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UPPER BOUND ESTIMATE FOR THE NORM OF REPEATED DE LA VALLÉE POUSSIN OPERATORS

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Abstract. The Lebesgue constant for the repeated de la Vallée Poussin operator, defined in the space of continuous periodic functions, is studied. An integral representation of the repeated de la Vallée Poussin means is obtained as a sum of Riemann integrals over finite domains. Based on this, an upper bound for the norm of the repeated de la Vallée Poussin operators is derived, expressed in terms of the well-studied Lebesgue constant of the Fourier operator.

1. Introduction

Let $\mathrm{L}(\mathbb{T})$, where $\mathbb{T}=[-\pi,\pi]$, be the space of summable 2π-periodic functions. Let

$$S[f]=\frac{a_0[f]}{2}+\sum_{k=1}^{\infty}(a_k[f]\cos kx+b_k[f]\sin kx)$$

be the Fourier series of the function $f\in\mathrm{L}(\mathbb{T})$, where

$$a_k[f]=\frac{1}{\pi}\int\limits_{\mathbb{T}} f(x)\cos kx\,dx,\quad b_k[f]=\frac{1}{\pi}\int\limits_{\mathbb{T}} f(x)\sin kx\,dx,\quad k\in\mathbb{Z}_+,$$

are the Fourier coefficients of the function f. Let

$$S_n[f](x)=\frac{a_0[f]}{2}+\sum_{k=1}^{n}(a_k[f]\cos kx+b_k[f]\sin kx)$$

be the n-th partial sum of the Fourier series of the function f.

The means

$$V_{n,p}[f](x)=\frac{1}{p}\sum_{k=n-p}^{n-1}S_k[f](x),\quad p\in\mathbb{N},\quad p<n$$

are called de la Vallée Poussin means of $S[f]$ [26, 27].

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These means are given by integrals of convolution type:

$$V_{n,p}[f](x) = \frac{1}{\pi}\int_{\mathbb{T}} f(x-t)\mathscr{V}_{n,p}(t)\,dt,$$

where $\mathscr{V}_{n,p}(t)$ is the de la Vallée Poussin kernel with parameters n and p. This kernel is given by [27]

$$\mathscr{V}_{n,p}(t) = \frac{\cos(n-p)t - \cos nt}{4p\sin^2\frac{t}{2}} = \frac{\sin\frac{pt}{2}\sin\frac{(2n-p)t}{2}}{2p\sin^2\frac{t}{2}}.$$

The de la Vallée Poussin means $V_{n,p}[f]$ are characterized by their simplicity of formulation and good approximation properties. This combination of qualities has led to these means, as well as their generalizations, being studied in various directions by many authors (see, e.g., [1, 5, 10, 12, 22, 24, 25]).

Let $\mathrm{C}(\mathbb{T})$ be the space of continuous 2π-periodic functions with the norm

$$\|f\|_{\mathrm{C}} = \max_{x\in\mathbb{T}}|f(x)|.$$

We consider $V_{n,p}[f]$ as linear operators in the space $\mathrm{C}(\mathbb{T})$. Regarding the norms of the de la Vallée Poussin means

$$L_{n,p} := \sup_{\|f\|\leqslant 1}\left\|V_{n,p}[f](x)\right\|,$$

Stechkin [23] proved

$$L_{n,p} = \frac{2}{\pi}\int_0^\infty |\sin rt\ \sin t|t^{-2}\,dt,$$

where

$$r = \frac{2n-p}{p} = 2\frac{n}{p} - 1.$$

This formula refines the asymptotic formula of Nikol'skii [17]. More detailed information on other representations of $L_{n,p}$ and its generalizations can be found in the works [6, 15].

If $p=1$, then $L_{n,1} = L_{n-1}$, which is the well-known Lebesgue constant (see, e.g., [8, 9], [29, Ch. II, § 12]).

The repeated de la Vallée Poussin means of $S[f]$, defined by

$$V^{(2)}_{n,p}[f](x) = \frac{1}{p}\sum_{k=n-p}^{n-1} V_{k+1,p}[f](x) = \frac{1}{p}\sum_{k=n-p}^{n-1}\frac{1}{p}\sum_{m=k-p+1}^{k} S_m[f](x), \quad p<\frac{n}{2},$$

were introduced in [20]. Recently, several works have appeared on the approximation properties of the repeated de la Vallée Poussin means [13, 14, 19, 21]. These works demonstrate the approximation capabilities of the operators $V^{(2)}_{n,p}[f]$. Motivated by these studies, we continue to explore the approximation properties of the repeated de la Vallée Poussin means. Specifically, we aim to derive upper estimates for the quantities

$$L_{n,p}^{(2)} := \sup_{\|f\|\leqslant 1} \left\| V_{n,p}^{(2)}[f](x) \right\|$$

in terms of the Lebesgue constants L_n. The properties of L_n, particularly upper, and less frequently, lower estimates, have been extensively studied in the mathematical literature (see, e.g., [2, 3, 4, 7, 16, 18, 28, 29], and the references therein).

