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CHARACTERIZATIONS OF LIPSCHITZ FUNCTIONS VIA THE COMMUTATORS OF FRACTIONAL MAXIMAL FUNCTION IN ORLICZ SPACES ON STRATIFIED LIE GROUPS

VAGIF S. GULIYEV

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Abstract. In this paper we obtain necessary and sufficient conditions for the boundedness of the fractional maximal commutators $M_{b,\alpha}$ and the commutators of the fractional maximal operator $[b,M_\alpha]$ in Orlicz spaces $L^\Phi(\mathbb{G})$ on any stratified Lie group $\mathbb{G}$ when b belongs to Lipschitz spaces $\dot{\Lambda}_\beta(\mathbb{G})$. We obtain some new characterizations for certain subclasses of Lipschitz spaces $\dot{\Lambda}_\beta(\mathbb{G})$.

1. Introduction

The aim of this paper is to study the boundedness of the fractional maximal commutators $M_{b,\alpha}$ and the commutators of the fractional maximal operator $[b,M_\alpha]$ in the Orlicz spaces $L^\Phi(\mathbb{G})$ on any stratified Lie group $\mathbb{G}$ when b belongs to Lipschitz spaces $\dot{\Lambda}_\beta(\mathbb{G})$. The commutator operators play an important role in studying the regularity of solutions of elliptic, parabolic and ultraparabolic partial differential equations of second order, and their boundedness can be used to characterize certain function spaces. (see [4, 7, 10, 16, 17, 21, 23]).

Let $\mathbb{G}$ be a stratified Lie group, $f \in L^1_{\rm loc}(\mathbb{G})$ and $0 \leqslant \alpha < Q$, where Q is the homogeneous dimension of $\mathbb{G}$. The fractional maximal function $M_\alpha f$ is defined by

$$M_\alpha f(x) = \sup_{B \ni x} |B|^{-1+\frac{\alpha}{Q}} \int_B |f(y)|dy,$$

where the supremum is taken over all balls $B \subset \mathbb{G}$ containing x, and $|B|$ is the Haar measure of the $\mathbb{G}$-ball B.

The fractional maximal commutator generated by $b \in L^1_{\rm loc}(\mathbb{G})$ and M_α is defined by

$$M_{b,\alpha}(f)(x) = \sup_{B \ni x} |B|^{-1+\frac{\alpha}{Q}} \int_B |b(x)-b(y)||f(y)|dy.$$

The commutators generated by $b \in L^1_{\rm loc}(\mathbb{G})$ and M_α is defined by

$$[b,M_\alpha]f(x) = b(x)M_\alpha f(x) - M_\alpha(bf)(x).$$

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When $\alpha = 0$, we simply denote by $[b,M] = [b,M_0]$ and $M_b = M_{b,0}$. Obviously, the $M_{b,\alpha}$ and $[b,M_\alpha]$ operators are essentially different from each other because $M_{b,\alpha}$ is positive and sublinear and $[b,M_\alpha]$ is neither positive nor sublinear.

The mapping properties of $M_{b,\alpha}$ and $[b,M_\alpha]$ have been studied extensively by many authors (see, for instance, [1, 2, 5, 9, 13, 14, 28, 29]). The operators M_α, $[b,M_\alpha]$ and $M_{b,\alpha}$ play an important role in real and harmonic analysis and applications (see, for instance, [3, 6, 8, 15, 26, 29]). In [28] Zhang and Wu studied the necessary and sufficient condition for the boundedness of $[b,M_\alpha]$ on $L^p(\mathbb{R}^n)$ spaces, see also [29]. In [18] Janson gave some characterizations of the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ via commutator $[b,T]$ and the author proved that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ if and only if $[b,T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1 < p < n/\beta$, $1/p - 1/q = \beta/n$ and T is the classical singular integral operator.

Stratified groups appear in quantum physics and many parts of mathematics, including Fourier analysis, several complex variables, geometry, and topology [3, 8, 25]. The geometric structure of stratified Lie groups is so good that they inherit many analysis properties from the Euclidean spaces [10]. Apart from this, the difference between the geometric structures of Euclidean spaces and stratified Lie groups makes the study of the function spaces on them more complicated. However, the study of Orlicz spaces on stratified Lie groups is quite a few, which makes it deserve a further investigation. In [27] Zhang considered some new characterizations of the Lipschitz spaces $\dot{\Lambda}_\beta(\mathbb{R}^n)$ via the boundedness of maximal commutator M_b and the (nonlinear) commutator $[b,M]$ in Lebesgue spaces and Morrey spaces on Euclidean spaces. In [29] Zhang et al gave necessary and sufficient conditions for the boundedness of the nonlinear commutator $[b,M_\alpha]$ on Orlicz spaces when the symbol b belongs to Lipschitz spaces $\dot{\Lambda}_\beta(\mathbb{R}^n)$ and obtained some new characterizations of non-negative Lipschitz functions. In [14] Guliyev et al. consider the boundedness of $M_{b,\alpha}$ and $[b,M_\alpha]$ in Orlicz spaces $L^\Phi(\mathbb{R}^n)$ when b belongs to the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$, by which some new characterizations of the Lipschitz spaces $\dot{\Lambda}_\beta(\mathbb{R}^n)$ are given. In [12] the author recently gave necessary and sufficient conditions for the boundedness of the fractional maximal commutators in the Orlicz spaces $L^\Phi(\mathbb{G})$ on any stratified Lie group $\mathbb{G}$ when b belongs to $BMO(\mathbb{G})$ spaces and obtained some new characterizations for certain subclasses of $BMO(\mathbb{G})$ spaces.

