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EXTENSION OF (m,C)–ISOMETRIC COMMUTING MULTI–OPERATORS

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Abstract. This article examines the structural characteristics of a new class of so-called $(n_1,\dots,n_d)$-quasi-(m,C)-isometric commuting multioperators related to a given conjugation operator C on a Hilbert space $\mathscr{Y}$.

1. Introduction

Let $\mathscr{B}[\mathscr{Y}]$ be the set of all bounded linear operators on a separable complex Hilbert space $\mathscr{Y}$ with inner product $\langle .\mid .\rangle$ and denote by $I_{\mathscr{Y}}$ be the identity of $\mathscr{B}[\mathscr{Y}]$. For an operator $\mathscr{U}\in\mathscr{B}[\mathscr{Y}]$, we denote by $\mathscr{R}(\mathscr{U})$ its range, $\ker(\mathscr{U})$ its kernel, and $\mathscr{U}^*$ its adjoint. Recall from [13] that a conjugation on $\mathscr{Y}$ is a map $C:\mathscr{Y}\longrightarrow\mathscr{Y}$ which is antilinear, involutive $(C^2=I_{\mathscr{Y}})$. Moreover C satisfies the following properties

$$\begin{cases} \langle C\psi;\mid C\varphi\rangle=\langle\varphi\mid\psi\rangle \ \text{ for all } \varphi,\psi\in\mathscr{Y}, \\ C\mathscr{U}C\in\mathscr{B}[\mathscr{Y}] \ \text{ for every } \mathscr{U}\in\mathscr{B}[\mathscr{Y}], \\ \big(C\mathscr{U}C\big)^r=C\mathscr{U}^rC \ \text{ for all } r\in\mathbb{N}, \\ \big(C\mathscr{U}C\big)^*=C\mathscr{U}^*C. \end{cases}$$

See [4, 10] for properties of conjugation operators.

In this work, $\mathbb{N}=\{1,2,\dots\}$, n and $m\in\mathbb{N}$. Let $\mathbb{N}_0=\{0,1,2,\dots\}$ and $\widetilde{\mathbf{n}}=(n_1,\dots,n_d)\in\mathbb{N}_0^d$.

During the past years, the m-isometric operators term has known a great interest on the part of researchers in the field of operator theory, by the works that has been published in this aspect. It should be noted that most of these works are dependent on the following definition, which is due to Agler. An operator $\mathscr{U}\in\mathscr{B}[\mathscr{Y}]$ is said to be m-isometric if

$$\sum_{0\leqslant k\leqslant m}(-1)^k\binom{m}{k}\mathscr{U}^{*m-k}\mathscr{U}^{m-k}=0, \tag{1.1}$$

or

$$\sum_{0\leqslant k\leqslant m}(-1)^k\binom{m}{k}\|\mathscr{U}^{m-k}x\|^2=0 \quad \forall\ x\in\mathscr{Y}. \tag{1.2}$$

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(For more detail, see [1, 2, 3, 6, 7] about the theory of m-isometries).

As extensions of the concepts of m-isometric operators on Hilbert spaces, some authors has introduced and study in different papers the following classes of operators.

(1) (m,C)-isometric operator that is an operator $\mathscr{U}\in\mathscr{B}[\mathscr{Y}]$ satisfies

$$\sum_{0\leqslant k\leqslant m}(-1)^k\binom{m}{k}\mathscr{U}^{*m-k}C\mathscr{U}^{m-k}C=0, \tag{1.3}$$

for some $m\in\mathbb{N}$ and some conjugation C ([8, 11, 12]).

(2) n-quasi-m-isometric operator that is an operator $\mathscr{U}\in\mathscr{B}[\mathscr{Y}]$ satisfies

$$\mathscr{U}^{*n}\left(\sum_{0\leqslant k\leqslant m}(-1)^{m-k}\binom{m}{k}\mathscr{U}^{*m-k}\mathscr{U}^{m-k}\right)\mathscr{U}^n=0, \tag{1.4}$$

for some $n\in\mathbb{N}$ and $m\in\mathbb{N}$ ([5, 18, 22]).

(3) n-quasi-(m,C)-isometric operator that is an operator $\mathscr{U}\in\mathscr{B}[\mathscr{Y}]$ satisfies

$$\mathscr{U}^{*n}\left(\sum_{0\leqslant k\leqslant m}(-1)^{m-k}\binom{m}{k}\mathscr{U}^{*m-k}C\mathscr{U}^{m-k}C\right)\mathscr{U}^n=0, \tag{1.5}$$

for some conjugation C and some $n\in\mathbb{N}$ and $m\in\mathbb{N}$ ([20, 21, 24]).

It is well known that the properties of powers, products and perturbations of the members of each of the classes cited above has been discussed ([8, 18, 22, 20, 21]).

For $d\in\mathbb{N}$, let $/=(\mathscr{U}_1,\dots,\mathscr{U}_d)\in\mathscr{B}[\mathscr{Y}]^d$ be a tuple of commuting multioperators (abbreviated c.m.o.). Let $\omega=(\omega_1,\dots,\omega_d)\in\mathbb{N}_0^d$ (multi-indices) and set $\omega!:=\omega_1!\dots\omega_d!$. Further, define $\mathbf{U}^\omega:=\mathscr{U}_1^{\omega_1}\mathscr{U}_2^{\omega_2}\dots\mathscr{U}_d^{\omega_d}$ where $\mathscr{U}_j^{\omega_j}=\mathscr{U}_j\dots\mathscr{U}_j$ (ω_j-times) and $\mathbf{U}^*=(\mathscr{U}_1^*,\dots,\mathscr{U}_d^*)$.

In [14] the authors extended the concept of m-isometric operators to the case of c.m.o. on a Hilbert space $\mathscr{Y}$. A c.m.o. $\mathbf{U}=(\mathscr{U}_1,\dots,\mathscr{U}_d)\in\mathscr{B}[\mathscr{Y}]^d$ is called m-isometric tuple if

$$\sum_{0\leqslant k\leqslant m}(-1)^{m-k}\binom{m}{k}\left(\sum_{|\omega|=k}\frac{k!}{\omega!}\mathbf{U}^{*\omega}\mathbf{U}^\omega\right)=0 \tag{1.6}$$

or

$$\sum_{0\leqslant k\leqslant m}(-1)^{m-k}\binom{m}{k}\left(\sum_{|\omega|=k}\frac{k!}{\omega!}\|\mathscr{U}^\omega x\|^2\right)=0 \quad \text{for all} \quad x\in\mathscr{Y}. \tag{1.7}$$

(See [15, 16, 23] for more information about m-isometric tuples).

$\mathbf{U}=(\mathscr{U}_1,\dots,\mathscr{U}_d)$ is called (m,C)-isometric tuple if

$$\sum_{0\leqslant k\leqslant m}(-1)^{m-k}\binom{m}{k}\left(\sum_{|\omega|=k}\frac{k!}{\omega!}\mathbf{U}^{*\omega}C\mathbf{U}^\omega C\right)=0. \tag{1.8}$$

2. $(n_1,\dots,n_d)$-quasi-(m,C)-isometric multioperators

This section is devoted to the study of a new concept of multivariable operators and some of the properties associated with the members of this class.

DEFINITION 2.1. Let $\mathbf{U}=(\mathscr{U}_1,\dots,\mathscr{U}_d)\in\mathscr{B}[\mathscr{Y}]^d$ be a c.m.o., $\widetilde{\mathbf{n}}=(n_1,\dots,n_d)\in\mathbb{N}^d$ and $m\in\mathbb{N}$. $\mathbf{U}$ is said to be $(n_1,\dots,n_d)$-quasi-(m,C)-isometric c.m.o. if

$$\mathbf{U}^{*\widetilde{\mathbf{n}}}\left(\sum_{0\leqslant k\leqslant m}(-1)^{m-k}\binom{m}{k}\left(\sum_{|\omega|=k}\frac{k!}{\omega!}\mathbf{U}^{*\omega}C\mathbf{U}^{\omega}C\right)\right)\mathbf{U}^{\widetilde{\mathbf{n}}}=0. \tag{2.1}$$

When $d=1$, we observe that Eq. (2.1) coincides with Eq. (1.5).

