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ON SEQUENCES SATISFYING FOURTH–ORDER DIFFERENCE INEQUALITIES

BESSEM SAMET

(*Communicated by M. Niezgoda*)

Abstract. We consider the class of sequences $a = (a_1, \cdots, a_n) \in \mathbb{R}^n$ satisfying the fourth-order difference inequality $\Delta^4 a_i \leqslant 0$, $i = 1, \cdots, n-4$. A Hermite-Hadamard-type inequality is established for this class of sequences. The proof of our result is based on the choice of an appropriate sequence which is the solution to a certain fourth-order difference equation. Moreover, if a is a convex sequence, we obtain an interesting refinement of the right discrete Hermite-Hadamard inequality. We next extend our study to the class of matrices satisfying a system of fourth-order difference inequalities. In particular, we obtain a trace inequality for the class of symmetric matrices.

1. Introduction

Convex functions are frequently applied to model many problems in engineering, economics, management, etc. Due to this fact, much effort has been devoted to the study of the properties of such functions, see e.g. [2, 3, 17, 28, 30–34, 40]. One of the important inequalities involving convex functions is the Hermite-Hadamard double inequality which can be stated as follows: If $f : [a, b] \to \mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The above double inequality dates back to an 1883 observation of Hermite [18] with an independent use by Hadamard [16] in 1893. Hermite-Hadamard double inequality is very useful in the study of the properties of convex functions and their applications in optimization and approximation theory, see e.g. [13–15]. This fact motivated the study of inequalities of type (1.1) in various directions. For more details, we refer to [1, 4–12, 19–22, 26, 27, 35, 36, 38] (see also the references therein).

The notion of convex sequences is a discrete version of the convexity concept. Namely, a sequence $a = (a_1, \cdots, a_n) \in \mathbb{R}^n$, where $n \geqslant 3$, is said to be convex, if a satisfies the second order difference inequality

$$\Delta^2 a_i \geqslant 0, \quad i = 1, \cdots, n-2,$$

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where $\Delta^2 a_i = a_{i+2} - 2a_{i+1} + a_i$. Several interesting inequalities involving convex sequences have been established, see e.g. [23–25, 29, 37, 39] and the references therein. In [23], the authors established (among many other results) a discrete version of Fejér double inequality [12]. In particular, they obtained the following discrete version of the Hermite-Hadamard double inequality (1.1): If $a = (a_1, \cdots, a_n) \in \mathbb{R}^n$ is a convex sequence, then

$$\frac{a_N + a_{n+1-N}}{2} \leqslant \frac{1}{n}\sum_{i=1}^{n} a_i \leqslant \frac{a_1 + a_n}{2}, \tag{1.2}$$

where $N = \left[\frac{n+1}{2}\right]$ is the integer part of $\frac{n+1}{2}$. We also refer to [29], where some extensions of the obtained results in [23] have been established using some matrix methods based on column stochastic and doubly stochastic matrices.

In this paper, we first consider the class of sequences $a = (a_1, \cdots, a_n) \in \mathbb{R}^n$, where $n \geqslant 5$, satisfying the fourth-order difference inequality

$$\Delta^4 a_i \leqslant 0, \quad i = 1, \cdots, n-4,$$

where $\Delta^4 a_i = \Delta^2(\Delta^2 a_i)$. For this class of sequences, we establish a Hermite-Hadamard-type inequality. If in addition the sequence a is convex, our obtained result provides an interesting refinement of the right inequality in (1.2). Our approach is completely different to that used in [23]. Namely, our method is based on the choice of an appropriate sequence, which is the solution to a certain fourth-order difference equation. We next consider the class of real matrices $A = (a_{i,j})_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}$, where $n, m \geqslant 5$, satisfying the system of fourth-order difference inequalities

(I) For all $j = 1, \cdots, m$,

$$\Delta_1^4 a_{i,j} \leqslant 0, \quad i = 1, \cdots, n-4,$$

(II) For all $i = 1, \cdots, n$,

$$\Delta_2^4 a_{i,j} \leqslant 0, \quad j = 1, \cdots, m-4,$$

where $\Delta_1^4 a_{i,j}$ (resp. $\Delta_2^4 a_{i,j}$) is the fourth-order partial difference operator with respect to the index i (resp. j). For this class of matrices, Hermite-Hadamard-type inequalities are proved.

Throughout this paper, we shall use the following notations. Let $n \geqslant 5$ and $a = (a_1, \cdots, a_n) \in \mathbb{R}^n$. We denote by Δ the first order difference operator defined by

$$\Delta a_i = a_{i+1} - a_i, \quad i = 1, \cdots, n-1.$$

By Δ^2, we mean the second order difference operator defined by

$$\Delta^2 a_i = \Delta(\Delta a_i) = a_{i+2} - 2a_{i+1} + a_i, \quad i = 1, \cdots, n-2.$$

The fourth-order difference operator Δ^4 is defined by

$$\Delta^4 a_i = \Delta^2(\Delta^2 a_i) = a_{i+4} - 4a_{i+3} + 6a_{i+2} - 4a_{i+1} + a_i, \quad i = 1, \cdots, n-4.$$

Let $n,m \geqslant 5$ and $A=(a_{i,j})_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}$ be a real matrix. For all $i=1,\cdots,n-1$, we denote by $\Delta_1 a_{i,j}$ the partial difference operator with respect to the index i, defined by

$$\Delta_1 a_{i,j} = a_{i+1,j} - a_{i,j}, \quad j=1,\cdots,m.$$

For all $i=1,\cdots,n-2$, we denote by $\Delta_1^2 a_{i,j}$ the second order partial difference operator with respect to the index i, defined by

$$\Delta_1^2 a_{i,j} = a_{i+2,j} - 2a_{i+1,j} + a_{i,j}, \quad j=1,\cdots,m.$$

For all $i=1,\cdots,n-4$, we denote by $\Delta_1^4 a_{i,j}$ the fourth-order partial difference operator with respect to the index i, defined by

$$\Delta_1^4 a_{i,j} = a_{i+4,j} - 4a_{i+3,j} + 6a_{i+2,j} - 4a_{i+1,j} + a_{i,j}, \quad j=1,\cdots,m.$$

Similarly, for all $j=1,\cdots,m-1$, we denote by $\Delta_2 a_{i,j}$ the partial difference operator with respect to the index j, defined by

$$\Delta_2 a_{i,j} = a_{i,j+1} - a_{i,j}, \quad i=1,\cdots,n.$$

For all $j=1,\cdots,m-2$, we denote by $\Delta_2^2 a_{i,j}$ the second order partial difference operator with respect to the index j, defined by

$$\Delta_2^2 a_{i,j} = a_{i,j+2} - 2a_{i,j+1} + a_{i,j}, \quad i=1,\cdots,n.$$

For all $j=1,\cdots,m-4$, we denote by $\Delta_2^4 a_{i,j}$ the fourth-order partial difference operator with respect to the index j, defined by

$$\Delta_2^4 a_{i,j} = a_{i,j+4} - 4a_{i,j+3} + 6a_{i,j+2} - 4a_{i,j+1} + a_{i,j}, \quad i=1,\cdots,n.$$