Inspired by the above literature, the purpose of this paper is to obtain the boundedness of the fractional maximal commutator $M_{b,\alpha}$ (Theorems 3.8, 3.12) and the nonlinear commutator $[b,M_\alpha]$ (Theorems 4.3, 4.7) in the Orlicz spaces $L^\Phi(\mathbb{G})$ on any stratified Lie group $\mathbb{G}$ when b belongs to Lipschitz spaces $\dot{\Lambda}_\beta(\mathbb{G})$. We give some new characterizations of the Lipschitz spaces.

By $A \lesssim B$ we mean that $A \leqslant CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Notations

We first recall some preliminaries concerning stratified Lie groups (or so-called Carnot groups). We refer the reader to the books [3, 8, 24] for analysis on stratified Lie groups. Let $\mathscr{G}$ be a finite-dimensional, stratified, nilpotent Lie algebra. Assume that there is a direct sum vector space decomposition

$$\mathscr{G} = V_1 \oplus \cdots \oplus V_m \tag{2.1}$$

so that each element of V_j, $2 \leqslant j \leqslant m$, is a linear combination of $(j-1)$th order commutator of elements of V_1. Equivalently, (2.1) is a stratification provided $[V_i, V_j] = V_{i+j}$ whenever $i+j \leqslant m$ and $[V_i, V_j] = 0$ otherwise. Let $X = \{X_1, \ldots, X_n\}$ be a basis for V_1 and X_{ij}, $1 \leqslant i \leqslant k_j$, for V_j consisting of commutators of length j. We set $X_{i1} = X_i$, $i = 1, \ldots, n$ and $k_1 = n$, and we call X_{i1} a commutator of length 1.

If $\mathbb{G}$ is the simply connected Lie group associated with $\mathscr{G}$, then the exponential mapping is a global diffeomorphism from $\mathscr{G}$ to $\mathbb{G}$. Thus, for each $g \in \mathbb{G}$, there is $x = (x_{ij}) \in \mathbb{R}^N$, $1 \leqslant i \leqslant k_j$, $1 \leqslant j \leqslant m$, $N = \sum\limits_{j=1}^{m} k_j$, such that $g = \exp\left(\sum x_{ij} X_{ij}\right)$. A homogeneous norm function $|\cdot|$ on $\mathbb{G}$ is defined by $|g| = \left(\sum |x_{ij}|^{2\cdot m!/j}\right)^{1/(2\cdot m!)}$, and $Q = \sum\limits_{j=1}^{m} j k_j$ is said to be the *homogeneous dimension* of $\mathbb{G}$, since $d(\delta_r x) = r^Q dx$ for $r > 0$. The dilation δ_r on $\mathbb{G}$ is defined by

$$\delta_r(g) = \exp\left(\sum r^j x_{ij} X_{ij}\right) \quad \text{if} \quad g = \exp\left(\sum x_{ij} X_{ij}\right).$$

Since $\mathbb{G}$ is nilpotent, the exponential map is diffeomorphism from $\mathbb{G}$ onto $\mathbb{G}$ which takes Lebesgue measure on $\mathbb{G}$ to a biinvariant Haar measure dx on $\mathbb{G}$. The group identity of $\mathbb{G}$ will be referred to as the origin and denoted by e.

A homogenous norm on $\mathbb{G}$ is a continuous function $x \to \rho(x)$ from $\mathbb{G}$ to $[0, \infty)$, which is C^∞ on $\mathbb{G} \backslash \{0\}$ and satisfies $\rho(x^{-1}) = \rho(x)$, $\rho(\delta_t x) = t\rho(x)$ for all $x \in \mathbb{G}$, $t > 0$; $\rho(e) = 0$ (the group identity). Moreover, there exists a constant $c_0 \geqslant 1$ such that $\rho(xy) \leqslant c_0(\rho(x) + \rho(y))$ for all $x, y \in \mathbb{G}$. With this norm, we define the ball centered at x with radius r by $B(x,r) = \{y \in \mathbb{G} : \rho(y^{-1}x) < r\}$, and we denote by $B_r = B(e,r) = \{y \in \mathbb{G} \ : \ \rho(y) < r\}$ the open ball centered at e, the identity element of $\mathbb{G}$, with radius r. By $^{\complement}B(x,r) = \mathbb{G} \setminus B(x,r)$ we denote the complement of $B(x,r)$. One easily recognizes that there exists $c_1 = c_1(\mathbb{G})$ such that

$$|B(x,r)| = c_1\, r^Q, \ \ x \in \mathbb{G}, \ r > 0.$$

The most basic partial differential operator in a stratified Lie group is the sub-Laplacian associated with X is the second-order partial differential operator on $\mathbb{G}$ given by $\mathscr{L} = \sum_{i=1}^n X_i^2$.

First, we recall the definition of Young functions.

DEFINITION 2.1. A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r \to +0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r \to \infty} \Phi(r) = \infty$.