REMARK 2.1. From Definition 2.1, we make the following remarks:

(1) If $\mathscr{R}(\mathscr{U}_i^{n_i})\subset\ker(\mathscr{U}_j^{n_j})$ for $n_in_j\neq 0$, then $\mathbf{U}=(\mathscr{U}_1,\dots,\mathscr{U}_d)$ is an $(n_1,\dots,n_d)$-quasi-(m,C)-isometric c.m.o.

In fact, under the assumption, it follows that $\mathbf{U}^{\widetilde{\mathbf{n}}}=0$ and hence

$$\mathbf{U}^{*\widetilde{\mathbf{n}}}\left(\sum_{0\leqslant k\leqslant m}(-1)^{m-k}\binom{m}{k}\left(\sum_{|\omega|=k}\frac{k!}{\omega!}\mathbf{U}^{*\omega}C\mathbf{U}^{\omega}C\right)\right)\mathbf{U}^{\widetilde{\mathbf{n}}}=0.$$

(2) If $\mathscr{U}_kC=C\mathscr{U}_k$ for $k=1,\dots,d$, then $\mathbf{U}=(\mathscr{U}_1,\dots,\mathscr{U}_d)$ is an $(n_1,\dots,n_d)$-quasi-(m,C)-isometric c.m.o. if and only if $\mathbf{U}=(\mathscr{U}_1,\dots,\mathscr{U}_d)$ is an $(n_1,\dots,n_d)$-quasi-m-isometric c.m.o. ([9]).

(3) It is obvious that if $\mathbf{U}$ is an (m,C)-isometric c.m.o., then $\mathbf{U}$ is an $(n_1,\dots,n_d)$-quasi-(m,C)-isometric c.o.m for any $(n_1,\dots,n_d)\in\mathbb{N}^d$. However the converse is not true in general as shown below.

REMARK 2.2. Note that, a $(1,\dots,1)$-quasi-(m,C)-isometric c.m.o. is a quasi-(m,C)-isometric c.m.o. and a $\widetilde{\mathbf{n}}$-quasi-$(1,C)$-isometric c.m.o. is $\widetilde{\mathbf{n}}$-quasi-C-isometric c.m.o.

REMARK 2.3. For an c.m.o. $\mathbf{U}=(\mathscr{U}_1,\mathscr{U}_2)\in\mathscr{B}[\mathscr{Y}]^2$, we make the following observations.

(i) $\mathbf{U}$ is an $(1,1)$-quasi-C-isometric pair if

$$\mathscr{U}_1^*\mathscr{U}_2^*\left(\mathscr{U}_1^*C\mathscr{U}_1C+\mathscr{U}_2^*C\mathscr{U}_2C-I_{\mathscr{Y}}\right)\mathscr{U}_1\mathscr{U}_2=0.$$

(ii) $\mathbf{U}$ is an $(1,1)$-quasi-$(2,C)$-isometric pair if

$$\mathscr{U}_1^*\mathscr{U}_2^*\Big(I_{\mathscr{Y}}-2\mathscr{U}_1^*C\mathscr{U}_1C-2\mathscr{U}_2^*C\mathscr{U}_2C+\mathscr{U}_1^{*2}C\mathscr{U}_1^2C+\mathscr{U}_2^{*2}C\mathscr{U}_2^2C$$
$$+2\mathscr{U}_1^*\mathscr{U}_2^*C\mathscr{U}_1\mathscr{U}_2C\Big)\mathscr{U}_1\mathscr{U}_2=0.$$

In the following example we show that there is a c.m.o. which is an $(n_1,\dots,n_d)$-quasi-(m,C)-isometric c.m.o., but not an (m,C)-isometric c.m.o. for some conjugation C, $(n_1,\dots,n_d)\in\mathbb{N}^d$ and $m\in\mathbb{N}$.

EXAMPLE 2.1. Let C be a conjugation given by $C(z_1,z_2,z_3)=(\overline{z_3},\overline{z_2},\overline{z_1})$. Consider $\mathbf{U}=(\mathscr{U}_1,\mathscr{U}_2)\in\mathscr{B}[\mathbb{C}^3]$ which admit the following representations:

$$\mathscr{U}_1=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}\quad\text{and}\quad\mathscr{U}_2=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}.$$

A simple calculation shows that

$$\mathscr{U}_2^*C\mathscr{U}_2C+\mathscr{U}_1^*C\mathscr{U}_1C-I_{\mathscr{Y}}\neq 0$$

and

$$(\mathscr{U}_2\mathscr{U}_1)^*\left(\mathscr{U}_2^*C\mathscr{U}_2C+\mathscr{U}_1^*C\mathscr{U}_1C-I_{\mathscr{Y}}\right)\mathscr{U}_1\mathscr{U}_2=0.$$

Therefore, $\mathbf{U}$ is a $(1,1)$-quasi-$(1,C)$-isometric c.m.o. but $\mathbf{U}$ is not $(1,C)$-isometric c.m.o.

REMARK 2.4. From the above example, we may conclude that the class of (m,C)-isometric c.m.o. is a proper subset of the class of $(n_1,\dots,n_d)$-quasi-(m,C)-isometric c.m.o.

In the next proposition we show that the concept of $(n_1,\dots,n_d)$-quasi-(m,C)-isometry for c.m.o. $\mathbf{U}=(\mathscr{U}_1,\dots,\mathscr{U}_d)$ is not affected by an permutation of the operators $\mathscr{U}_j$.

PROPOSITION 2.1. *Let $J_d=\{1,\dots,d\}$ and $\mathbb{S}_d$ be the set of all permutations on J_d. Let $\mathbf{U}=(U_1,\dots,U_d)\in\mathscr{B}[\mathscr{Y}]^d$ be a c.m.o. If $\mathbf{U}$ is an $(n_1,\dots,n_d)$-quasi-(m,C)-isometric c.m.o. for some conjugation C on $\mathscr{Y}$, then for every $\varphi\in\mathbb{S}_d$, $\mathbf{U}_\varphi:=(\mathscr{U}_{\varphi(1)},\dots,\mathscr{U}_{\varphi(d)})$ is an $(n_{\varphi(1)},\dots,n_{\varphi(d)})$-quasi-$(m,C)$-isometric c.m.o.*

Proof. By noting that $\mathscr{U}_1.\dots.\mathscr{U}_d=\mathscr{U}_{\varphi(1)}.\dots.\mathscr{U}_{\varphi(d)}$, one can immediately deduce the proof. □

REMARK 2.5. We notice that if $\mathbf{U}$ is an $(n_1,\dots,n_d)$-quasi-(m,C)-isometric c.m.o., one can easily check that $\mathbf{U}$ is a $(n_1+r_1,\dots,n_d+r_d)$-quasi-(m,C)-isometric c.m.o. for every $\widetilde{\mathbf{r}}=(r_1,\dots,r_d)\in\mathbb{N}_0^d$.

Recall from [21] that if $\mathscr{U}$ is an n-quasi-(m,C)-isometric single operator such that $[\mathscr{U},C\mathscr{U}C]=0$, then $\mathscr{U}$ is an n-quasi-(k,C)-isometric operator for all integer $k\geqslant m$.

The following proposition extended this result to the multidimensional case.

PROPOSITION 2.2. *Let* $\mathbf{U}=(\mathscr{U}_1,\dots,\mathscr{U}_d)$ *be an* $(n_1,\dots,n_d)$*-quasi-*(m,C)*-isometric c.m.o. If* $\left[C\mathscr{U}_jC,\mathscr{U}_l\right]=0$ *for all* $j,l=1,\dots,d$*, then* $\mathbf{U}$ *is an* $(n_1,\dots,n_d)$*-quasi-*$(m+k,C)$*-isometric c.m.o. for any positive integer* k*.*

Proof. This follows immediately from the identity

$$\mathscr{Q}_{m+1}(\mathbf{U},C)=\sum_{1\leqslant j\leqslant d}\mathscr{U}_j^*\mathscr{Q}_m(\mathbf{U},C)(C\mathscr{U}_jC)-\mathscr{Q}_m(\mathbf{U},C),$$

(see [19, Proposition 2.1]) and the conditions $\left[C\mathscr{U}_jC,\mathscr{U}_l\right]=0$ for $j,l=1,\dots,d$. □

In the next proposition, we present some conditions for which a $(n_1,\dots,n_d)$-quasi-(m,C)-isometric c.m.o. becomes (m,C)-isometric c.m.o.