The rest of the paper is organized as follows. Section 2 is devoted to the main results and their proofs. Namely, in Subsection 2.1, we establish an auxiliary result that will be used later in the proofs of the main results. In Subsection 2.2, we establish a Hermite-Hadamard-type inequality for the class of sequences

$$\mathscr{S}_n = \left\{a=(a_1,\cdots,a_n)\in\mathbb{R}^n : \Delta^4 a_i \leqslant 0, i=1,\cdots,n-4\right\}.$$

In Subsection 2.3, a Hermite-Hadamard-type inequality is derived for the class of sequences

$${}^C\mathscr{S}_n = \{a=(a_1,\cdots,a_n)\in\mathscr{S}_n : a \text{ is convex}\}.$$

Finally, in Subsection 2.4, we obtain a Hermite-Hadamard-type inequality for systems of fourth-order difference inequalities.

2. Main results and proofs

2.1. An auxiliary result

For $n \geqslant 5$, let us consider the fourth-order difference equation

$$\Delta^4 b_{i-2} = 1, \quad i=3,\cdots,n-2 \tag{2.1}$$

under the boundary conditions

$$b_1 = b_2 = b_{n-1} = b_n = 0. \tag{2.2}$$

The following lemma will be useful later.

LEMMA 2.1. *Let $n \geqslant 5$. The sequence $b = (b_1, \cdots, b_n) \in \mathbb{R}^n$ defined by*

$$b_i = \frac{1}{24}(i-1)(i-2)(i-(n-1))(i-n), \quad i = 1, \cdots, n, \tag{2.3}$$

satisfies (2.1)–(2.2).

Proof. Let us write the possible solutions to problem (2.1)–(2.2) in the form $b = (b_1, \cdots, b_n) \in \mathbb{R}^n$, where

$$b_i = c_0 + c_1 i + c_2 i^2 + c_3 i^3 + c_4 i^4, \quad i = 1, \cdots, n,$$

where c_j, $j = 0, \cdots, 4$, are constants. Elementary calculations show that for all $i = 3, \cdots, n-2$,

$$\begin{aligned}\Delta^4 b_{i-2} &= b_{i+2} - 4b_{i+1} + 6b_i - 4b_{i-1} + b_{i-2} \\ &= 24c_4.\end{aligned}$$

Hence, taking $c_4 = \frac{1}{24}$, we obtain

$$\Delta^4 b_{i-2} = 1, \quad i = 3, \cdots, n-2.$$

On the other hand, b satisfies the boundary conditions (2.2) if and only if 1, 2, $n-1$ and n are roots of b_i for all $i = 1, \cdots, n$. Consequently, we get

$$b_i = c_4(i-1)(i-2)(i-(n-1))(i-n), \quad i = 1, \cdots, n,$$

which proves the desired result. □

2.2. The class of sequences satisfying $\Delta^4 a_i \leqslant 0$

For $n \geqslant 5$, we consider the class of sequences

$$\mathscr{S}_n = \left\{ a = (a_1, \cdots, a_n) \in \mathbb{R}^n : \Delta^4 a_i \leqslant 0, i = 1, \cdots, n-4 \right\}.$$

Our first main result is the following Hermite-Hadamard-type inequality.

THEOREM 2.2. *Let $n \geqslant 5$. If $a = (a_1, \cdots, a_n) \in \mathscr{S}_n$, then*

$$\frac{1}{n}\sum_{i=1}^{n} a_i \leqslant \frac{a_1 + a_n}{2} - \frac{n-1}{12}\left(\Delta a_{n-1} - \Delta a_1\right). \tag{2.4}$$

Proof. Let $a=(a_1,\cdots,a_n)\in\mathscr{S}_n$ and $b=(b_1,\cdots,b_n)\in\mathbb{R}^n$ be a given sequence. We have

$$\begin{aligned}
\sum_{i=3}^{n-2} b_i\Delta^4 a_{i-2} &= \sum_{i=3}^{n-2} b_i\left(a_{i+2}-4a_{i+1}+6a_i-4a_{i-1}+a_{i-2}\right)\\
&= \sum_{i=3}^{n-2} b_ia_{i+2}-4\sum_{i=3}^{n-2} b_ia_{i+1}+6\sum_{i=3}^{n-2} a_ib_i-4\sum_{i=3}^{n-2} b_ia_{i-1}+\sum_{i=3}^{n-2} b_ia_{i-2}\\
&= \sum_{i=5}^{n} a_ib_{i-2}-4\sum_{i=4}^{n-1} a_ib_{i-1}+6\sum_{i=3}^{n-2} a_ib_i-4\sum_{i=2}^{n-3} a_ib_{i+1}+\sum_{i=1}^{n-4} a_ib_{i+2}\\
&= -a_3b_1-a_4b_2+a_{n-1}b_{n-3}+a_nb_{n-2}+\sum_{i=3}^{n-2} a_ib_{i-2}\\
&\quad -4\left(-a_3b_2+a_{n-1}b_{n-2}+\sum_{i=3}^{n-2} a_ib_{i-1}\right)+6\sum_{i=3}^{n-2} a_ib_i\\
&\quad -4\left(a_2b_3-a_{n-2}b_{n-1}+\sum_{i=3}^{n-2} a_ib_{i+1}\right)\\
&\quad +a_1b_3+a_2b_4-a_{n-3}b_{n-1}-a_{n-2}b_n+\sum_{i=3}^{n-2} a_ib_{i+2},
\end{aligned}$$

that is,

$$\begin{aligned}
\sum_{i=3}^{n-2} b_i\Delta^4 a_{i-2} &= \sum_{i=3}^{n-2} a_i\left(b_{i+2}-4b_{i+1}+6b_i-4b_{i-1}+b_{i-2}\right)+\xi(a,b)\\
&= \sum_{i=3}^{n-2} a_i\Delta^4 b_{i-2}+\xi(a,b),
\end{aligned}$$

where

$$\begin{aligned}
\xi(a,b) &= a_3(4b_2-b_1)+a_{n-1}(b_{n-3}-4b_{n-2})+a_2(b_4-4b_3)+a_{n-2}(4b_{n-1}-b_n)\\
&\quad -a_4b_2+a_nb_{n-2}+a_1b_3-a_{n-3}b_{n-1}.
\end{aligned}$$

Now, let us consider the sequence b defined by (2.3). By Lemma 2.1, we obtain

$$\sum_{i=3}^{n-2} a_i = \sum_{i=3}^{n-2} b_i\Delta^4 a_{i-2}-a_{n-1}(b_{n-3}-4b_{n-2})-a_2(b_4-4b_3)-a_nb_{n-2}-a_1b_3,$$

which implies that

$$\begin{aligned}
\sum_{i=1}^{n} a_i &= \sum_{i=3}^{n-2} b_i\Delta^4 a_{i-2}-a_{n-1}(b_{n-3}-4b_{n-2}-1)-a_2(b_4-4b_3-1)\\
&\quad -a_n(b_{n-2}-1)-a_1(b_3-1).
\end{aligned}$$