From the convexity and $\Phi(0)=0$ it follows that any Young function is increasing. If there exists $s\in(0,\infty)$ such that $\Phi(s)=\infty$, then $\Phi(r)=\infty$ for $r\geqslant s$. The set of Young functions such that

$$0<\Phi(r)<\infty \qquad \text{for} \qquad 0<r<\infty$$

will be denoted by $\mathscr{Y}$. If $\Phi\in\mathscr{Y}$, then Φ is absolutely continuous on every closed interval in $[0,\infty)$ and bijective from $[0,\infty)$ to itself.

For a Young function Φ and $0\leqslant s\leqslant\infty$, let

$$\Phi^{-1}(s)=\inf\{r\geqslant 0:\Phi(r)>s\}.$$

If $\Phi\in\mathscr{Y}$, then Φ^{-1} is the usual inverse function of Φ.

It is well known that

$$r\leqslant\Phi^{-1}(r)\widetilde{\Phi}^{-1}(r)\leqslant 2r \qquad \text{for } r\geqslant 0, \tag{2.2}$$

where $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r)=\begin{cases}\sup\{rs-\Phi(s):s\in[0,\infty)\}, & r\in[0,\infty),\\ \infty, & r=\infty.\end{cases}$$

A Young function Φ is said to satisfy the Δ_2-condition, written $\Phi\in\Delta_2$, if $\Phi(2r)\leqslant C\Phi(r)$, $r>0$ for some $C>1$. If $\Phi\in\Delta_2$, then $\Phi\in\mathscr{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi\in\nabla_2$, if $\Phi(r)\leqslant\frac{1}{2C}\Phi(Cr)$, $r\geqslant 0$ for some $C>1$.

DEFINITION 2.2. (Orlicz Space). For a Young function Φ, the set

$$L^{\Phi}(\mathbb{G})=\left\{f\in L^1_{\rm loc}(\mathbb{G}):\int_{\mathbb{G}}\Phi(k|f(x)|)dx<\infty \text{ for some } k>0\right\}$$

is called Orlicz space. The space $L^{\Phi}_{\rm loc}(\mathbb{G})$ is defined as the set of all functions f such that $f\chi_B\in L^{\Phi}(\mathbb{G})$ for all balls $B\subset\mathbb{G}$.

$L^{\Phi}(\mathbb{G})$ is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi}(\mathbb{G})}=\inf\left\{\lambda>0:\int_{\mathbb{G}}\Phi\Big(\frac{|f(x)|}{\lambda}\Big)dx\leqslant 1\right\}.$$

If $\Phi(r)=r^p$, $1\leqslant p<\infty$, then $L^{\Phi}(\mathbb{G})=L^p(\mathbb{G})$. If $\Phi(r)=0$, $0\leqslant r\leqslant 1$ and $\Phi(r)=\infty$, $r>1$, then $L^{\Phi}(\mathbb{G})=L^{\infty}(\mathbb{G})$.

For a measurable set $D\subset\mathbb{G}$, a measurable function f and $t>0$, let $m(D,\,f,\,t)=|\{x\in D:|f(x)|>t\}|$. In the case $D=\mathbb{G}$, we shortly denote it by $m(f,\,t)$.

DEFINITION 2.3. The weak Orlicz space

$$WL^{\Phi}(\mathbb{G})=\{f\in L^1_{\rm loc}(\mathbb{G}):\|f\|_{WL^{\Phi}}<\infty\}$$

is defined by the norm

$$\|f\|_{WL^{\Phi}}=\inf\Big\{\lambda>0\,:\,\sup_{t>0}\Phi(t)m\Big(\frac{f}{\lambda},\,t\Big)\leqslant 1\Big\}.$$

We note that $\|f\|_{WL^\Phi} \leqslant \|f\|_{L^\Phi}$, $\|f\|_{L^\Phi(D)} = \|f\chi_D\|_{L^\Phi}$ and $\|f\|_{WL^\Phi(D)} = \|f\chi_D\|_{WL^\Phi}$. Let us define the Φ-average of f over a ball B of $\mathbb{G}$ by

$$\|f\|_{\Phi,B} = \inf\left\{\lambda > 0 : \frac{1}{|B|}\int_B \Phi\Big(\frac{|f(x)|}{\lambda}\Big)dx \leqslant 1\right\}.$$

The following analogue of the Hölder's inequality is well known (see, for example, [22]).

LEMMA 2.4. *Let $D \subset \mathbb{G}$ be a measurable set and f,g be measurable functions on D. For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid*

$$\int_D |f(x)g(x)|dx \leqslant 2\|f\|_{L^\Phi(D)}\|g\|_{L^{\tilde{\Phi}}(D)}. \tag{2.3}$$

By elementary calculations we have the following property.

LEMMA 2.5. *Let Φ be a Young function and D be a set in $\mathbb{G}$ with finite Haar measure. Then*

$$\|\chi_D\|_{L^\Phi(\mathbb{G})} = \|\chi_D\|_{WL^\Phi(\mathbb{G})} = \frac{1}{\Phi^{-1}\left(|D|^{-1}\right)}.$$

By Lemmas 2.4, 2.5 and (2.2) we get the following estimate.