PROPOSITION 2.3. *Let* $\mathbf{U}=(\mathscr{U}_1,\dots,\mathscr{U}_d)\in\mathscr{B}[\mathscr{Y}]^d$ *be an* $\widetilde{\mathbf{n}}$*-quasi-*(m,C)*-isometric c.m.o. for* $\widetilde{\mathbf{n}}=(n_1,\dots,n_d)\in\mathbb{N}_0^d$ *and* $m\in\mathbb{N}$*. If* $\ker\mathscr{U}_{j_0}^*=\{0\}$ *for some* $j_0\in\{1,\dots,d\}$*, then* $\mathbf{U}$ *is an* (m,C)*-isometric c.m.o.*

Proof. According to the fact that $\mathbf{U}$ is an $\widetilde{\mathbf{n}}$-quasi-(m,C)-isometric c.m.o., it follows that

$$\mathbf{U}^{*\widetilde{\mathbf{n}}}\left(\sum_{0\leqslant k\leqslant m}(-1)^{m-k}\binom{m}{k}\sum_{|\omega|=k}\frac{k!}{\omega!}\mathbf{U}^{*\omega}C\mathbf{U}^{\omega}C\right)\mathbf{U}^{\widetilde{\mathbf{n}}}=0.$$

Since $\ker\left(\mathbf{U}^{*\widetilde{\mathbf{n}}}\right)=\bigcap_{1\leqslant j\leqslant d}\ker(U_j^{*n_j})\subset\ker(\mathscr{U}_{j_0}^*)$ and $\mathscr{U}\,{j_0}^*$ is injective it follows that the above equation becomes

$$\left(\sum_{0\leqslant k\leqslant m}(-1)^{m-k}\binom{m}{k}\sum_{|\omega|=k}\frac{k!}{\omega!}\mathbf{U}^{*\omega}C\mathbf{U}^{\omega}C\right)\mathbf{U}^{\widetilde{\mathbf{n}}}=0,$$

and so that

$$\mathbf{U}^{*\widetilde{\mathbf{n}}}\left(\sum_{0\leqslant k\leqslant m}(-1)^{m-k}\binom{m}{k}\sum_{|\omega|=k}\frac{k!}{\omega!}\mathbf{U}^{*\omega}C\mathbf{U}^{\omega}C\right)=0.$$

By using the item $\ker\left(\mathbf{U}^{*\widetilde{\mathbf{n}}}\right)=\{0\}$, we get

$$\sum_{0\leqslant k\leqslant m}(-1)^{m-k}\binom{m}{k}\left(\sum_{|\omega|=k}\frac{k!}{\omega!}\mathbf{U}^{*\omega}C\mathbf{U}^{\omega}C\right)=0.$$

We may conclude that $\mathbf{U}$ satisfies the identity (1.8). □

THEOREM 2.1. *Let* $\mathbf{U}=(\mathscr{U}_1,\dots,\mathscr{U}_d)\in\mathscr{B}[\mathscr{Y}]^d$ *is a* $\widetilde{\mathbf{n}}$*-quasi-*(m,C)*-isometric c.m.o. for some conjugation* C *on* $\mathscr{Y}$*. If* $\ker(\mathscr{U}_{j_0}^*)=\ker(\mathscr{U}_{j_0}^{*2})$ *for some* $j_0\in\{1,\dots,d\}$*, then* $\mathbf{u}$ *is an quasi-*(m,C)*-isometric c.m.o.*

Proof. From the condition that $\ker(\mathscr{U}_{j_0}^*) = \ker(\mathscr{U}_{j_0}^{*2})$ for some $j_0 \in \{1,\ldots,d\}$, it follows that $\ker(\mathscr{U}_{j_0}^*) = \ker(\mathscr{U}_{j_0}^{*n_{j_0}})$ and however $\ker(\mathscr{U}^{*\widetilde{\mathbf{n}}}) = \ker(\prod_{1\leqslant j\leqslant d} \mathscr{U}_j^*)$. Since $\mathbf{U}$ is a $\widetilde{\mathbf{n}}$-quasi-(m,C)-isometric c.m.o., we have

$$\prod_{1\leqslant j\leqslant d} \mathscr{U}_j^{*n_j}\left(\sum_{0\leqslant k\leqslant m}(-1)^{m-k}\binom{m}{k}\sum_{|\omega|=k}\frac{k!}{\omega!}\mathbf{U}^{*\omega}C\mathbf{U}^{\omega}C\right)\prod_{1\leqslant j\leqslant d}\mathscr{U}_j^{n_j}=0.$$

From the fact that $\ker(\mathbf{U}^{*\widetilde{\mathbf{n}}}) = \ker\left(\prod_{1\leqslant j\leqslant p} \mathscr{U}_j^*\right)$, which yields

$$\prod_{1\leqslant j\leqslant d} \mathscr{U}_j^{*}\left(\sum_{0\leqslant k\leqslant m}(-1)^{m-k}\binom{m}{k}\sum_{|\omega|=k}\frac{k!}{\omega}\mathbf{U}^{*\omega}C\mathbf{U}^{\omega}C\right)\prod_{1\leqslant j\leqslant d}\mathscr{U}_j^{n_j}=0.$$

Applying the adjoint operation, we obtain

$$\prod_{1\leqslant j\leqslant d} \mathscr{U}_j^{*n_j}\left(\sum_{0\leqslant k\leqslant m}(-1)^{m-k}\binom{m}{k}\sum_{|\omega|=k}\frac{k!}{\omega!}\mathbf{U}^{*\omega}C\mathbf{U}^{\omega}C\right)\prod_{1\leqslant j\leqslant d}\mathscr{U}_j=0.$$

Similarly, we have

$$\prod_{1\leqslant j\leqslant d} \mathscr{U}_j^{*}\left(\sum_{0\leqslant k\leqslant m}(-1)^{m-k}\binom{m}{k}\sum_{|\omega|=k}\frac{k!}{\omega!}\mathbf{U}^{*\omega}C\mathbf{U}^{\omega}C\right)\prod_{1\leqslant j\leqslant d}\mathscr{U}_j=0.$$

Therefore $\mathbf{U}$ is an quasi-(m,C)-isometric c.m.o. □

COROLLARY 2.1. *If $\mathbf{U} = (\mathscr{U}_1,\ldots,\mathscr{U}_d) \in \mathscr{B}[\mathscr{Y}]^d$ be an $\widetilde{\mathbf{n}}$-quasi-(m,C)-isometric c.m.o. for some conjugation C. Assume that there exists $j_0 \in \{1,\ldots,d\}$ such that $\mathscr{U}_{j_0}$ is idempotent, then $\mathbf{U}$ is a quasi-(m,C)-isometric c.m.o.*

Proof. Since $\mathscr{U}_{j_0}^2 = \mathscr{U}_{j_0}$ it follows that $\mathscr{U}_{j_0}^{*2} = \mathscr{U}_{j_0}^*$ and therefore $\ker(\mathscr{U}_{j_0}^*) = \ker(\mathscr{U}_{j_0}^{*2})$. Using Theorem 2.1 we may see that $\mathbf{U}$ is an quasi-(m,C)-isometric c.m.o. □

For the next result, we need the following lemma.