On the other hand, by (2.3), we have $b_i \geqslant 0$ for all $i = 3, \cdots, n-2$. Furthermore, since $a \in \mathscr{S}_n$, we have $\Delta^4 a_{i-2} \leqslant 0$ for all $i = 3, \cdots, n-2$. Consequently, we get

$$\sum_{i=3}^{n-2} b_i \Delta^4 a_{i-2} \leqslant 0.$$

Therefore, it holds that

$$\sum_{i=1}^{n} a_i \leqslant -a_{n-1}(b_{n-3} - 4b_{n-2} - 1) - a_2(b_4 - 4b_3 - 1) - a_n(b_{n-2} - 1) - a_1(b_3 - 1). \tag{2.5}$$

Elementary calculations give us that

$$b_3 = b_{n-2} = \frac{1}{12}(n-4)(n-3),\ b_4 = b_{n-3} = \frac{1}{4}(n-5)(n-4),$$

which implies that

$$b_{n-3} - 4b_{n-2} - 1 = b_4 - 4b_3 - 1 = -\frac{1}{12}(n-1)n$$

and

$$b_{n-2} - 1 = b_3 - 1 = \frac{n(n-7)}{12} = \frac{n(n-1)}{12} - \frac{n}{2}.$$

Then, the right hand side of (2.5) can be written as

$$\begin{aligned}
&-a_{n-1}(b_{n-3} - 4b_{n-2} - 1) - a_2(b_4 - 4b_3 - 1) - a_n(b_{n-2} - 1) - a_1(b_3 - 1)\\
&= \frac{1}{12}(n-1)n(a_{n-1} + a_2) + \left(\frac{n}{2} - \frac{n(n-1)}{12}\right)(a_n + a_1)\\
&= \frac{n}{2}(a_1 + a_n) - \frac{n(n-1)}{12}\left((a_n - a_{n-1}) - (a_2 - a_1)\right)\\
&= \frac{n}{2}(a_1 + a_n) - \frac{n(n-1)}{12}\left(\Delta a_{n-1} - \Delta a_1\right).
\end{aligned}$$

Finally, (2.4) follows from the above identity and (2.5). □

2.3. The class of convex sequences satisfying $\Delta^4 a_i \leqslant 0$

For $n \geqslant 5$, we now consider the class of sequences

$${}^{C}\mathscr{S}_n = \{a = (a_1, \cdots, a_n) \in \mathscr{S}_n : a \text{ is convex}\}.$$

Observe that, if $a = (a_1, \cdots, a_n) \in \mathbb{R}^n$ is convex, that is,

$$\Delta^2 a_i \geqslant 0, \quad i = 1, \cdots, n-2,$$

then

$$a_{i+2} - 2a_{i+1} + a_i = (a_{i+2} - a_{i+1}) - (a_{i+1} - a_i) \geqslant 0, \quad i = 1, \cdots, n-2,$$

which means that the sequence $(\Delta a_i)_{i=1,\cdots,n-1}$ is nondecreasing. Consequently, if $a = (a_1,\cdots,a_n) \in \mathbb{R}^n$ is convex, then

$$\Delta a_{n-1} - \Delta a_1 \geqslant 0.$$

Hence, from Theorem 2.2, we deduce the following interesting refinement of the right inequality in (1.2).

COROLLARY 2.3. *Let $n \geqslant 5$. If $a = (a_1,\cdots,a_n) \in {}^{C}\mathscr{S}_n$, then*

$$\frac{1}{n}\sum_{i=1}^{n} a_i \leqslant \frac{a_1+a_n}{2} - \frac{n-1}{12}(\Delta a_{n-1} - \Delta a_1) \leqslant \frac{a_1+a_n}{2}.$$

2.4. Systems of fourth-order difference inequalities

We now consider the class of real matrices $A = (a_{i,j})_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}$, where $n,m \geqslant 5$, satisfying the system of fourth-order difference inequalities

(I) For all $j = 1,\cdots,m$,
$$\Delta_1^4 a_{i,j} \leqslant 0, \quad i = 1,\cdots,n-4,$$

(II) For all $i = 1,\cdots,n$,
$$\Delta_2^4 a_{i,j} \leqslant 0, \quad j = 1,\cdots,m-4.$$

We have the following result.

THEOREM 2.4. *Let $n,m \geqslant 5$. If $A = (a_{i,j})_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}$ satisfies (I) and (II), then*

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{m} a_{i,j} \\
&\leqslant \frac{n}{4}\left(\sum_{j=1}^{m} a_{1,j} + \sum_{j=1}^{m} a_{n,j}\right) + \frac{m}{4}\left(\sum_{i=1}^{n} a_{i,1} + \sum_{i=1}^{n} a_{i,m}\right) \\
&\quad - \frac{n(n-1)}{24}\sum_{j=1}^{m}\left(\Delta_1 a_{n-1,j} - \Delta_1 a_{1,j}\right) - \frac{m(m-1)}{24}\sum_{i=1}^{n}\left(\Delta_2 a_{i,m-1} - \Delta_2 a_{i,1}\right) \\
&\leqslant \frac{mn}{4}\left(a_{1,1} + a_{1,m} + a_{n,1} + a_{n,m}\right) \\
&\quad - \frac{m(m-1)n}{48}\left(\Delta_2 a_{1,m-1} - \Delta_2 a_{1,1} + \Delta_2 a_{n,m-1} - \Delta_2 a_{n,1}\right) \\
&\quad - \frac{n(n-1)m}{48}\left(\Delta_1 a_{n-1,1} - \Delta_1 a_{1,1} + \Delta_1 a_{n-1,m} - \Delta_1 a_{1,m}\right) \\
&\quad - \frac{n(n-1)}{24}\sum_{j=1}^{m}\left(\Delta_1 a_{n-1,j} - \Delta_1 a_{1,j}\right) - \frac{m(m-1)}{24}\sum_{i=1}^{n}\left(\Delta_2 a_{i,m-1} - \Delta_2 a_{i,1}\right).
\end{aligned} \tag{2.6}$$

Proof. Let $j \in \{1,\cdots,m\}$ be fixed. By (I), the sequence $(a_{1,j},\cdots,a_{n,j}) \in \mathscr{S}_n$. Hence, by Theorem 2.2, we obtain

$$\frac{1}{n}\sum_{i=1}^{n} a_{i,j} \leqslant \frac{a_{1,j}+a_{n,j}}{2} - \frac{n-1}{12}\left(\Delta_1 a_{n-1,j} - \Delta_1 a_{1,j}\right).$$