LEMMA 2.6. *For a Young function Φ and any ball B, the following inequality is valid:*

$$\int_B |f(y)|dy \leqslant 2|B|\Phi^{-1}\left(|B|^{-1}\right)\|f\|_{L^\Phi(B)}. \tag{2.4}$$

3. Characterization of Lipschitz spaces via commutators

For a given ball B and $0 \leqslant \alpha < Q$, we define the following maximal function:

$$M_{\alpha,B}f(x) = \sup_{B \supseteq B' \ni x} |B'|^{-1+\frac{\alpha}{Q}} \int_{B'} |f(y)|dy,$$

where the supremum is taken over all balls B' such that $x \in B' \subseteq B$. Moreover, we denote by $M_B = M_{0,B}$ when $\alpha = 0$.

LEMMA 3.1. [12,28] *Let $0 \leqslant \alpha < Q$, $f : \mathbb{G} \to \mathbb{R}$ be a locally integrable function and B is a ball on $\mathbb{G}$. Then for every $x \in B \subset \mathbb{G}$*

$$M_\alpha\big(f\chi_B\big)(x) = M_{\alpha,B}(f)(x),$$
$$M_\alpha\big(\chi_B\big)(x) = M_{\alpha,B}\big(\chi_B\big)(x) = |B|^{\frac{\alpha}{Q}}.$$

The following result completely characterizes the boundedness of M_α in Orlicz spaces.

THEOREM 3.2. [12] *Let $0<\alpha<Q$, Φ,Ψ be Young functions and $\Phi\in\mathscr{Y}$. The condition*

$$r^{\alpha}\Phi^{-1}\big(r^{-Q}\big)\leqslant C\Psi^{-1}\big(r^{-Q}\big) \tag{3.1}$$

for all $r>0$, where $C>0$ does not depend on r, is necessary and sufficient for the boundedness of M_α from $L^{\Phi}(\mathbb{G})$ to $WL^{\Psi}(\mathbb{G})$. Moreover, if $\Phi\in\nabla_2$, then the condition (3.1) is necessary and sufficient for the boundedness of M_α from $L^{\Phi}(\mathbb{G})$ to $L^{\Psi}(\mathbb{G})$.

REMARK 3.3. Note that in the case $\mathbb{G}=\mathbb{R}^n$, Theorem 3.2 was proved in [14].

In this section, as an application of Theorem 3.2 we consider the boundedness of $M_{b,\alpha}$ in Orlicz spaces when b belongs to the Lipschitz space, we give some new characterizations of the Lipschitz spaces.

Next we give the definition of the Lipschitz spaces on $\mathbb{G}$, and state some basic properties and useful lemmas.

DEFINITION 3.4. (Lipschitz-type spaces on $\mathbb{G}$)

(1) [19] Let $0<\beta<1$, we say a function b belongs to the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{G})$ if there exists a constant C such that for all $x,y\in\mathbb{G}$,

$$|b(x)-b(y)|\leqslant C\rho(y^{-1}x)^{\beta}.$$

The smallest such constant C is called the $\dot{\Lambda}_\beta(\mathbb{G})$ norm of b and is denoted by $\|b\|_{\dot{\Lambda}_\beta(\mathbb{G})}$.

(2) [20] Let $0<\beta<1$. The space $\mathrm{Lip}_\beta(\mathbb{G})$ is defined to be the set of all locally integrable functions b, i.e., there exists a positive constant C, such that

$$\sup_B\frac{1}{|B|^{1+\beta/Q}}\int_B|b(x)-b_B|dx\leqslant C,$$

where the supremum is taken over every ball $B\subset\mathbb{G}$ containing x and $b_B=\frac{1}{|B|}\int_B b(y)dy$. The smallest such constant C is called the $\mathrm{Lip}_\beta(\mathbb{G})$ norm of b and is denoted by $\|b\|_{\mathrm{Lip}_\beta(\mathbb{G})}$.

Since stratified Lie groups can be regarded as a special case of spaces of homogeneous type in the sense of Coifman-Weiss, hence, the following characterization of Lipschitz space (see [20]).

LEMMA 3.5. *Let $0<\beta<1$ and $b\in L^1_{\mathrm{loc}}(\mathbb{G})$, then*

$$\|b\|_{\dot{\Lambda}_\beta(\mathbb{G})}\approx\|b\|_{\mathrm{Lip}_\beta(\mathbb{G})}.$$

LEMMA 3.6. *Let $0<\beta<1$, $0\leqslant\alpha<Q$, $0<\alpha+\beta<Q$ and $b\in\dot{\Lambda}_\beta(\mathbb{G})$, then the following pointwise estimate holds:*

$$M_{b,\alpha}f(x)\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{G})}M_{\alpha+\beta}f(x).$$

Proof. If $b \in \dot{\Lambda}_\beta(\mathbb{G})$, then

$$\begin{aligned} M_{b,\alpha}(f)(x) &= \sup_{B \ni x} |B|^{-1+\frac{\alpha}{Q}} \int_B |b(x)-b(y)||f(y)|dy \\ &\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{G})} \sup_{B \ni x} |B|^{-1+\frac{\alpha+\beta}{Q}} \int_B |f(y)|dy \\ &= C\|b\|_{\dot{\Lambda}_\beta(\mathbb{G})} M_{\alpha+\beta} f(x). \quad \square \end{aligned}$$