LEMMA 2.1. ([19]) *Let $\mathbf{U} = (\mathscr{U}_1,\ldots,\mathscr{U}_d) \in \mathscr{B}[\mathscr{Y}]^d$ and $\mathbf{A} = (\mathscr{A}_1,\ldots,\mathscr{A}_d) \in \mathscr{B}[\mathscr{Y}]^d$ be c.m.o. such that $[\mathscr{A}_j^*, C\mathscr{A}_kC] = [\mathscr{U}_j,\mathscr{A}_k] = [\mathscr{A}_k^*, C\mathscr{U}_jC] = 0$ for all j,k. Then the identity holds:*

$$\mathscr{Q}_l(\mathbf{U}\cdot\mathbf{A},C) = \sum_{0\leqslant k\leqslant l}\sum_{|\omega|=k}\binom{l}{k}\binom{k}{\omega}\mathbf{U}^{*\omega}\mathscr{Q}_{l-k}(\mathbf{U},C))C\mathbf{U}^{\omega}C\prod_{1\leqslant i\leqslant d}\mathscr{Q}_{\omega_i}(\mathscr{A}_i,C). \quad (2.2)$$

THEOREM 2.2. *Let* $\mathbf{U}=(\mathscr{U}_1,\ldots,\mathscr{U}_d)\in\mathscr{B}[\mathscr{Y}]^d$ *and* $\mathbf{A}=(\mathscr{A}_1,\ldots,\mathscr{A}_d)\in\mathscr{B}[\mathscr{Y}]^d$ *be c.m.o. such that*

$$\left[\mathscr{A}_j^*,C\mathscr{A}_kC\right]=\left[\mathscr{U}_j,\mathscr{A}_k\right]=\left[\mathscr{A}_k^*,C\mathscr{U}_jC\right]=\left[\mathscr{U}_k,C\mathscr{U}_jC\right]=\left[\mathscr{A}_k,C\mathscr{A}_jC\right]=0 \quad all \;\; j,k.$$

Let $\widetilde{\mathbf{n}}=(n_1,\ldots,n_d),\widetilde{\mathbf{p}}=(p_1,\ldots,p_d)\in\mathbb{N}_0^d$ *and* $(m_1,\ldots,m_d)\in\mathbb{N}^d$. *If* $\mathbf{U}$ *is a* $\widetilde{\mathbf{n}}$*-quasi-*(m,C)*-isometric c.m.o. and each* $\mathscr{A}_k$ *is a* p_k*-quasi* (m_k,C)*-isometric operator for* $k=1,\ldots,d$. *Then* $\mathbf{U}\cdot\mathbf{A}$ *is a* $(\widetilde{\mathbf{n}}+\widetilde{\mathbf{p}})$*-quasi-*$\left(m+\sum\limits_{1\leqslant k\leqslant d}m_k-d,C\right)$*-isometric c.m.o.*

Proof. It is enough to show that

$$\left(\mathbf{UA}\right)^{*\widetilde{\mathbf{n}}+\widetilde{\mathbf{p}}}\mathscr{Q}_{m+\sum\limits_{1\leqslant k\leqslant d}m_k-d}\left(\mathbf{UA},C\right)\left(\mathbf{UA}\right)^{\widetilde{\mathbf{n}}+\widetilde{\mathbf{p}}}=0.$$

Indeed, set $r=\sum\limits_{1\leqslant k\leqslant d}m_k$ and $l=m+r-d$. Following Lemma 2.1 and the conditions $\left[\mathscr{U}_k,C\mathscr{U}_jC\right]=\left[\mathscr{A}_k,C\mathscr{A}_jC\right]=0$ all $j,k,$ we get

$$\begin{aligned}
&\left(\mathbf{UA}\right)^{*\widetilde{\mathbf{n}}+\widetilde{\mathbf{p}}}\mathscr{Q}_l\left(\mathbf{UA},C\right)\left(\mathbf{UA}\right)^{\widetilde{\mathbf{n}}+\widetilde{\mathbf{p}}}\\
&=\left(\mathbf{UA}\right)^{*\widetilde{\mathbf{n}}+\widetilde{\mathbf{p}}}\left(\sum_{0\leqslant k\leqslant l}\sum_{|\omega|=k}\binom{l}{k}\binom{k}{\omega}\mathbf{U}^{*\omega}\mathscr{Q}_{l-k}(\mathbf{U},C))C\mathbf{U}^{\omega}C\prod_{1\leqslant i\leqslant d}\mathscr{Q}_{\omega_i}\left(A_i,C\right)\right)\left(\mathbf{UA}\right)^{\widetilde{\mathbf{n}}+\widetilde{\mathbf{p}}}\\
&=\sum_{0\leqslant k\leqslant l}\sum_{|\omega|=k}\binom{l}{k}\binom{k}{\omega}\mathbf{U}^{*\omega}\left(\mathbf{U}\right)^{*\widetilde{\mathbf{n}}+\widetilde{\mathbf{p}}}\mathscr{Q}_{l-k}\left(\mathbf{U},C\right))C\mathbf{U}^{\omega}C\left(\mathbf{U}\right)^{\widetilde{\mathbf{n}}+\widetilde{\mathbf{p}}}\\
&\quad\times\prod_{1\leqslant i\leqslant d}\mathscr{A}_i^{*n_i+p_i}\mathscr{Q}_{\omega_i}\left(\mathscr{A}_i,C\right)\mathscr{A}_i^{n_i+p_i}\\
&=\sum_{0\leqslant k\leqslant l}\sum_{|\omega|=k}\binom{l}{k}\binom{k}{\omega}\mathbf{U}^{*\omega}\left(\mathbf{U}\right)^{*\widetilde{\mathbf{n}}+\widetilde{\mathbf{p}}}\mathscr{Q}_{l-k}\left(\mathbf{U},C\right)\mathbf{U}^{\widetilde{\mathbf{n}}+\widetilde{\mathbf{p}}}C\mathbf{U}^{\omega}C\\
&\quad\times\prod_{1\leqslant i\leqslant d}\mathscr{A}_i^{*n_i+p_i}\mathscr{Q}_{\omega_i}\left(\mathscr{A}_i,C\right)\mathscr{A}_i^{n_i+p_i}.
\end{aligned}$$

Thus, we can deduce from the previous relations that for $k\in\{0,\ldots,r-d\}$, then $l-k\geqslant m$ and so $\mathbf{U}^{*\widetilde{\mathbf{n}}+\widetilde{\mathbf{p}}}\mathscr{Q}_{l-k}\left(\mathbf{U},C\right)\mathbf{U}^{\widetilde{\mathbf{n}}+\widetilde{\mathbf{p}}}=0$ by Proposition 2.2.

However, if $k>r-d$, then one of the ω_j satisfies $\omega_j\geqslant m_j$ since $|\omega|=k$ and therefore, $\mathscr{A}_i^{*n_i+p_i}\mathscr{Q}_{\omega_i}\left(\mathscr{A}_i,C\right)\mathscr{A}_i^{n_i+p_i}=0$ for $i=1,\ldots,d$ by [21, Proposition 2.1].

Finally, $\left(\mathbf{UA}\right)^{*\widetilde{\mathbf{n}}+\widetilde{\mathbf{p}}}\mathscr{Q}_l\left(\mathbf{UA},C\right)\left(\mathbf{UA}\right)^{\widetilde{\mathbf{n}}+\widetilde{\mathbf{p}}}=0$ gives the result. □

Now, we proceed to the computation of the tensor product of two c.m.o. Let $\mathbf{U}=(U_1,\ldots,U_d)\in\mathscr{B}[\mathscr{Y}]^d$ and $\mathbf{A}=(\mathscr{A}_1,\ldots,\mathscr{A}_d)\in\mathscr{B}[\mathscr{Y}]^d$ be c.m.o. Note that

$$\begin{aligned}
\mathbf{U}\otimes\mathbf{A}&=\left(\mathscr{U}_1\otimes\mathscr{A}_1,\ldots,\mathscr{U}_d\otimes\mathscr{A}_d\right)\\
&=\Big((\mathscr{U}_1\otimes I)(I\otimes\mathscr{A}_1),\ldots,(\mathscr{U}_d\otimes I)(I\otimes\mathscr{A}_d)\Big).
\end{aligned}$$

Then we can write $\mathbf{U}\otimes\mathbf{I}=\big(\mathscr{U}_1\otimes I,\dots,\mathscr{U}_d\otimes I\big)$ and

$$\mathbf{I}\otimes\mathbf{A}=\big(I\otimes\mathscr{A}_1,\dots,I\otimes\mathscr{A}_d\big).$$