Summing over $j=1,\cdots,m$, we get

$$\sum_{j=1}^{m}\sum_{i=1}^{n} a_{i,j} \leqslant n\sum_{j=1}^{m} \frac{a_{1,j}+a_{n,j}}{2} - \frac{n(n-1)}{12}\sum_{j=1}^{m}\left(\Delta_1 a_{n-1,j} - \Delta_1 a_{1,j}\right). \tag{2.7}$$

Similarly, let $i \in \{1,\cdots,n\}$ be fixed. By (II), the sequence $(a_{i,1},\cdots,a_{i,m}) \in \mathscr{S}_m$. Hence, by Theorem 2.2, we obtain

$$\frac{1}{m}\sum_{j=1}^{m} a_{i,j} \leqslant \frac{a_{i,1}+a_{i,m}}{2} - \frac{m-1}{12}\left(\Delta_2 a_{i,m-1} - \Delta_2 a_{i,1}\right),$$

which implies after summation over $i=1,\cdots,n$ that

$$\sum_{i=1}^{n}\sum_{j=1}^{m} a_{i,j} \leqslant m\sum_{i=1}^{n} \frac{a_{i,1}+a_{i,m}}{2} - \frac{m(m-1)}{12}\sum_{i=1}^{n}\left(\Delta_2 a_{i,m-1} - \Delta_2 a_{i,1}\right). \tag{2.8}$$

From (2.7) and (2.8), we deduce that

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m} a_{i,j} \leqslant{}& \frac{n}{4}\left(\sum_{j=1}^{m} a_{1,j} + \sum_{j=1}^{m} a_{n,j}\right) + \frac{m}{4}\left(\sum_{i=1}^{n} a_{i,1} + \sum_{i=1}^{n} a_{i,m}\right) \\ & - \frac{n(n-1)}{24}\sum_{j=1}^{m}\left(\Delta_1 a_{1,j} - \Delta_1 a_{1,j}\right) - \frac{m(m-1)}{24}\sum_{i=1}^{n}\left(\Delta_2 a_{i,m-1} - \Delta_2 a_{i,1}\right).\end{aligned} \tag{2.9}$$

On the other hand, by (II), the sequence $(a_{1,1},\cdots,a_{1,m}) \in \mathscr{S}_m$. Then, Theorem 2.2 yields

$$\frac{1}{m}\sum_{j=1}^{m} a_{1,j} \leqslant \frac{a_{1,1}+a_{1,m}}{2} - \frac{m-1}{12}\left(\Delta_2 a_{1,m-1} - \Delta_2 a_{1,1}\right). \tag{2.10}$$

Similarly, we have

$$\frac{1}{m}\sum_{j=1}^{m} a_{n,j} \leqslant \frac{a_{n,1}+a_{n,m}}{2} - \frac{m-1}{12}\left(\Delta_2 a_{n,m-1} - \Delta_2 a_{n,1}\right). \tag{2.11}$$

From (2.10) and (2.11), we deduce that

$$\begin{aligned}\frac{n}{4}\left(\sum_{j=1}^{m} a_{1,j} + \sum_{j=1}^{m} a_{n,j}\right) \leqslant{}& \frac{mn}{8}\left(a_{1,1}+a_{1,m}+a_{n,1}+a_{n,m}\right) \\ & - \frac{m(m-1)n}{48}\left(\Delta_2 a_{1,m-1} - \Delta_2 a_{1,1} + \Delta_2 a_{n,m-1} - \Delta_2 a_{n,1}\right).\end{aligned} \tag{2.12}$$

Proceeding as above, we obtain

$$\frac{1}{n}\sum_{i=1}^{n} a_{i,1} \leqslant \frac{a_{1,1}+a_{n,1}}{2} - \frac{n-1}{12}\left(\Delta_1 a_{n-1,1} - \Delta_1 a_{1,1}\right) \tag{2.13}$$

and

$$\frac{1}{n}\sum_{i=1}^{n} a_{i,m} \leqslant \frac{a_{1,m}+a_{n,m}}{2} - \frac{n-1}{12}\left(\Delta_1 a_{n-1,m} - \Delta_1 a_{1,m}\right). \tag{2.14}$$

Then, (2.13) and (2.14) yield

$$\frac{m}{4}\left(\sum_{i=1}^{n} a_{i,1} + \sum_{i=1}^{n} a_{i,m}\right) \leqslant \frac{mn}{8}\left(a_{1,1}+a_{n,1}+a_{1,m}+a_{n,m}\right) - \frac{n(n-1)m}{48}\left(\Delta_1 a_{n-1,1} - \Delta_1 a_{1,1} + \Delta_1 a_{n-1,m} - \Delta_1 a_{1,m}\right). \tag{2.15}$$

Next, making use of (2.9), (2.12) and (2.15), we get

$$\begin{aligned}
&\frac{n}{4}\left(\sum_{j=1}^{m} a_{1,j} + \sum_{j=1}^{m} a_{n,j}\right) + \frac{m}{4}\left(\sum_{i=1}^{n} a_{i,1} + \sum_{i=1}^{n} a_{i,m}\right) \\
&\quad - \frac{n(n-1)}{24}\sum_{j=1}^{m}\left(\Delta_1 a_{n-1,j} - \Delta_1 a_{1,j}\right) - \frac{m(m-1)}{24}\sum_{i=1}^{n}\left(\Delta_2 a_{i,m-1} - \Delta_2 a_{i,1}\right) \\
&\leqslant \frac{mn}{4}\left(a_{1,1}+a_{1,m}+a_{n,1}+a_{n,m}\right) \\
&\quad - \frac{m(m-1)n}{48}\left(\Delta_2 a_{1,m-1} - \Delta_2 a_{1,1} + \Delta_2 a_{n,m-1} - \Delta_2 a_{n,1}\right) \\
&\quad - \frac{n(n-1)m}{48}\left(\Delta_1 a_{n-1,1} - \Delta_1 a_{1,1} + \Delta_1 a_{n-1,m} - \Delta_1 a_{1,m}\right) \\
&\quad - \frac{n(n-1)}{24}\sum_{j=1}^{m}\left(\Delta_1 a_{n-1,j} - \Delta_1 a_{1,j}\right) - \frac{m(m-1)}{24}\sum_{i=1}^{n}\left(\Delta_2 a_{i,m-1} - \Delta_2 a_{i,1}\right).
\end{aligned} \tag{2.16}$$