LEMMA 3.7. *If $b \in L^1_{\mathrm{loc}}(\mathbb{G})$ and $B_0 := B(x_0, r_0)$, then*

$$|B_0|^{\frac{\alpha}{Q}} |b(x) - b_{B_0}| \leqslant M_{b,\alpha}\chi_{B_0}(x) \ \textit{for every} \ x \in B_0.$$

Proof. For $x \in B_0$, we get

$$\begin{aligned} &M_{b,\alpha}\chi_{B_0}(x) = \sup_{B \ni x} |B|^{-1+\frac{\alpha}{Q}} \int_B |b(x)-b(y)|\chi_{B_0}(y)dy \\ &= \sup_{B \ni x} |B|^{-1+\frac{\alpha}{Q}} \int_{B\cap B_0} |b(x)-b(y)|dy \geqslant |B_0|^{-1+\frac{\alpha}{Q}} \int_{B_0 \cap B_0} |b(x)-b(y)|dy \\ &\geqslant |B_0|^{\frac{\alpha}{Q}} \left| |B_0|^{-1} \int_{B_0} (b(x)-b(y))dy \right| = |B_0|^{\frac{\alpha}{Q}} |b(x) - b_{B_0}|. \quad \square \end{aligned}$$

The following theorem is valid.

THEOREM 3.8. *Let $0<\beta<1$, $0 \leqslant \alpha < Q$, $0<\alpha+\beta<Q$, $b \in L^1_{\mathrm{loc}}(\mathbb{G})$, Φ, Ψ be Young functions and $\Phi \in \mathscr{Y}$.*

1. *If $\Phi \in \nabla_2$ and the condition*

$$t^{-\frac{\alpha+\beta}{Q}} \Phi^{-1}\big(t\big) \leqslant C\Psi^{-1}\big(t\big) \tag{3.2}$$

holds for all $t>0$, where $C>0$ does not depend on t, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{G})$ is sufficient for the boundedness of $M_{b,\alpha}$ from $L^\Phi(\mathbb{G})$ to $L^\Psi(\mathbb{G})$.

2. *If the condition*

$$\Psi^{-1}(t) \leqslant C\Phi^{-1}(t)t^{-\frac{\alpha+\beta}{Q}} \tag{3.3}$$

holds for all $t>0$, where $C>0$ does not depend on t, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{G})$ is necessary for the boundedness of $M_{b,\alpha}$ from $L^\Phi(\mathbb{G})$ to $L^\Psi(\mathbb{G})$.

3. *If $\Phi \in \nabla_2$ and $\Psi^{-1}(t) \approx \Phi^{-1}(t)t^{-\frac{\alpha+\beta}{Q}}$, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{G})$ is necessary and sufficient for the boundedness of $M_{b,\alpha}$ from $L^\Phi(\mathbb{G})$ to $L^\Psi(\mathbb{G})$.*

Proof. (1) The first statement of the theorem follows from Theorem 3.2 and Lemma 3.6.

(2) We shall now prove the second part. Suppose that $\Psi^{-1}(t) \lesssim \Phi^{-1}(t)t^{-(\alpha+\beta)/Q}$ and $M_{b,\alpha}$ is bounded from $L^{\Phi}(\mathbb{G})$ to $L^{\Psi}(\mathbb{G})$. Choose any ball B in $\mathbb{G}$, by Lemmas 2.5 and 2.6

$$\begin{aligned}
\frac{1}{|B|^{1+\frac{\beta}{Q}}}\int_B |b(y)-b_B|dy &= \frac{1}{|B|^{1+\frac{\alpha+\beta}{Q}}}\int_B \left|\frac{1}{|B|^{1-\frac{\alpha}{Q}}}\int_B (b(y)-b(z))dz\right|dy \\
&\leqslant \frac{1}{|B|^{1+\frac{\alpha+\beta}{Q}}}\int_B M_{b,a}(\chi_B)(y)dy \\
&\leqslant \frac{2\Psi^{-1}(|B|^{-1})}{|B|^{\frac{\alpha+\beta}{Q}}}\|M_{b,\alpha}(\chi_B)\|_{L^{\Psi}(B)} \\
&\leqslant \frac{C}{|B|^{\frac{\alpha+\beta}{Q}}}\,\frac{\Psi^{-1}(|B|^{-1})}{\Phi^{-1}(|B|^{-1})} \leqslant C.
\end{aligned}$$

Thus by Lemma 3.5 we get $b \in \dot{\Lambda}_{\beta}(\mathbb{G})$.

(3) The third statement of the theorem follows from the first and second parts of the theorem. □

If we take $\alpha = 0$ at Theorem 3.8, we have the following result.

COROLLARY 3.9. *Let* $0 < \beta < 1$, $b \in L^1_{\mathrm{loc}}(\mathbb{G})$, Φ, Ψ *be Young functions and* $\Phi \in \mathscr{Y}$.