COROLLARY 2.2. *Let $\mathbf{U}=(\mathscr{U}_1,\dots,\mathscr{U}_d)\in\mathscr{B}[\mathscr{Y}]^d$ and $\mathbf{A}=(\mathscr{A}_1,\dots,\mathscr{A}_d)\in\mathscr{B}[\mathscr{Y}]^d$ be c.m.o. and C be a conjugation on $\mathscr{Y}$. Assume that*

$$\big[\mathscr{A}_j^*,C\mathscr{A}_kC\big]=\big[\mathscr{U}_k,C\mathscr{U}_jC\big]=\big[\mathscr{A}_k,C\mathscr{A}_jC\big]=0 \ \ \textit{for all} \ \ j,k.$$

Let $\widetilde{\mathbf{n}}=(n_1,\dots,n_d)$ and $\widetilde{\mathbf{p}}=(p_1,\dots,p_d)\in\mathbb{N}_0^d$ and $m,m_1,\dots,m_d\in\mathbb{N}$. If $\mathbf{U}$ is an $(n_1,\dots,n_d)$-quasi-(m,C)-isometric c.m.o. and each $\mathscr{A}_j$ is an p_k-quasi-(m_k,C)-isometric operator for $k=1,\dots,d$. Then $\mathbf{U}\otimes\mathbf{A}$ is an $(\widetilde{\mathbf{n}}+\widetilde{\mathbf{p}})$-quasi-$\big(m+\sum_{1\leqslant k\leqslant d}m_k-d,C\otimes C\big)$-isometric c.m.o.

Proof. Since $\mathbf{U}\otimes\mathbf{A}=\Big((\mathscr{U}_1\otimes I)(I\otimes\mathscr{A}_1),\dots,(\mathscr{U}_d\otimes I)(I\otimes\mathscr{A}_d)\Big)=\widetilde{\mathbf{U}}\widetilde{\mathbf{A}}$, where $\widetilde{\mathbf{U}}=\big(\mathscr{U}_1\otimes I,\dots,\mathscr{U}_d\otimes I\big)$ and $\widetilde{\mathbf{A}}=\big(I\otimes\mathscr{A}_1,\dots,I\otimes\mathscr{A}_d\big)$. According to the conditions $\big[\mathscr{A}_j^*,C\mathscr{A}_kC\big]=\big[\mathscr{U}_k,C\mathscr{U}_jC\big]=\big[\mathscr{A}_k,C\mathscr{A}_jC\big]=0$ for all j,k, one can check that $\widetilde{\mathbf{U}}$ and $\widetilde{\mathbf{A}}$ satisfying the required conditions in Theorem 2.2 and we easily find that $\widetilde{\mathbf{U}}\cdot\widetilde{\mathbf{A}}$ is an $(\widetilde{\mathbf{n}}+\widetilde{\mathbf{p}})$-quasi-$\big(m+\sum_{1\leqslant k\leqslant d}m_k-d,C\otimes C\big)$-c.m.o. Hence the desired assertion follows since we have $\mathbf{U}\otimes\mathbf{A}=\widetilde{\mathbf{U}}\cdot\widetilde{\mathbf{A}}$. $\square$

Let $\mathbf{U}=(\mathscr{U}_1,\dots,\mathscr{U}_d)\in\mathscr{B}[\mathscr{Y}]^d$ be a c.m.o., we say that $\mathbf{U}$ is q-nilpotent, $q>0$, if $\mathbf{U}^\omega=\mathscr{U}_1^{\omega_1}\dots\mathscr{U}_d^{\omega_d}=0$ for all $\omega=(\omega_1,\dots,\omega_d)\in\mathbb{N}_0^d$ such that $|\omega|=q$.

For $\omega=(\omega_1,\dots,\omega_d)$, $\omega'=(\omega_1',\dots,\omega_d')\in\mathbb{N}_0^d$, we will say that

$$\omega'\geqslant\omega\Longleftrightarrow\omega_k'\geqslant\omega_k,\ \text{for all}\ \ k=1,\dots,d.$$

For the next result, we need the following lemma.

LEMMA 2.2. ([19]) *Let $\mathbf{U}=(\mathscr{U}_1,\dots,\mathscr{U}_d)\in\mathscr{B}[\mathscr{Y}]^d$ and $\mathbf{A}=(\mathscr{A}_1,\dots,\mathscr{A}_d)\in\mathscr{B}[\mathscr{Y}]^d$ be two c.m.o. such that $\big[\mathscr{A}_j,\mathscr{U}_k\big]=0$ for all j,k and let C be a conjugation on $\mathscr{Y}$. Then the following identity holds:*

$$\mathscr{Q}_n(\mathbf{U}+\mathbf{A})=\sum_{|\omega|+|\mu|+k=n}\binom{n}{\omega,\mu,k}(\mathbf{U}+\mathbf{A})^{*\omega}\cdot\mathbf{A}^{*\mu}\cdot\mathscr{Q}_k(\mathbf{U})\cdot C\mathbf{U}^\mu C\cdot C\mathbf{A}^\omega C, \quad (2.3)$$

where $\displaystyle\binom{n}{\omega,\mu,k}=\frac{n!}{\omega!\mu!k!}$.

THEOREM 2.3. *Let $\mathbf{U}:=(\mathscr{U}_1,\dots,\mathscr{U}_d)\in\mathscr{B}[\mathscr{Y}]^d$, $\mathbf{A}:=(\mathscr{A}_1,\dots,\mathscr{A}_d)\in\mathscr{B}[\mathscr{Y}]^d$ be c.m.o. Assume that $\big[\mathscr{U}_i,\mathscr{A}_j\big]=\big[\mathscr{U}_i,C\mathscr{U}_jC\big]=0$ for all j,i. If $\mathbf{U}$ is $(n_1,\dots,n_d)$-quasi-(m,C)-isometric c.m.o. and $\mathbf{A}$ is nilpotent of order q, then $\mathbf{U}+\mathbf{A}$ is $(n_1+q,\ \dots,n_d+q)$-quasi-$(m+2q-2,C)$-isometric c.m.o.*

Proof. Let $\ell = m+2q-2$ and $\widetilde{\mathbf{m}} = (n_1+q, \ldots, n_d+q) = \widetilde{\mathbf{n}} + (q,\ldots,q)$. We have to prove that

$$\big(\mathbf{U}+\mathbf{A}\big)^{*\widetilde{\mathbf{m}}}\mathscr{Q}_\ell\big(\mathbf{U}+\mathbf{A},C\big)\big(\mathbf{U}+\mathbf{A}\big)^{\widetilde{\mathbf{m}}} = 0.$$

By Lemma 2.3, we can write,

$$\begin{aligned}
&\big(\mathbf{U}+\mathbf{A}\big)^{*\widetilde{\mathbf{m}}}\mathscr{Q}_\ell\big(\mathbf{U}+\mathbf{A},C\big)\big(\mathbf{U}+\mathbf{A}\big)^{\widetilde{\mathbf{m}}}\\
=&\Big(\sum_{0\leqslant\gamma\leqslant\widetilde{\mathbf{m}}}\binom{\widetilde{\mathbf{m}}}{\gamma}\mathbf{U}^{*\widetilde{\mathbf{m}}-\gamma}\mathbf{A}^{*\gamma}\Big)\\
&\Big(\sum_{|\omega|+|\mu|+k=n}\binom{n}{\omega,\mu,k}(\mathbf{U}+\mathbf{A})^{*\omega}\cdot\mathbf{A}^{*\mu}\cdot\mathscr{Q}_k(\mathbf{U})\cdot C\mathbf{U}^{\mu}C\cdot C\mathbf{A}^{\omega}C\Big)\\
&\Big(\sum_{0\leqslant\beta\leqslant\widetilde{\mathbf{m}}}\binom{\widetilde{\mathbf{m}}}{\beta}\mathbf{U}^{\widetilde{\mathbf{m}}-\beta}\mathbf{A}^{\beta}\Big).
\end{aligned}$$