2. Upper bound estimate for the norm of repeated de la Vallée Poussin means

The main result is as follows.

THEOREM 1. *Let $f \in \mathrm{L}(\mathbb{T})$, $p,\ n \in \mathbb{N}$, $p < \frac{n}{2}$. Then*

$$L_{n,p}^{(2)} \leqslant \frac{16}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{4k^2-1} \int_0^{\infty} \frac{\sin^2 x \sin^2 2krx}{x^3}\, dx + \frac{1}{2p^2} L_{n-p}, \tag{1}$$

where $r = \frac{2(n-p)+1}{p}$, and L_n is the Lebesgue constant.

The proof of Theorem 1 is based on the following lemma.

LEMMA 1. *Let $z \in \mathbb{C}$. Then*

$$\frac{1}{\sin^3 z} = \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(z-k\pi)^3} + \frac{1}{2}\frac{1}{\sin z}. \tag{2}$$

Proof. Applying the Mittag-Leffler theorem to the function

$$F(z) = \frac{1}{\sin^3 z} - \frac{1}{z^3} - \frac{1}{2z},$$

we obtain

$$F(z) = F(0) + \sum_{k=-\infty}^{\infty} \left(G_k\left(\frac{1}{z-k\pi}\right) - G_k\left(\frac{1}{-k\pi}\right)\right),$$

where

$$G_k\left(\frac{1}{z-k\pi}\right) = \frac{(-1)^k}{(z-\pi k)^3} + \frac{1}{2}\frac{(-1)^k}{z-\pi k}$$

represents the principal part of the Laurent series of the function $F(z)$ at the poles $z_k = k\pi,\ k \in \mathbb{N}$.

Thus,

$$\frac{1}{\sin^3 z} = \frac{1}{z^3} + \frac{1}{2z} + \sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} \left((-1)^k \left[\frac{1}{(z-k\pi)^3} + \frac{1/2}{z-k\pi} \right] - (-1)^k \left[\frac{1}{(-k\pi)^3} + \frac{1/2}{-k\pi} \right] \right)$$
$$= \sum_{k=-\infty}^{\infty} (-1)^k \left(\frac{1}{(z-k\pi)^3} + \frac{1/2}{z-k\pi} \right). \tag{3}$$

A key motivation for Mittag-Leffler expansions was a result derived by Euler in *Introductio in analysin infinitorum* (1748):

$$\frac{\pi}{\sin \pi z} = \frac{1}{z} + 2z \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 - z^2}.$$

Based on this formula (see also (1.422 (3)) and (1.422 (6)) in [11]), we obtain

$$\frac{1}{\sin z} = \frac{1}{z} + \sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} (-1)^k \left[\frac{1}{z-k\pi} + \frac{1}{\pi k} \right]. \tag{4}$$

Combining (3) and (4), we can immediately derive formula (2). Thus, Lemma 1 is proved. □

Proof of Theorem 1. We have (see [14])

$$V_{n,p}^{(2)}[f](x) = \frac{1}{\pi} \int\limits_{\mathbb{T}} f(x-t) \mathscr{V}_{n,p}^{(2)}(t)\, dt,$$

where

$$\mathscr{V}_{n,p}^{(2)}(t) = \frac{\sin^2 \frac{pt}{2} \sin \frac{(2(n-p)+1)t}{2}}{2p^2 \sin^3 \frac{t}{2}}.$$

Based on Lemma 1, we have

$$V_{n,p}^{(2)}[f](x) = \frac{4}{\pi p^2} \int\limits_{\mathbb{T}} f(x-t) \sum_{k=-\infty}^{\infty} (-1)^k \sin^2 \frac{pt}{2} \sin \frac{(2(n-p)+1)t}{2} \frac{dt}{(t+2k\pi)^3}$$
$$+ \frac{1}{4\pi p^2} \int\limits_{\mathbb{T}} f(x-t) \sin^2 \frac{pt}{2} \sin \frac{(2(n-p)+1)t}{2} \frac{dt}{\sin \frac{t}{2}}$$
$$:= J_{n,p}^{(1)}[f](x) + J_{n,p}^{(2)}[f](x). \tag{5}$$