1. *If* $\Phi \in \nabla_2$ *and the condition* $\Phi^{-1}(t)t^{-\beta/Q} \lesssim \Psi^{-1}(t)$ *holds, then the condition* $b \in \dot{\Lambda}_{\beta}(\mathbb{G})$ *is sufficient for the boundedness of* M_b *from* $L^{\Phi}(\mathbb{G})$ *to* $L^{\Psi}(\mathbb{G})$.
2. *If* $\Psi^{-1}(t) \lesssim \Phi^{-1}(t)t^{-\beta/Q}$, *then the condition* $b \in \dot{\Lambda}_{\beta}(\mathbb{G})$ *is necessary for the boundedness of* M_b *from* $L^{\Phi}(\mathbb{G})$ *to* $L^{\Psi}(\mathbb{G})$.
3. *If* $\Phi \in \nabla_2$ *and* $\Psi^{-1}(t) \approx \Phi^{-1}(t)t^{-\beta/Q}$, *then the condition* $b \in \dot{\Lambda}_{\beta}(\mathbb{G})$ *is necessary and sufficient for the boundedness of* M_b *from* $L^{\Phi}(\mathbb{G})$ *to* $L^{\Psi}(\mathbb{G})$.

If we take $\Phi(t) = t^p$ and $\Psi(t) = t^q$ with $1 \leqslant p < \infty$ and $1 \leqslant q \leqslant \infty$ at Theorem 3.8, we have the following result.

COROLLARY 3.10. *Let* $0 < \beta < 1$, $0 \leqslant \alpha < Q$, $0 < \alpha+\beta < Q$, $b \in L^1_{\mathrm{loc}}(\mathbb{G})$, $1 < p < q \leqslant \infty$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha+\beta}{Q}$. *Then the condition* $b \in \dot{\Lambda}_{\beta}(\mathbb{G})$ *is necessary and sufficient for the boundedness of* $M_{b,\alpha}$ *from* $L^p(\mathbb{G})$ *to* $L^q(\mathbb{G})$.

REMARK 3.11. Note that in the case $\mathbb{G} = \mathbb{R}^n$, Theorem 3.8 was proved in [14].

The following theorem is valid.

THEOREM 3.12. *Let* $0 < \beta < 1$, $0 \leqslant \alpha < Q$, $0 < \alpha+\beta < Q$, $b \in L^1_{\mathrm{loc}}(\mathbb{G})$, Φ, Ψ *be Young functions and* $\Phi \in \mathscr{Y}$.

1. *If condition* (3.2) *holds, then the condition* $b \in \dot{\Lambda}_{\beta}(\mathbb{G})$ *is sufficient for the boundedness of* $M_{b,\alpha}$ *from* $L^{\Phi}(\mathbb{G})$ *to* $WL^{\Psi}(\mathbb{G})$.

2. *If condition* (3.3) *holds and* $\frac{t^{1+\varepsilon}}{\Psi(t)}$ *is almost decreasing for some* $\varepsilon > 0$, *then the condition* $b \in \dot{\Lambda}_\beta(\mathbb{G})$ *is necessary for the boundedness of* $M_{b,\alpha}$ *from* $L^\Phi(\mathbb{G})$ *to* $WL^\Psi(\mathbb{G})$.

3. *If* $\Psi^{-1}(t) \approx \Phi^{-1}(t)t^{-(\alpha+\beta)/Q}$ *and* $\frac{t^{1+\varepsilon}}{\Psi(t)}$ *is almost decreasing for some* $\varepsilon > 0$, *then the condition* $b \in \dot{\Lambda}_\beta(\mathbb{G})$ *is necessary and sufficient for the boundedness of* $M_{b,\alpha}$ *from* $L^\Phi(\mathbb{G})$ *to* $WL^\Psi(\mathbb{G})$.

Proof. (1) The first statement of the theorem follows from Theorem 3.2 and Lemma 3.6.

(2) For any fixed ball B_0 such that $x \in B_0$ by Lemma 3.7 we have $|B_0|^{\alpha/Q}|b(x) - b_{B_0}| \leqslant M_{b,\alpha}\chi_{B_0}(x)$. This together with the boundedness of $M_{b,\alpha}$ from $L^\Phi(\mathbb{G})$ to $WL^\Psi(\mathbb{G})$ and Lemma 2.5

$$\begin{aligned} |\{x \in B_0 : |B_0|^{\alpha/Q}|b(x) - b_{B_0}| > \lambda\}| &\leqslant |\{x \in B_0 : M_{b,\alpha}\chi_{B_0}(x) > \lambda\}| \\ &\leqslant \frac{1}{\Psi\left(\frac{\lambda}{C\|\chi_{B_0}\|_{L^\Phi}}\right)} = \frac{1}{\Psi\left(\frac{\lambda\Phi^{-1}(|B_0|^{-1})}{C}\right)}. \end{aligned}$$

Let $t > 0$ be a constant to be determined later, then

$$\begin{aligned} \int_{B_0} |b(x) - b_{B_0}|dx &= |B_0|^{-\alpha/Q} \int_0^\infty |\{x \in B_0 : |b(x) - b_{B_0}| > |B_0|^{-\alpha/Q}\lambda\}|d\lambda \\ &= |B_0|^{-\alpha/Q} \int_0^t \{x \in B_0 : |b(x) - b_{B_0}| > |B_0|^{-\alpha/Q}\lambda\}|d\lambda \\ &\quad + |B_0|^{-\alpha/Q} \int_t^\infty |\{x \in B_0 : |b(x) - b_{B_0}| > |B_0|^{-\alpha/Q}\lambda\}|d\lambda \\ &\leqslant t|B_0|^{1-\alpha/Q} + |B_0|^{-\alpha/Q} \int_t^\infty \frac{1}{\Psi\left(\frac{\lambda\Phi^{-1}(|B_0|^{-1})}{C}\right)} d\lambda \\ &\lesssim t\,|B_0|^{1-\alpha/Q} + \frac{|B_0|^{-\alpha/Q}t}{\Psi\left(\frac{t\Phi^{-1}(|B_0|^{-1})}{C}\right)}, \end{aligned}$$

where we use almost decreasingness of $\frac{t^{1+\varepsilon}}{\Psi(t)}$ in the last step.