In view of the above, if $\max\{|\omega|,|\mu|\}\geqslant q$, then $\mathbf{A}^{*\mu}=0$ or $\mathbf{A}^{\omega}=0$ and therefore,

$$\big(\mathbf{U}+\mathbf{A}\big)^{*\widetilde{\mathbf{m}}}\mathscr{Q}_\ell\big(\mathbf{U}+\mathbf{A},C\big)\big(\mathbf{U}+\mathbf{A}\big)^{\widetilde{\mathbf{m}}} = 0.$$

Similarly, if $\max\{|\omega|,|\mu|\}\leqslant q-1$, then we have $m+2q-2 = |\omega|+|\mu|+k \leqslant 2q-2+k$ and so $k\geqslant m$. In view of Proposition 2.2, it holds

$$\mathbf{U}^{*\widetilde{\mathbf{m}}-\gamma}\mathscr{Q}_k\big(\mathbf{U},C\big)\mathbf{U}^{\widetilde{\mathbf{m}}-\beta} = 0, \text{for } (\gamma_1,\ldots,\gamma_d) \text{ and } (\beta_1,\ldots,\beta_d)\in\{0,\ldots,q\}^d.$$

However, if $(\gamma_1,\ldots,\gamma_d)$ and $(\beta_1,\ldots,\beta_d)$ which satisfy one of the following conditions

(i) $\begin{cases}(\gamma_1,\ldots,\gamma_d)\in\{0,\ldots,q\}^d\\ (\beta_1,\ldots,\beta_d)\in\{q+1,\ldots,n_d+q\}^d,\end{cases}$

(ii) $\begin{cases}(\gamma_1,\ldots,\gamma_d)\in\{q+1,\ldots,n_d+q\}^d\\ (\beta_1,\ldots,\beta_d)\in\{0,\ldots,q\}^d\end{cases}$

or

(iii) $\begin{cases}(\gamma_1,\ldots,\gamma_d)\in\{q+1,\ldots,n_d+q\}^d\\ (\beta_1,\ldots,\beta_d)\in\{q+1,\ldots,n_d+q\}^d,\end{cases}$

it then follows $\mathbf{A}^{*\gamma}=0$ or $\mathbf{A}^{\beta}=0$ and therefore

$$\mathbf{A}^{*\gamma}\Big((\mathbf{U}+\mathbf{A})^{*\omega}\cdot\mathbf{A}^{*\mu}\cdot\mathscr{Q}_k(\mathbf{N})\cdot C\mathbf{U}^{\mu}C\cdot C\mathbf{A}^{\omega}C\Big)\mathbf{A}^{\beta} = 0.$$

By considering the different cases discuss above we can conclude the result of the theorem. □

A particularly interesting consequence of Theorem 2.3 is the following corollary.

COROLLARY 2.3. *Let* $\mathbf{U}=(\mathscr{U}_1,\ldots,\mathscr{U}_d)\in\mathscr{B}[\mathscr{Y}]^d$ *be an* $(n_1,\ldots,n_d)$*-quasi-*(m,C)*-isometric c.m.o. for some conjugation* C *and let* $A=(\mathscr{A}_1,\ldots,\mathscr{A}_d)\in\mathscr{B}[\mathscr{Y}]^d$ *be a* q*-nilpotent c.m.o. If* $\left[\mathscr{U}_i,C\mathscr{U}_jC\right]=0$ *for all* i,j*, then* $\mathbf{U}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{A}:=(\mathscr{U}_1\otimes I+I\otimes\mathscr{A}_1,\ldots,\mathscr{U}_d\otimes I+I\otimes\mathscr{A}_d)$ *is an* $(n_1+q,\ldots,n_d+q)$*-quasi-*$(m+2q-2,C\otimes C)$*-isometric c.m.o.*

Proof. Using similar notations to that used in the proof of Corollary 2.2 for $\widetilde{\mathbf{U}}$ and $\widetilde{\mathbf{A}}$. It can be shown that

$$\left[\mathscr{U}_i\otimes I,I\otimes\mathscr{A}_j\right]=\left[\mathscr{U}_i,(C\otimes C)(\mathscr{U}_j\otimes I)(C\otimes C)\right]=0\ \forall\ i,j=1\ldots,d.$$

However we easily see that $\widetilde{\mathbf{U}}$ is an $(n_1,\ldots,n_d)$-quasi-$(m,C\otimes C)$-isometric c.m.o. and $\widetilde{\mathbf{A}}$ is a nilpotent c.m.o. of order q. We conclude that $\widetilde{\mathbf{U}}$ and $\widetilde{\mathbf{A}}$ satisfy the conditions of Theorem 2.3. This, in turn, implies that $\mathbf{U}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{A}$ is an $(n_1+q,\ldots,n_d+q)$-quasi-$\left(m+2q-2,C\otimes C\right)$-c.m.o. □

3. Spectral properties of $(n_1,\ldots,n_d)$-quasi-m-isometries

Since the spectral properties of $(n_1,\ldots,n_d)$-quasi-m-isometric c.m.o. have not yet received any previous study as far as we know, we devoted this section to the study of these properties. However the question relating to the spectral properties of $(n_1,\ldots,n_d)$-quasi-(m,C)-isometric c.m.o. will be examined in future work.

DEFINITION 3.1. ([17]) Let $\mathbf{U}=(\mathscr{U}_1,\ldots,\mathscr{U}_d)$ be a c.m.o. on $\mathscr{Y}$.

(1) A point $\mu=(\mu_1,\ldots,\mu_d)\in\mathbb{C}^d$ is called a joint point eigenvalue of $\mathbf{U}$ if there exists a non zero vector $x\in\mathscr{Y}$ such that

$$\left(\mathscr{U}_j-\mu_jI_{\mathscr{Y}})\right)x=0\ \text{ for }\ j=1,\ldots,d.$$

Or equivalently if

$$\bigcap_{1\leqslant j\leqslant d}\ker(\mathscr{U}_j-\mu_j)\neq\{0\}.$$

The joint point spectrum, denoted by $\sigma_{jp}(\mathbf{U})$ is the set of all joint eigenvalues of $\mathbf{U}$.

(2) $\mu=(\mu_1,\ldots,\mu_d)\in\mathbb{C}^d$ is in the joint approximate point spectrum $\sigma_{jap}(\mathbf{U})$ if and only if there exists a sequence of unit vector $(x_k)_k$ such that

$$\lim_{k\longrightarrow\infty}\|\left(\mathscr{U}_j-\mu_j\right)x_k\|=0\ \ j=1,\ldots,d.$$

The following theorem generalize [22, Theorem 2.2].

THEOREM 3.1. *Let* $\mathbf{U}=(\mathscr{U}_1,\ldots,\mathscr{U}_d)\in\mathscr{B}[\mathscr{Y}]^d$ *be a c.m.o. If* $\mathbf{U}$ *is an* $(n_1,\ldots,n_d)$*-quasi-*m*-isometric c.m.o., then*

$$\sigma_{jap}(\mathbf{U})\subset\left\{\ \mu=(\mu_1,\ldots,\mu_d)\in\mathbb{C}^d:\ \|\mu\|_2:=\left(\sum_{1\leqslant i\leqslant d}|\mu_i|^2\right)^{\frac{1}{2}}=1\ \right\}\cup[0],$$

where

$$[0] := \{(\mu_1,\ldots,\mu_d)\in\mathbb{C}^d : \prod_{1\leqslant i\leqslant d}\mu_i = 0\}.$$