Finally, (2.6) follows from (2.9) and (2.16). □

Let us consider the special case of Theorem 2.4, where A is a square and symmetric matrix, that is,

$$m = n \geqslant 5,\ a_{i,j} = a_{j,i},\ i,j = 1,\cdots,n.$$

In this case, we obtain

$$\sum_{i=1}^{n}\sum_{j=1}^{m} a_{i,j} = \sum_{i=1}^{n} a_{i,i} + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_{i,j}, \tag{2.17}$$

$$\begin{aligned}
\frac{n}{4}\left(\sum_{j=1}^{m} a_{1,j} + \sum_{j=1}^{m} a_{n,j}\right) &= \frac{n}{4}\left(\sum_{i=1}^{n} a_{1,i} + \sum_{i=1}^{n} a_{n,i}\right) \\
&= \frac{n}{4}\left(a_{1,1}+a_{n,1}\right) + \frac{n}{4}\sum_{i=2}^{n}\left(a_{1,i}+a_{n,i}\right),
\end{aligned} \tag{2.18}$$

$$\begin{aligned}
&\frac{n(n-1)}{24}\sum_{j=1}^{m}\left(\Delta_1 a_{n-1,j}-\Delta_1 a_{1,j}\right)+\frac{m(m-1)}{24}\sum_{i=1}^{n}\left(\Delta_2 a_{i,m-1}-\Delta_2 a_{i,1}\right)\\
&=\frac{n(n-1)}{24}\sum_{i=1}^{n}\left(\Delta_1 a_{n-1,i}-\Delta_1 a_{1,i}+\Delta_2 a_{i,n-1}-\Delta_2 a_{i,1}\right)\\
&=\frac{n(n-1)}{24}\sum_{i=1}^{n}\left[(a_{n,i}-a_{n-1,i})-(a_{2,i}-a_{1,i})+(a_{i,n}-a_{i,n-1})-(a_{i,2}-a_{i,1})\right] \qquad (2.19)\\
&=\frac{n(n-1)}{12}\sum_{i=1}^{n}\left[(a_{i,n}-a_{i,n-1})-(a_{i,2}-a_{i,1})\right]\\
&=\frac{n(n-1)}{12}\sum_{i=1}^{n}\left(\Delta_2 a_{i,n-1}-\Delta_2 a_{i,1}\right),
\end{aligned}$$

$$\begin{aligned}
&\frac{m(m-1)n}{48}\left(\Delta_2 a_{1,m-1}-\Delta_2 a_{1,1}+\Delta_2 a_{n,m-1}-\Delta_2 a_{n,1}\right)\\
&\quad+\frac{n(n-1)m}{48}\left(\Delta_1 a_{n-1,1}-\Delta_1 a_{1,1}+\Delta_1 a_{n-1,m}-\Delta_1 a_{1,m}\right) \qquad (2.20)\\
&=\frac{n^2(n-1)}{24}\left(\Delta_2 a_{1,n-1}-\Delta_2 a_{1,1}+\Delta_2 a_{n,n-1}-\Delta_2 a_{n,1}\right),
\end{aligned}$$

and

$$\frac{mn}{4}\left(a_{1,1}+a_{1,m}+a_{n,1}+a_{n,m}\right)=\frac{n^2}{4}\left(a_{1,1}+2a_{1,n}+a_{n,n}\right). \qquad (2.21)$$

On the other hand, since A is symmetric, for all $i\in\{1,\cdots,n\}$ and $j\in\{1,\cdots,n-4\}$, we have

$$\begin{aligned}
\Delta_2^4 a_{i,j}&=a_{i,j+4}-4a_{i,j+3}+6a_{i,j+2}-4a_{i,j+1}+a_{i,j}\\
&=a_{j+4,i}-4a_{j+3,i}+6a_{j+2,i}-4a_{j+1,i}+a_{j,i}\\
&=\Delta_1^4 a_{j,i},
\end{aligned}$$

which shows that (I) and (II) are equivalent.

Hence, from Theorem 2.4, (2.17), (2.18), (2.19), (2.20), and (2.21), we deduce the following result.

COROLLARY 2.5. *Let $n\geqslant 5$. If $A=(a_{i,j})_{1\leqslant i,j\leqslant n}$ is a symmetric matrix satisfying (I) with $m=n$, then*

$$\begin{aligned}
&\frac{1}{n}\mathrm{Tr}(A)+\frac{2}{n}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}a_{i,j}\\
&\leqslant\frac{1}{2}\left(a_{1,1}+a_{n,1}+\sum_{i=2}^{n}(a_{1,i}+a_{n,i})\right)-\frac{n-1}{12}\sum_{i=1}^{n}(\Delta_2 a_{i,n-1}-\Delta_2 a_{i,1})\\
&\leqslant\frac{n}{4}\left(a_{1,1}+2a_{1,n}+a_{n,n}\right)-\frac{n(n-1)}{24}\left(\Delta_2 a_{1,n-1}-\Delta_2 a_{1,1}+\Delta_2 a_{n,n-1}-\Delta_2 a_{n,1}\right)\\
&\quad-\frac{n-1}{12}\sum_{i=1}^{n}(\Delta_2 a_{i,n-1}-\Delta_2 a_{i,1}),
\end{aligned}$$

where $\mathrm{Tr}(A)=\sum_{i=1}^{n} a_{i,i}$.

Assume that in addition of the assumptions of Corollary 2.5, the matrix A satisfies

(III) For all $i=1,\cdots,n$,

$$\Delta_2^2 a_{i,j}\geqslant 0,\quad j=1,\cdots,n-2.$$

In this case, for all $i=1,\cdots,n$, one has

$$\begin{aligned}\Delta_2^2 a_{i,j}&=a_{i,j+2}-2a_{i,j+1}+a_{i,j}\\&=(a_{i,j+2}-a_{i,j+1})-(a_{i,j+1}-a_{i,j})\\&=\Delta_2 a_{i,j+1}-\Delta_2 a_{i,j}\geqslant 0,\end{aligned}$$

which implies that the sequence $\left\{\Delta_2 a_{i,j}\right\}_{j=1,\cdots,n-1}$ is decreasing (with respect to j). Consequently, we have

$$\Delta_2 a_{i,n-1}-\Delta_2 a_{i,1}\geqslant 0,\quad i=1,\cdots,n.$$

Then, from Corollary 2.5, we deduce the following trace inequalities.

COROLLARY 2.6. *Let $n\geqslant 5$. If $A=(a_{i,j})_{1\leqslant i,j\leqslant n}$ is a symmetric matrix satisfying (I) with $m=n$, and (III), then*

$$\begin{aligned}\mathrm{Tr}(A)\leqslant{}&\frac{n}{2}\left(a_{1,1}+a_{n,1}+\sum_{i=2}^{n}(a_{1,i}+a_{n,i})\right)-2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}a_{i,j}\\&-\frac{n(n-1)}{12}\sum_{i=1}^{n}(\Delta_2 a_{i,n-1}-\Delta_2 a_{i,1})\\\leqslant{}&\frac{n}{2}\left(a_{1,1}+a_{n,1}+\sum_{i=2}^{n}(a_{1,i}+a_{n,i})\right)-2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}a_{i,j}.\end{aligned}$$

We provide below an example to illustrate the above result.

EXAMPLE 2.1. In this example, we will construct a symmetric matrix A with $m=n=5$ such that

$$\Delta_1^4 a_{1,j}=0,\quad j=1,\cdots,5. \tag{2.22}$$

Clearly, the above condition implies that (I) is satisfied.