Set $t = C|B_0|^{\frac{\alpha+\beta}{Q}}$ in the above estimate, we have

$$\int_{B_0} |b(x) - b_{B_0}|dx \lesssim |B_0|^{1+\beta/Q}.$$

Thus by Lemma 3.5 we get $b \in \dot{\Lambda}_\beta(\mathbb{G})$ since B_0 is an arbitrary ball in $\mathbb{G}$.

(3) The third statement of the theorem follows from the first and second parts of the theorem. $\square$

If we take $\alpha = 0$ at Theorem 3.12, we have the following result.

COROLLARY 3.13. [11] *Let* $0<\beta<1$, $b\in L^1_{\rm loc}(\mathbb{G})$, Φ,Ψ *be Young functions and* $\Phi\in\mathscr{Y}$.

1. *If the condition* $\Phi^{-1}(t)t^{-\beta/Q}\lesssim\Psi^{-1}(t)$ *holds, then the condition* $b\in\dot\Lambda_\beta(\mathbb{G})$ *is sufficient for the boundedness of* M_b *from* $L^\Phi(\mathbb{G})$ *to* $WL^\Psi(\mathbb{G})$.

2. *If* $\Psi^{-1}(t)\lesssim\Phi^{-1}(t)t^{-\beta/Q}$ *and* $\frac{t^{1+\varepsilon}}{\Psi(t)}$ *is almost decreasing for some* $\varepsilon>0$, *then the condition* $b\in\dot\Lambda_\beta(\mathbb{G})$ *is necessary for the boundedness of* M_b *from* $L^\Phi(\mathbb{G})$ *to* $WL^\Psi(\mathbb{G})$.

3. *If* $\Psi^{-1}(t)\approx\Phi^{-1}(t)t^{-\beta/Q}$ *and* $\frac{t^{1+\varepsilon}}{\Psi(t)}$ *is almost decreasing for some* $\varepsilon>0$, *then the condition* $b\in\dot\Lambda_\beta(\mathbb{G})$ *is necessary and sufficient for the boundedness of* M_b *from* $L^\Phi(\mathbb{G})$ *to* $WL^\Psi(\mathbb{G})$.

If we take $\Phi(t)=t^p$ and $\Psi(t)=t^q$ with $1\leqslant p<\infty$ and $1\leqslant q\leqslant\infty$ at Theorem 3.12, we have the following result.

COROLLARY 3.14. *Let* $0<\beta<1$, $0\leqslant\alpha<Q$, $0<\alpha+\beta<Q$, $b\in L^1_{\rm loc}(\mathbb{G})$, $1\leqslant p<q\leqslant\infty$ *and* $\frac{1}{p}-\frac{1}{q}=\frac{\alpha+\beta}{Q}$. *Then the condition* $b\in\dot\Lambda_\beta(\mathbb{G})$ *is necessary and sufficient for the boundedness of* $M_{b,\alpha}$ *from* $L^p(\mathbb{G})$ *to* $WL^q(\mathbb{G})$.

REMARK 3.15. Note that in the case $\mathbb{G}=\mathbb{R}^n$, Theorem 3.12 was proved in [14].

4. Commutators of fractional maximal function in Orlicz spaces

For a function b defined on $\mathbb{G}$, we denote

$$b^-(x):=\begin{cases}0\,, & \text{if } b(x)\geqslant 0\\ |b(x)|, & \text{if } b(x)<0\end{cases}$$

and $b^+(x):=|b(x)|-b^-(x)$. Obviously, $b^+(x)-b^-(x)=b(x)$.

The following relations between $[b,M_\alpha]$ and $M_{b,\alpha}$ are valid.

Let b be any non-negative locally integrable function. Then for all $f\in L^1_{\rm loc}(\mathbb{G})$ and $x\in\mathbb{G}$ the following inequality is valid

$$\begin{aligned}\big|[b,M_\alpha]f(x)\big| &= \big|b(x)M_\alpha f(x)-M_\alpha(bf)(x)\big|\\ &= \big|M_\alpha(b(x)f)(x)-M_\alpha(bf)(x)\big|\\ &\leqslant M_\alpha(|b(x)-b|f)(x)=M_{b,\alpha}(f)(x).\end{aligned}$$

If b is any locally integrable function on $\mathbb{G}$, then

$$|[b,M_\alpha]f(x)|\leqslant M_{b,\alpha}(f)(x)+2b^-(x)M_\alpha f(x),\qquad x\in\mathbb{G} \tag{4.1}$$

holds for all $f\in L^1_{\rm loc}(\mathbb{G})$ (see, for example, [6,29]).