Proof. Let $\mu=(\mu_1,\ldots,\mu_d)\in\sigma_{jap}(\mathbf{U})$, we want to show that $\prod_{1\leqslant i\leqslant d}\mu_i=0$ or $\sum_{1\leqslant i\leqslant d}|\mu_i|^2=1$. It is well known that there exists a sequence $(x_k)_{k\geqslant 1}\subset\mathscr{Y}$, with $||x_k||=1$ such that $(\mathscr{U}_i-\mu_i I)x_k\longrightarrow 0$ for all $i=1,\ldots,d$ and we find $\lim_{k\to\infty}\big(\mathbf{U}^{\omega}-\mu^{\omega}\big)x_k=0$. According to that $\mathbf{U}$ is an $(n_1,\ldots,n_d)$-quasi-m-isometric c.m.o., it follows that

$$\sum_{0\leqslant j\leqslant m}(-1)^{m-j}\binom{m}{j}\left(\sum_{|\omega|=j}\frac{j!}{\omega!}\|\mathbf{U}^{\omega+\widetilde{\mathbf{n}}}x\|^2\right)=0 \quad\text{for all } x\in\mathscr{Y},$$

and we find

$$\begin{aligned}
0 &= \sum_{0\leqslant j\leqslant m}(-1)^{m-j}\binom{m}{j}\left(\sum_{|\omega|=j}\frac{j!}{\omega!}\|\left(\mathbf{U}^{\omega+\widetilde{\mathbf{n}}}-\mu^{\omega+\widetilde{\mathbf{n}}}+\mu^{\omega+\widetilde{\mathbf{n}}}\right)x_k\|^2\right)\\
&= \sum_{0\leqslant j\leqslant m}(-1)^{m-j}\binom{m}{j}\sum_{|\omega|=j}\frac{j!}{\omega!}\Big\{\|\big(\mathbf{U}^{\omega+\widetilde{\mathbf{n}}}-\mu^{\omega+\widetilde{\mathbf{n}}}\big)x_k\|^2\\
&\quad +2Re\Big\langle\big(\mathbf{U}^{\omega+\widetilde{\mathbf{n}}}-\mu^{\omega+\widetilde{\mathbf{n}}}\big)x_k \mid \mu^{\omega+\widetilde{\mathbf{n}}}x_k\Big\rangle+\big\|\mu^{\omega+\widetilde{\mathbf{n}}}x_k\big\|^2\Big\}\\
&= \sum_{0\leqslant j\leqslant m}(-1)^{m-j}\binom{m}{j}\sum_{|\omega|=j}\frac{j!}{\omega!}\Big\{\|\big(\mathbf{U}^{\omega+\widetilde{\mathbf{n}}}-\mu^{\omega+\widetilde{\mathbf{n}}}\big)x_k\|^2\\
&\quad +2Re\Big\langle\big(\mathbf{U}^{\omega+\widetilde{\mathbf{n}}}-\mu^{\omega+\widetilde{\mathbf{n}}}\big)x_k \mid \mu^{\omega+\widetilde{\mathbf{n}}}x_k\Big\rangle+\big(|\mu|^2\big)^{\omega+\widetilde{\mathbf{n}}}\Big\}.
\end{aligned}$$

By taking the limit as $k\longrightarrow\infty$ we find

$$\begin{aligned}
0 &= \sum_{0\leqslant j\leqslant m}(-1)^{m-j}\binom{m}{k}\sum_{|\omega|=j}\frac{j!}{\omega!}\big(|\mu|^2\big)^{\omega+\widetilde{\mathbf{n}}}\\
&= \big(|\mu|^2\big)^{\widetilde{\mathbf{n}}}\sum_{0\leqslant j\leqslant m}(-1)^{m-j}\binom{m}{j}\sum_{|\omega|=j}\frac{k!}{\omega!}\big(|\mu|^2\big)^{\omega}\\
&= \big(|\mu|^2\big)^{\widetilde{\mathbf{n}}}\left(1-\sum_{1\leqslant j\leqslant d}|\mu_k|^2\right)^m.
\end{aligned}$$

In this case, we still have

$$\prod_{1\leqslant j\leqslant d}\mu_j=0 \ \text{ or } \sum_{1\leqslant j\leqslant d}|\mu_j|^2=1. \quad\square$$

The authors S. Mecheri et al. has proved in [22, Theorem 2.3] that if $\lambda\neq 0$ is an approximate eigenvalue of a n-quasi-m-isometric single operator U, then $\overline{\lambda}-U^*$ is not bounded from below. In the next theorem, we extend this result to multidimensional case.

THEOREM 3.2. *Let* $\mathbf{U}=(\mathscr{U}_1,\dots,\mathscr{U}_d)\in\mathscr{B}[\mathscr{Y}]^d$ *be an* $(n_1,\dots,n_d)$*-quasi-*m*-isometric c.m.o. If* $\mu=(\mu_1,\dots,\mu_d)\in\sigma_{jap}(\mathbf{U})\diagdown[0]$*, then* $\sum_{1\leqslant i\leqslant d}\mu_i\big(\overline{\mu_i}-\mathscr{U}_i^*\big)$ *is not bounded from below. In particular if* $\mu=(\mu_1,\dots,\mu_d)\in\sigma_{jp}(\mathbf{U})\diagdown[0]$*,then* $\sum_{1\leqslant i\leqslant d}\mu_i\big(\overline{\mu_i}-\mathscr{U}_i^*\big)$ *is not bounded from below.*

Proof. Consider $\mu=(\mu_1,\dots,\mu_d)\in\sigma_{jap}(\mathbf{U})\diagdown[0]$. It is well known that there exists a sequence $(x_k)_{k\geqslant1}\subset\mathscr{Y}$, with $||x_k||=1$ such that $\big(\mathscr{U}_j-\mu_jI_{\mathscr{Y}}\big)x_k\longrightarrow0$ for all $j=1,\dots,d$ and we find $\lim_{k\to\infty}\big(\mathbf{U}^\omega-\mu^\omega\big)x_k=0$. If $\mathbf{U}$ is an $(n_1,\dots,n_d)$-quasi-m-isometric c.m.o. we can write

$$\mathbf{U}^{*\widetilde{\mathbf{n}}}\left(\sum_{0\leqslant j\leqslant m}(-1)^{m-j}\binom{m}{j}\Big(\sum_{|\omega|=j}\frac{j!}{\omega!}\mathbf{U}^{*\omega}\mathbf{U}^{\omega}\right)\mathbf{U}^{\widetilde{\mathbf{n}}}=0.$$

Thus we have

$$\begin{aligned}0&=\mathbf{U}^{*\widetilde{\mathbf{n}}}\left(\sum_{0\leqslant j\leqslant m}(-1)^{m-j}\binom{m}{j}\Big(\sum_{|\omega|=j}\frac{j!}{\omega!}\mathbf{U}^{*\omega}\right)\mathbf{U}^{\omega+\widetilde{\mathbf{n}}}x_k\\&=\mathbf{U}^{*\widetilde{\mathbf{n}}}\left(\sum_{0\leqslant j\leqslant m}(-1)^{m-j}\binom{m}{j}\sum_{|\omega|=j}\frac{j!}{\omega!}\mathbf{U}^{*\omega}\right)\left[\mathbf{U}^{\omega+\widetilde{\mathbf{n}}}-\mu^{\omega+\widetilde{\mathbf{n}}}+\mu^{\omega+\widetilde{\mathbf{n}}}\right]x_k\\&=\mathbf{U}^{*\widetilde{\mathbf{n}}}\left(\sum_{0\leqslant j\leqslant m}(-1)^{m-j}\binom{m}{j}\sum_{|\omega|=j}\frac{k!}{\omega!}\mathbf{U}^{*\omega}\right)\left[\mathbf{U}^{\omega+\widetilde{\mathbf{n}}}-\mu^{\omega+\widetilde{\mathbf{n}}}\right]x_k\\&\quad+\mathbf{U}^{*\widetilde{\mathbf{n}}}\left(\sum_{0\leqslant j\leqslant m}(-1)^{m-j}\binom{m}{j}\sum_{|\omega|=j}\frac{j!}{\omega!}\mathbf{U}^{*\omega}\right)\mu^{\omega+\widetilde{\mathbf{n}}}x_k.\end{aligned}$$

By letting $k\longrightarrow\infty$ we find,

$$\mathbf{U}^{*\widetilde{\mathbf{n}}}\left(1-\sum_{1\leqslant i\leqslant d}\mu_i\mathscr{U}_i^*\right)^m x_k\longrightarrow0.$$