For $J^{(1)}_{n,p}[f](x)$, we have

$$J^{(1)}_{n,p}[f](x) = \frac{4}{\pi p^2}\int_{\mathbb{T}} f(x-t)\sum_{k=-\infty}^{\infty}(-1)^k \sin^2\frac{pt}{2}\sin\frac{(2(n-p)+1)t}{2}\frac{dt}{(t+2k\pi)^3}$$

$$=\frac{4}{\pi p^2}\sum_{k=-\infty}^{\infty}\int_{\mathbb{T}} f(x-t-2k\pi)\sin^2\frac{p(t+2k\pi)}{2}\sin\frac{(2(n-p)+1)(t+2k\pi)}{2}\frac{dt}{(t+2k\pi)^3}$$

$$=\frac{4}{\pi p^2}\sum_{k=-\infty}^{\infty}\int_{-\pi+2k\pi}^{\pi+2k\pi} f(x-t)\sin^2\frac{pt}{2}\sin\frac{(2(n-p)+1)t}{2}\frac{dt}{t^3}$$

$$=\frac{4}{\pi p^2}\int_{-\infty}^{\infty} f(x-t)\sin^2\frac{pt}{2}\sin\frac{(2(n-p)+1)t}{2}\frac{dt}{t^3}.$$

Therefore,

$$\sup_{\|f\|\leqslant 1}\left\|J^{(1)}_{n,p}[f](x)\right\| = \frac{4}{\pi p^2}\int_{-\infty}^{\infty}\sin^2\frac{pt}{2}\left|\sin\frac{(2(n-p)+1)t}{2}\frac{1}{t^3}\right| dt$$

$$= \frac{2}{\pi p^2}\int_0^{\infty}\frac{\sin^2 pu|\sin(2(n-p)+1)u|}{u^3}\,du.$$

Based on the Fourier series expansion of the function $|\sin x|$:

$$|\sin x| = \frac{2}{\pi}\left(1-2\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\cos 2kx\right) = \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\sin^2 2kx,$$

we obtain

$$\sup_{\|f\|\leqslant 1}\left\|J^{(1)}_{n,p}[f](x)\right\| = \frac{16}{\pi^2 p^2}\int_0^{\infty}\frac{\sin^2 pu}{u^3}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\sin^2 2k(2(n-p)+1)u\,du$$

$$= \frac{16}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\int_0^{\infty}\frac{\sin^2 pu\sin^2 2k(2(n-p)+1)u}{u^3}\,du.$$

Using the notation $r=\frac{2(n-p)+1}{p}$, we obtain

$$\sup_{\|f\|\leqslant 1}\left\|J^{(1)}_{n,p}[f](x)\right\| = \frac{16}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\int_0^{\infty}\frac{\sin^2 x\sin^2 2krx}{x^3}\,dx. \tag{6}$$

For $J^{(2)}_{n,p}[f](x)$, we have

$$\sup_{\|f\|\leqslant 1}\left\|J^{(2)}_{n,p}[f](x)\right\| dt \leqslant \frac{1}{4\pi p^2}\int_{\mathbb{T}}\left|\frac{\sin\frac{(2n-2p+1)t}{2}}{\sin\frac{t}{2}}\right| dt = \frac{1}{2p^2}L_{n-p}, \tag{7}$$

where L_n is the Lebesgue constant.

Combining (5), (6), and (7), we obtain (1). Theorem 1 is proved. □

REMARK 1. For the improper integral in (6), we use the following formula (see (3.828 (1)), (3.828 (4)), and (3.827 (6)) in [11]):

$$\int_0^\infty \frac{\sin^2 \alpha x \sin^2 \beta x}{x^3}\, dx = \begin{cases} \frac{\alpha^2}{4} \ln \frac{|\alpha^2-\beta^2|}{\alpha^2} + \frac{\beta^2}{4} \ln \frac{|\alpha^2-\beta^2|}{\beta^2} + \frac{\alpha\beta}{2} \ln \frac{|\alpha^2-\beta^2|}{(\alpha-\beta)^2}, & \alpha \neq \beta, \\ \alpha^2 \ln 2, & \alpha = \beta. \end{cases}$$

3. Conclusion

A representation of the Lebesgue constants for the repeated de la Vallée Poussin operators, $L_{n,p}^{(2)}$, in terms of the Lebesgue constants for Fourier operators, L_n, is obtained. Consequently, this representation can be used to apply known estimates for L_n (see, e.g., [2, 4, 18, 28], etc.) to study the properties of the norms of the repeated de la Vallée Poussin operators.

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