A!_\tau B). \tag{2.10}$$

(ii) *If* $B\geqslant A$*, then*

$$A\sharp_{p,v}B-A\sharp_vB\leqslant\frac{v(1-v)}{\tau(1-\tau)}(A\sharp_{p,\tau}B-A\sharp_\tau B) \tag{2.11}$$

and

$$A\sharp_{p,v}B-A!_vB\leqslant\frac{v(1-v)}{\tau(1-\tau)}(A\sharp_{p,\tau}B-A!_\tau B). \tag{2.12}$$

Proof. By a standard functional calculus in the inequality (2.1) and (2.5) with $a=I$ and $b=A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, respectively, we obtain

$$\begin{aligned}&\left((1-v)I+v\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)^p\right)^{\frac{1}{p}}-\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)^v\\&\geqslant\frac{v(1-v)}{\tau(1-\tau)}\left(\left((1-\tau)I+\tau\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)^p\right)^{\frac{1}{p}}-\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)^\tau\right)\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}&\left((1-v)I+v\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)^p\right)^{\frac{1}{p}}-\left((1-v)I+v\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)^{-1}\right)^{-1}\\&\geqslant\frac{v(1-v)}{\tau(1-\tau)}\left(\left((1-\tau)I+\tau\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)^p\right)^{\frac{1}{p}}-\left((1-\tau)I+\tau\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)^{-1}\right)^{-1}\right).\end{aligned} \tag{2.14}$$

Multiplying $A^{\frac{1}{2}}$ to both sides of (2.13) and (2.14), we can get (2.9) and (2.10), respectively.

Similarly, we can obtain (2.11) and (2.12) by (2.2) and (2.6), so we omit it. □

At the end of this paper, we give some determinant inequalities using the following lemma.

LEMMA 2.8. [4] *Let* $a=[a_i]$, $b=[b_i]$, $i=1,2,\cdots,n$ *be such that* a_i,b_i *positive real numbers. Then*

$$\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} + \left(\prod_{i=1}^{n} b_i\right)^{\frac{1}{n}} \leqslant \left(\prod_{i=1}^{n} (a_i+b_i)\right)^{\frac{1}{n}}.$$

THEOREM 2.9. *Let* $A,B \in M_n^{++}(\mathbb{C})$ *be such that* $A \geqslant B$. *If* $0<v\leqslant \tau <1$, $p\in[\frac{1}{2},1]$ *and* $m\in\mathbb{N}^+$, *then we have*

$$\det(A\sharp_{p,v}B)^{\frac{m}{n}} - \det(A\sharp_{v}B)^{\frac{m}{n}} \geqslant \frac{v(1-v)}{\tau(1-\tau)} \det(A\sharp_{p,\tau}B - A\sharp_{\tau}B)^{\frac{m}{n}} \tag{2.15}$$

and

$$\det(A\sharp_{p,v}B)^{\frac{m}{n}} - \det(A!_{v}B)^{m} \geqslant \frac{v(1-v)}{\tau(1-\tau)} \det(A\sharp_{p,\tau}B - A!_{\tau}B)^{\frac{m}{n}}. \tag{2.16}$$

Proof. We denote the positive definite matrix $T=A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. By the inequality (2.3), we have

$$\frac{(1\sharp_{p,v}s_i(T))^m - (1\sharp_{v}s_i(T))^m}{(1\sharp_{p,\tau}s_i(T))^m - (1\sharp_{\tau}s_i(T))^m} \geqslant \frac{v(1-v)}{\tau(1-\tau)}, \quad i=1,2,\cdots,n. \tag{2.17}$$

Since the determinant of a positive definite matrix is product of its singular values. Thus,

$$\begin{aligned}
&\det(I\sharp_{p,v}T)^{\frac{m}{n}} \\
&= \left(\prod_{i=1}^{n} 1\sharp_{p,v}s_i(T)\right)^{\frac{m}{n}} \\
&\geqslant \left(\prod_{i=1}^{n}\left[\frac{v(1-v)}{\tau(1-\tau)}\left((1\sharp_{p,\tau}s_i(T))^m - (1\sharp_{\tau}s_i(T))^m\right) + (1\sharp_{v}s_i(T))^m\right]\right)^{\frac{1}{n}} \quad (\text{by } 2.17) \\
&\geqslant \prod_{i=1}^{n}\left[\frac{v(1-v)}{\tau(1-\tau)}\left((1\sharp_{p,\tau}s_i(T))^m - (1\sharp_{\tau}s_i(T))^m\right)\right]^{\frac{1}{n}} + \prod_{i=1}^{n}(1\sharp_{v}s_i(T))^{\frac{m}{n}} \quad (\text{by Lemma } 2.8) \\
&\geqslant \frac{v(1-v)}{\tau(1-\tau)} \prod_{i=1}^{n}(1\sharp_{p,\tau}s_i(T) - 1\sharp_{\tau}s_i(T))^{\frac{m}{n}} + \prod_{i=1}^{n}(1\sharp_{v}s_i(T))^{\frac{m}{n}} \\
&= \frac{v(1-v)}{\tau(1-\tau)} \det(I\sharp_{p,\tau}T - I\sharp_{\tau}T)^{\frac{m}{n}} + \det(I\sharp_{v}T)^{\frac{m}{n}},
\end{aligned}$$

where the last inequality is by the fact $a^m - b^m \geqslant (a-b)^m$ for $a \geqslant b > 0$ and $m \in \mathbb{N}^+$. Then multiply both sides of the inequalities above by $\left(\det A^{\frac{1}{2}}\right)^{\frac{m}{n}}$, we complete the proof of (2.15).

Using the same method, we can get (2.16) by (2.7). $\square$

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