If $\big(1-\sum_{1\leqslant i\leqslant d}\mu_i\mathscr{U}_i^*\big)$ is bounded from below so is $\left(1-\sum_{1\leqslant i\leqslant d}\mu_i\mathscr{U}_i^*\right)^m$ and therefore

$$\|\big(1-\sum_{1\leqslant i\leqslant d}\mu_i\mathscr{U}_i^*\big)^m x\|\geqslant M\|x\|\qquad\forall\,x\in\mathscr{Y},$$

for some $M>0$. This implies that

$$\left\|\mathbf{U}^{*\widetilde{\mathbf{n}}}\left(1-\sum_{1\leqslant i\leqslant d}\mu_i\mathscr{U}_i^*\right)^m x_k\right\|\geqslant M\left\|\mathbf{U}^{*\widetilde{\mathbf{n}}}x_k\right\|.$$

By letting $k\longrightarrow\infty$ we find, $\mathbf{U}^{*\widetilde{\mathbf{n}}}x_k\longrightarrow0$. By observing that

$$\begin{aligned}\Big\langle\mathbf{U}^{*\widetilde{\mathbf{n}}}x_k\mid x_k\Big\rangle=\Big\langle x_k\mid\mathbf{U}^{\widetilde{\mathbf{n}}}x_k\Big\rangle&=\Big\langle x_k\mid\Big(\mathbf{U}^{\widetilde{\mathbf{n}}}-\mu^{\widetilde{\mathbf{n}}}+\mu^{\widetilde{\mathbf{n}}}\Big)x_k\Big\rangle\\&=\Big\langle x_k\mid\Big(\mathbf{U}^{\widetilde{\mathbf{n}}}-\mu^{\widetilde{\mathbf{n}}}\Big)x_k\Big\rangle+\overline{\mu^{\widetilde{\mathbf{n}}}}\,\langle x_k\mid x_k\rangle,\end{aligned}$$

we obtain $\prod_{1\leqslant j\leqslant d}\mu_j^{n_j}=0$ which is impossible since $\mu\notin[0]$.

Consequently, $\big(1-\sum_{1\leqslant j\leqslant d}\mu_j\mathscr{U}_j^*\big)$ is not bounded from below.

Taking into account the hypothesis that $\mu=(\mu_1,\ldots,\mu_d)\in\sigma_{jap}(\mathbf{U})\setminus[0]$, if follow from Theorem 3.1 that $\sum_{1\leqslant j\leqslant d}|\mu_j|^2=1$. This implies

$$1-\sum_{1\leqslant j\leqslant d}\mu_j\mathscr{U}_j^*=\sum_{1\leqslant j\leqslant d}|\mu_j|^2-\sum_{1\leqslant j\leqslant d}\mu_j\mathscr{U}_j^*=\sum_{1\leqslant j\leqslant d}\mu_j\big(\overline{\mu_j}-\mathscr{U}_j^*\big)$$

and the proof is complete. □

In [5], the authors has proved that if $\mathscr{U}$ is an quasi-m-isometric operator and $\lambda,\mu\in\sigma_p(N)$ such that $\lambda\overline{\mu}\neq 1$, then

$$\ker\big(\mathscr{U}-\lambda\big)\perp\ker\big(\mathscr{U}-\mu\big).$$

PROPOSITION 3.1. *Let* $\mathbf{U}=(\mathscr{U}_1,\ldots,\mathscr{U}_d)\in\mathscr{B}[\mathscr{Y}]^d$ *be an* $(n_1,\ldots,n_d))$*-quasi-*m*-isometric c.m.o. The following statements are true.*

(1) *If* $\lambda=(\lambda_1,\ldots,\lambda_d)\in\sigma_{jp}(\mathbf{U})\setminus[0]$ *and* $\mu=(\mu_1,\ldots,\mu_d)\in\sigma_{jp}(\mathbf{U})\setminus[0]$ *such that* $\sum_{1\leqslant j\leqslant d}\overline{\mu_j}\lambda_j\neq 1$*,then*

$$\bigcap_{1\leqslant j\leqslant d}\ker\big(\mathscr{U}_j-\mu_j\big)\perp\bigcap_{1\leqslant j\leqslant d}\ker\big(\mathscr{U}_j-\lambda_j\big).$$

(2) *Let* $\lambda=(\lambda_1,\ldots,\lambda_d)$ *and* $\mu=(\mu_1,\ldots,\mu_d)\in\sigma_{jap}(\mathbf{U})\setminus[0]$ *such that* $\sum_{1\leqslant j\leqslant d}\lambda_j\overline{\mu_j}$ $\neq 1$*. If* $\{x_k\}_k$ *and* $\{y_k\}_k$ *are two sequences of unit vectors in* $\mathscr{Y}$ *for which* $\lim_{k\to\infty}\|(\mathscr{U}_j-\lambda_j)x_k\|=\lim_{k\to\infty}\|(\mathscr{U}_j-\mu_j)y_k\|=0$*,for all* $j=1,\ldots,d$*, then* $\lim_{k\to\infty}\langle x_k\,|\,y_k\rangle=0$.

Proof. (1) Let $x\in\bigcap_{1\leqslant j\leqslant d}\ker\big(\mathscr{U}_j-\mu_j\big)$ and $y\in\bigcap_{1\leqslant j\leqslant d}\ker\big(\mathscr{U}_j-\lambda_j\big)$, it follows that

$$\mathscr{U}_jx=\mu_jx\ \text{ and }\ \mathscr{U}_jy=\lambda_jy,\ \ j=1,\ldots,d.$$

Since $\mathbf{N}$ is an $(n_1,\ldots,n_d)$-quasi-m-isometric c.m.o. we have

$$\begin{aligned}0&=\sum_{0\leqslant j\leqslant m}(-1)^{m-j}\binom{m}{j}\Big(\sum_{|\omega|=j}\frac{j!}{\omega!}\Big\langle\mathbf{U}^{\omega+\widetilde{\mathbf{n}}}x\,|\,\mathbf{U}^{\omega+\widetilde{\mathbf{n}}}y\Big\rangle\\&=\big(\lambda\overline{\mu}\big)^{\widetilde{\mathbf{n}}}\sum_{0\leqslant j\leqslant m}(-1)^{m-j}\binom{m}{j}\sum_{|\omega|=j}\frac{k!}{\omega!}\big(\lambda\overline{\mu}\big)^{\omega}\langle x\,|\,y\rangle\\&=\big(\lambda\widetilde{\mu}\big)^{\widetilde{\mathbf{n}}}\Big(1-\sum_{1\leqslant j\leqslant d}\overline{\mu_j}\lambda_j\Big)^m\langle x\,|\,y\rangle,\end{aligned}$$

where $\lambda.\overline{\mu} = (\lambda_1\overline{\mu}_1,\ldots,\lambda_d\overline{\mu}_d)$.

Then from $\lambda,\mu \notin [0]$ and $1 - \sum_{1\leqslant j\leqslant d} \overline{\mu_j}\lambda_j \neq 0$ we obtain $\langle x \mid y\rangle = 0$.

By observing that

$$\begin{aligned}&\left\langle \mathbf{U}^{\omega+\widetilde{\mathbf{n}}}x_k \mid \mathbf{U}^{\omega+\widetilde{\mathbf{n}}}y_k \right\rangle\\ &= \left\langle (\mathbf{U}^{\omega+\widetilde{\mathbf{n}}} - \lambda^{\widetilde{\mathbf{n}}})x_k + \lambda^{\widetilde{\mathbf{n}}}x_k \mid (\mathbf{U}^{\omega+\widetilde{\mathbf{n}}} - \mu^{\widetilde{\mathbf{n}}})y_k + \mu^{\widetilde{\mathbf{n}}}y_k \right\rangle,\end{aligned}$$

and using similar argument as in the proof of the statement (1) we can infer that

$$(\lambda\widetilde{\mu})^{\widetilde{\mathbf{n}}}\left(1 - \sum_{1\leqslant j\leqslant d}\overline{\mu_j}\lambda_j\right)^m \lim_{k\to\infty}\langle x_k \mid y_k\rangle = 0.$$

Following the conditions $\lambda,\mu \notin [0]$ and $1 - \sum_{1\leqslant j\leqslant d} \overline{\mu_j}\lambda_j \neq 0$, it follows that

$$\lim_{k\to\infty}\langle x_k \mid y_k\rangle = 0.$$

This completes the proof. □

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