Let us consider the symmetric matrix $A=(a_{i,j})_{1\leqslant i,j\leqslant 5}$ given by

$$\begin{pmatrix}-10 & -\frac{15}{2} & -4 & \frac{1}{2} & 6\\ -\frac{15}{2} & -\frac{103}{12} & -\frac{23}{3} & -\frac{19}{4} & \frac{1}{6}\\ -4 & -\frac{23}{3} & -\frac{25}{3} & -6 & -\frac{2}{3}\\ \frac{1}{2} & -\frac{19}{4} & -6 & -\frac{13}{4} & \frac{7}{2}\\ 6 & \frac{1}{6} & -\frac{2}{3} & \frac{7}{2} & \frac{38}{3}\end{pmatrix}.$$

- We first show that (2.22) holds.

  For $j=1$, we have

$$\begin{aligned}\Delta_1^4 a_{1,1} &= a_{5,1}-4a_{4,1}+6a_{3,1}-4a_{2,1}+a_{1,1}\\ &= 6-4\times\frac{1}{2}+6\times(-4)-4\times\left(-\frac{15}{2}\right)-10\\ &= 0.\end{aligned}$$

For $j=2$, we have

$$\begin{aligned}\Delta_1^4 a_{1,2} &= a_{5,2}-4a_{4,2}+6a_{3,2}-4a_{2,2}+a_{1,2}\\ &= \frac{1}{6}-4\times\left(-\frac{19}{4}\right)+6\times\left(-\frac{23}{3}\right)-4\times\left(-\frac{103}{12}\right)-\frac{15}{2}\\ &= 0.\end{aligned}$$

For $j=3$, we have

$$\begin{aligned}\Delta_1^4 a_{1,3} &= a_{5,3}-4a_{4,3}+6a_{3,3}-4a_{2,3}+a_{1,3}\\ &= -\frac{2}{3}-4\times(-6)+6\times\left(-\frac{25}{3}\right)-4\times\left(-\frac{23}{3}\right)-4\\ &= 0.\end{aligned}$$

For $j=4$, we have

$$\begin{aligned}\Delta_1^4 a_{1,4} &= a_{5,4}-4a_{4,4}+6a_{3,4}-4a_{2,4}+a_{1,4}\\ &= \frac{7}{2}-4\times\left(-\frac{13}{4}\right)+6\times(-6)-4\times\left(-\frac{19}{4}\right)+\frac{1}{2}\\ &= 0.\end{aligned}$$

For $j=5$, we have

$$\begin{aligned}\Delta_1^4 a_{1,5} &= a_{5,5}-4a_{4,5}+6a_{3,5}-4a_{2,5}+a_{1,5}\\ &= \frac{38}{3}-4\times\left(\frac{7}{2}\right)+6\times\left(-\frac{2}{3}\right)-4\times\left(\frac{1}{6}\right)+6\\ &= 0.\end{aligned}$$

Consequently, (2.22) holds.

- Next, we show that (III) is satisfied.

For $i=1$, we have

$$\begin{pmatrix} \Delta_2^2 a_{1,1} \\ \Delta_2^2 a_{1,2} \\ \Delta_2^2 a_{1,3} \end{pmatrix} = \begin{pmatrix} a_{1,3}-2a_{1,2}+a_{1,1} \\ a_{1,4}-2a_{1,3}+a_{1,2} \\ a_{1,5}-2a_{1,4}+a_{1,3} \end{pmatrix} = \begin{pmatrix} a_{1,3} & a_{1,2} & a_{1,1} \\ a_{1,4} & a_{1,3} & a_{1,2} \\ a_{1,5} & a_{1,4} & a_{1,3} \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & -\frac{15}{2} & -10 \\ \frac{1}{2} & -4 & -\frac{15}{2} \\ 6 & \frac{1}{2} & -4 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1>0 \\ 1>0 \\ 1>0 \end{pmatrix}.$$

For $i=2$, we have

$$\begin{pmatrix} \Delta_2^2 a_{2,1} \\ \Delta_2^2 a_{2,2} \\ \Delta_2^2 a_{2,3} \end{pmatrix} = \begin{pmatrix} a_{2,3}-2a_{2,2}+a_{2,1} \\ a_{2,4}-2a_{2,3}+a_{2,2} \\ a_{2,5}-2a_{2,4}+a_{2,3} \end{pmatrix} = \begin{pmatrix} a_{2,3} & a_{2,2} & a_{2,1} \\ a_{2,4} & a_{2,3} & a_{2,2} \\ a_{2,5} & a_{2,4} & a_{2,3} \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{23}{3} & -\frac{103}{12} & -\frac{15}{2} \\ -\frac{19}{4} & -\frac{23}{3} & -\frac{103}{12} \\ \frac{1}{6} & -\frac{19}{4} & -\frac{23}{3} \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2>0 \\ 2>0 \\ 2>0 \end{pmatrix}.$$

For $i=3$, we have

$$\begin{pmatrix} \Delta_2^2 a_{3,1} \\ \Delta_2^2 a_{3,2} \\ \Delta_2^2 a_{3,3} \end{pmatrix} = \begin{pmatrix} a_{3,3}-2a_{3,2}+a_{3,1} \\ a_{3,4}-2a_{3,3}+a_{3,2} \\ a_{3,5}-2a_{3,4}+a_{3,3} \end{pmatrix} = \begin{pmatrix} a_{3,3} & a_{3,2} & a_{3,1} \\ a_{3,4} & a_{3,3} & a_{3,2} \\ a_{3,5} & a_{3,4} & a_{3,3} \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{25}{3} & -\frac{23}{3} & -4 \\ -6 & -\frac{25}{3} & -\frac{23}{3} \\ -\frac{2}{3} & -6 & -\frac{25}{3} \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3>0 \\ 3>0 \\ 3>0 \end{pmatrix}.$$

For $i=4$, we have

$$\begin{pmatrix}\Delta_2^2 a_{4,1}\\ \Delta_2^2 a_{4,2}\\ \Delta_2^2 a_{4,3}\end{pmatrix}=\begin{pmatrix}a_{4,3}-2a_{4,2}+a_{4,1}\\ a_{4,4}-2a_{4,3}+a_{4,2}\\ a_{4,5}-2a_{4,4}+a_{4,3}\end{pmatrix}=\begin{pmatrix}a_{4,3}&a_{4,2}&a_{4,1}\\ a_{4,4}&a_{4,3}&a_{4,2}\\ a_{4,5}&a_{4,4}&a_{4,3}\end{pmatrix}\begin{pmatrix}1\\-2\\1\end{pmatrix}$$
$$=\begin{pmatrix}-6&-\frac{19}{4}&\frac{1}{2}\\ -\frac{13}{4}&-6&-\frac{19}{4}\\ \frac{7}{2}&-\frac{13}{4}&-6\end{pmatrix}\begin{pmatrix}1\\-2\\1\end{pmatrix}=\begin{pmatrix}4>0\\4>0\\4>0\end{pmatrix}.$$

For $i=5$, we have

$$\begin{pmatrix}\Delta_2^2 a_{5,1}\\ \Delta_2^2 a_{5,2}\\ \Delta_2^2 a_{5,3}\end{pmatrix}=\begin{pmatrix}a_{5,3}-2a_{5,2}+a_{5,1}\\ a_{5,4}-2a_{5,3}+a_{5,2}\\ a_{5,5}-2a_{5,4}+a_{5,3}\end{pmatrix}=\begin{pmatrix}a_{5,3}&a_{5,2}&a_{5,1}\\ a_{5,4}&a_{5,3}&a_{5,2}\\ a_{5,5}&a_{5,4}&a_{5,3}\end{pmatrix}\begin{pmatrix}1\\-2\\1\end{pmatrix}$$
$$=\begin{pmatrix}-\frac{2}{3}&\frac{1}{6}&6\\ \frac{7}{2}&-\frac{2}{3}&\frac{1}{6}\\ \frac{38}{3}&\frac{7}{2}&-\frac{2}{3}\end{pmatrix}\begin{pmatrix}1\\-2\\1\end{pmatrix}=\begin{pmatrix}5>0\\5>0\\5>0\end{pmatrix}.$$

The above calculations confirm that (III) is satisfied. Then, from Corollary 2.6, we have

$$\mathrm{Tr}(A)\leqslant X+Y+Z\leqslant X+Y, \tag{2.23}$$

where

$$X=\frac{5}{2}\left(a_{1,1}+a_{5,1}+\sum_{i=2}^{5}(a_{1,i}+a_{5,i})\right),\quad Y=-2\sum_{i=1}^{4}\sum_{j=i+1}^{5}a_{i,j},$$

and

$$Z=-\frac{5}{3}\sum_{i=1}^{5}(\Delta_2 a_{i,4}-\Delta_2 a_{i,1}).$$

Let us check the validity of (2.23). From the definition of the matrix A, we have

$$\begin{aligned}X&=\frac{5}{2}\left[a_{1,1}+a_{1,5}+(a_{1,2}+a_{5,2}+a_{1,3}+a_{5,3}+a_{1,4}+a_{5,4}+a_{1,5}+a_{5,5})\right]\\ &=\frac{5}{2}\left(a_{1,1}+2a_{1,5}+a_{1,2}+a_{5,2}+a_{1,3}+a_{5,3}+a_{1,4}+a_{5,4}+a_{5,5}\right)\\ &=\frac{5}{2}\left(-10+2\times 6-\frac{15}{2}+\frac{1}{6}-4-\frac{2}{3}+\frac{1}{2}+\frac{7}{2}+\frac{38}{3}\right)\\ &=\frac{50}{3},\end{aligned}$$

$$\begin{aligned}
Y &= -2\left(\sum_{j=2}^{5} a_{1,j} + \sum_{j=3}^{5} a_{2,j} + \sum_{j=4}^{5} a_{3,j} + a_{4,5}\right) \\
&= -2\left(a_{1,2}+a_{1,3}+a_{1,4}+a_{1,5}+a_{2,3}+a_{2,4}+a_{2,5}+a_{3,4}+a_{3,5}+a_{4,5}\right) \\
&= -2\left(-\frac{15}{2}-4+\frac{1}{2}+6-\frac{23}{3}-\frac{19}{4}+\frac{1}{6}-6-\frac{2}{3}+\frac{7}{2}\right) \\
&= \frac{245}{6},
\end{aligned}$$

and

$$\begin{aligned}
Z = &-\frac{5}{3}\left[(\Delta_2 a_{1,4}-\Delta_2 a_{1,1})+(\Delta_2 a_{2,4}-\Delta_2 a_{2,1})+(\Delta_2 a_{3,4}-\Delta_2 a_{3,1})+(\Delta_2 a_{4,4}-\Delta_2 a_{4,1})\right. \\
&\quad \left.+(\Delta_2 a_{5,4}-\Delta_2 a_{5,1})\right] \\
= &-\frac{5}{3}\left[(a_{1,5}-a_{1,4}-a_{1,2}+a_{1,1})+(a_{2,5}-a_{2,4}-a_{2,2}+a_{2,1})\right. \\
&\quad +(a_{3,5}-a_{3,4}-a_{3,2}+a_{3,1})+(a_{4,5}-a_{4,4}-a_{4,2}+a_{4,1}) \\
&\quad \left.+(a_{5,5}-a_{5,4}-a_{5,2}+a_{5,1})\right] \\
= &-\frac{5}{3}\left(2a_{1,5}+a_{1,1}-2a_{2,4}-a_{2,2}+a_{3,5}-a_{3,4}-a_{3,2}+a_{3,1}-a_{4,4}+a_{5,5}\right) \\
= &-\frac{5}{3}\left(2\times 6-10-2\times\left(-\frac{19}{4}\right)-\left(-\frac{103}{12}\right)-\frac{2}{3}+6+\frac{23}{3}-4+\frac{13}{4}+\frac{38}{3}\right) \\
= &-75.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\mathrm{Tr}(A) &= -10-\frac{103}{12}-\frac{25}{3}-\frac{13}{4}+\frac{38}{3} \\
&= -\frac{35}{2}.
\end{aligned}$$

Finally, the above calculations show the validity of (2.23).

Conclusion

Various classes of sequences are studied. We first considered the class of sequences $a=(a_1,\cdots,a_n)\in\mathscr{S}_n$. For this class, a Hermite-Hadamard-type inequality is proved (see Theorem 2.2). The proof of this result makes use of an appropriate choice of a sequence which is the solution to a certain fourth-order difference equation (see Lemma 2.1). We next considered the sub-class ${}^{C}\mathscr{S}_n$ of $\mathscr{S}_n$ containing all the convex sequences belonging to $\mathscr{S}_n$. For this sub-class, an interesting refinement of the right inequality in (1.2) is obtained (see Corollary 2.3). We also studied the class of real matrices $A=(a_{i,j})_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}$ satisfying the system of difference inequalities (I) and (II). For this class of matrices, some Hermite-Hadamard-type inequalities are derived (see Theorem 2.4). In the special case when A is symmetric and satisfies (III), a trace inequality is obtained (see Corollary 2.6).

In this paper, we only considered some classes of sequences satisfying fourth-order difference inequalities. It will be interesting to extend the present study to the set of sequences $a = (a_1, \cdots, a_n) \in \mathbb{R}^n$ satisfying the k-th order difference inequality

$$(-1)^k \Delta^{2k} a_i \leqslant 0, \quad i = 1, \cdots, n-2k,$$

where $n \geqslant 2k+1$.

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REFERENCES

[1] S. ABRAMOVICH, L. E. PERSSON, *Fejér and Hermite-Hadamard type inequalities for N-quasiconvex functions*, Math Notes. **102** (2017) 599–609.
[2] G. ADILOV, I. YESILCE, *Some important properties of B-convex functions*, J. Nonlinear Convex Anal. **19** (4) (2018) 669–680.
[3] D. BERTSEKAS, A. NEDI, A. OZDAGLAR, *Convex Analysis and Optimization*, Athena Scientific, Belmont, 2003.
[4] H. CHEN, U. N. KATUGAMPOLA, *Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for generalized fractional integrals*, J. Math. Anal. Appl. **446** (2) (2017) 1274–1291.
[5] M. R. DELAVAR, M. DE LA SEN, *A mapping associated to h-convex version of the Hermite-Hadamard inequality with applications*, J. Math. Inequal. **14** (2) (2020) 329–335.
[6] M. R. DELAVAR, *On Fejér's inequality: generalizations and applications*, J. Inequal. Appl. **2023** (1) (2023) 42.
[7] S. S. DRAGOMIR, *Reverses of the first Hermite-Hadamard type inequality for the square operator modulus in Hilbert spaces*, Journal of Linear and Topological Algebra. **11** (1) (2022) 1–13.
[8] S. S. DRAGOMIR, R. P. AGARWAL, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett. **11** (5) (1998) 91–95.
[9] S. S. DRAGOMIR, Y. J. CHO, S. S. KIM, *Inequalities of Hadamard's type for Lipschitzian mappings and their applications*, J. Math. Anal. Appl. **245** (2) (2000) 489–501.
[10] S. S. DRAGOMIR, C. E. M. PEARCE, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, Melbourne, 2000.
[11] T. DU, Y. PENG, *Hermite-Hadamard type inequalities for multiplicative Riemann-Liouville fractional integrals*, J. Comput. Appl. Math. **440** (2024) 115582.
[12] L. FEJÉR, *Über die Fourierreihen, II*, Math. Naturwiss, Anz. Ungar. Akad. Wiss. **24** (1906) 369–390.
[13] A. GUESSAB, G. SCHMEISSER, *Sharp integral inequalities of the Hermite-Hadamard type*, J. Approx. Theory. **115** (2002) 260–288.
[14] A. GUESSAB, G. SCHMEISSER, *Sharp error estimates for interpolatory approximation on convex polytopes*, SIAM J. Numer. Anal. **43** (2005) 909–923.
[15] A. GUESSAB, B. SEMISALOV, *Optimal general Hermite-Hadamard-type inequalities in a ball and their applications in multidimensional numerical integration*, Appl. Numer. Math. **170** (2021) 83–108.
[16] J. HADAMARD, *Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann*, J. Math. Pures Appl. **58** (4) (1893) 171–215.
[17] N. HADJISAVVAS, S. KOMLOSI, S. SCHAIBLE, *Handbook of Generalized Convexity and Generalized Monotonicity*, Berlin, Springer-Verlag, 2005.
[18] C. HERMITE, *Sur deux limites d'une intégrale défine*, Mathesis. **3** (1) (1983) 1–82.

[19] A. IQBAL, M. ADIL KHAN, M. SULEMAN, Y.M. CHU, *The right Riemann-Liouville fractional Hermite-Hadamard type inequalities derived from Green's function*, AIP Adv. **10** (2020), Article ID 045032.

[20] A. IQBAL, M. ADIL KHAN, S. ULLAH, A. KASHURI, Y.M. CHU, *Hermite-Hadamard type inequalities pertaining conformable fractional integrals and their applications*, AIP Adv. **8** (2018), Article ID 075101.

[21] R. JAKŠIĆ, L. KVESIĆ, J. E. PEČARIĆ, *On weighted generalization of the Hermite-Hadamard inequality*, Math. Inequal. Appl. **18** (2) (2015) 649–665.

[22] M. JLELI, B. SAMET, *On Hermite-Hadamard-type inequalities for subharmonic functions over circular ring domains*, Numer. Funct. Anal. Optim. **44** (13) (2023) 1395–1408.

[23] Z. LATREUCH, B. BELAIDI, *New inequalities for convex sequences with applications*, Int. J. Open Problems Comput. Math. **5** (3) (2012) 15–27.

[24] A. M. MERCER, *Polynomials and convex sequence inequalities*, J. Inequal. Pure Appl. Math. **6** (1) (2005) Art. 8.

[25] D. S. MITRINOVIĆ, P. M. VASIĆ, *Analytic Inequalities*, Springer-Verlag, New York, 1970.

[26] C. P. NICULESCU, *The Hermite-Hadamard inequality for convex functions of a vector variable*, Math. Inequal. Appl. **5** (2002) 619–623.

[27] C. P. NICULESCU, L. E. PERSSON, *Old and new on the Hermite-Hadamard inequality*, Real Anal. Exch. **29** (2) (2003) 663–685.

[28] C. P. NICULESCU, L. E. PERSSON, *Convex Functions and Their Applications. A Contemporary Approach*, Springer-Verlag, New York, 2006.

[29] M. NIEZGODA, *Sherman, Hermite-Hadamard and Fejér like inequalities for convex sequences and nondecreasing convex functions*, Filomat. **31** (8) (2017) 2321–2335.

[30] J. E. PEČARIĆ, F. PROSCHAN, Y. L. TONG, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Boston, 1992.

[31] A. W. ROBERTS, D. E. VARBERG, *Convex functions*, Academic Press, New York, 1973.

[32] R. T. ROCKAFELLAR, *Convex analysis*, vol. 11, Princeton university press, 1997.

[33] B. SAMET, *On an implicit convexity concept and some integral inequalities*, J. Inequal. Appl. **308** (2016) (2016) 1–16.

[34] B. SAMET, *A convexity concept with respect to a pair of functions*, Numer. Funct. Anal. Optim. **43** (2022) 522–540.

[35] T. SZOSTOK, *Inequalities of Hermite-Hadamard type for higher order convex functions, revisited*, Communications on Pure & Applied Analysis. **20** (2) (2021) 903–912.

[36] A. TAGHAVI, V. DARVISH, H. M. NAZARI, S. S. DRAGOMIR, *Hermite-Hadamard type inequalities for operator geometrically convex functions*, Monatsh. Math. **181** (1) (2016) 187–203.

[37] S. WU, *The generalization of an inequality for convex sequence*, J. Chengdu Univ. (Natural Science). **23** (3) (2004) 11–15.

[38] Y. WU, F. QI, *Discussions on two integral inequalities of Hermite-Hadamard type for convex functions*, J. Comput. Appl. Math. **406** (2022) 114049.

[39] S. WU, H. N. SHI, *Majorization proofs of inequalities for convex sequences*, Math. Practice Theory. **33** (12) (2003) 132–137.

[40] X. M. YANG, *A characterization of convex function*, Appl. Math. Lett. **13** (1) (2000) 27–30.

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Bessem Samet
Department of Mathematics
College of Science, King Saud University
Riyadh 11451, Saudi Arabia
e-mail: bsamet@ksu.edu.sa