LEMMA 4.1. *Let $0<\beta<1$, $b\in L^1_{\mathrm{loc}}(\mathbb{G})$ and Φ be a Young function. Then the following statements are equivalent:*

1. *$b\in\dot{\Lambda}_\beta(\mathbb{G})$ and $b\geqslant 0$.*
2. *For all Φ we have*

$$\sup_B |B|^{-\frac{\beta}{Q}}\Phi^{-1}\big(|B|^{-1}\big)\,\|b(\cdot)-M_B(b)(\cdot)\|_{L^\Phi(B)}\leqslant C. \tag{4.2}$$

3. *There exists Φ such that (4.2) is valid.*

Proof. $(1)\Rightarrow(2)$: Let b be any non-negative locally integrable function. Then

$$|[b,M_\alpha]f(x)|\leqslant M_{b,\alpha}(f)(x),\qquad x\in\mathbb{G} \tag{4.3}$$

holds for all $f\in L^1_{\mathrm{loc}}(\mathbb{G})$.

By Lemmas 3.1 and 3.6, for all $x\in B$, we have

$$\big|[b,M_\alpha]\big(\chi_B\big)(x)\big|\leqslant\|b\|_{\dot{\Lambda}_\beta}M_{\alpha+\beta}\big(\chi_B\big)(x)\lesssim\|b\|_{\dot{\Lambda}_\beta}|B|^{\frac{\alpha+\beta}{Q}}, \tag{4.4}$$

and

$$\big|[b,M]\big(\chi_B\big)(x)\big|\leqslant\|b\|_{\dot{\Lambda}_\beta}M_\beta\big(\chi_B\big)(x)\lesssim\|b\|_{\dot{\Lambda}_\beta}|B|^{\frac{\beta}{Q}}. \tag{4.5}$$

For any fixed ball B,

$$\begin{aligned}
&|B|^{-\frac{\beta}{Q}}\Psi^{-1}\big(|B|^{-1}\big)\,\|b(\cdot)-M_B(b)(\cdot)\|_{L^\Psi(B)}\\
&\leqslant|B|^{-\frac{\beta}{Q}}\Psi^{-1}\big(|B|^{-1}\big)\,\Big\|b(\cdot)-|B|^{-\frac{\alpha}{Q}}M_{\alpha,B}(b)(\cdot)\Big\|_{L^\Psi(B)}\\
&\quad+|B|^{-\frac{\beta}{Q}}\Psi^{-1}\big(|B|^{-1}\big)\,\Big\|M_B(b)(\cdot)-|B|^{-\frac{\alpha}{Q}}M_{\alpha,B}(b)(\cdot)\Big\|_{L^\Psi(B)}:=I_1+I_2.
\end{aligned} \tag{4.6}$$

For I_1. By Lemma 3.1, for any $x\in B$,

$$\begin{aligned}
b(x)-|B|^{-\frac{\alpha}{Q}}M_{\alpha,B}(b)(x)&=|B|^{-\frac{\alpha}{Q}}\Big(b(x)|B|^{\frac{\alpha}{Q}}-M_{\alpha,B}(b)(x)\Big)\\
&=|B|^{-\frac{\alpha}{Q}}\Big(b(x)M_{\alpha,B}\big(\chi_B\big)(x)-M_\alpha\big(b\chi_B\big)(x)\Big)=|B|^{-\frac{\alpha}{Q}}[b,M_\alpha]\big(\chi_B\big)(x).
\end{aligned}$$

Therefore, from (4.4) we obtain

$$\begin{aligned}
I_1&=|B|^{-\frac{\beta}{Q}}\Psi^{-1}\big(|B|^{-1}\big)\|b(\cdot)-|B|^{-\frac{\alpha}{Q}}M_{\alpha,B}(b)(\cdot)\|_{L^\Psi(B)}\\
&=|B|^{-\frac{\alpha+\beta}{Q}}\Psi^{-1}\big(|B|^{-1}\big)\|b(\cdot)M_\alpha(\chi_B)(\cdot)-M_\alpha(b\chi_B)(\cdot)\|_{L^\Psi(B)}\\
&=|B|^{-\frac{\alpha+\beta}{Q}}\Psi^{-1}\big(|B|^{-1}\big)\|[b,M_\alpha](\chi_B)\|_{L^\Psi(B)}\\
&\lesssim|B|^{-\frac{\alpha+\beta}{Q}}\Psi^{-1}\big(|B|^{-1}\big)\,\|b\|_{\dot{\Lambda}_\beta}|B|^{\frac{\alpha+\beta}{Q}}\,\|\chi_B\|_{L^\Psi}\lesssim\|b\|_{\dot{\Lambda}_\beta}.
\end{aligned} \tag{4.7}$$

Next, we estimate I_2. By Lemma 3.1, for any $x \in B$, we have

$$\begin{aligned}
&\big||B|^{-\frac{\alpha}{Q}} M_{\alpha,B}(b)(x) - M_B(b)(x)\big| = |B|^{-\frac{\alpha}{Q}} \big|M_{\alpha,B}(b)(x) - |B|^{\frac{\alpha}{Q}} M_B(b)(x)\big| \\
&= |B|^{-\frac{\alpha}{Q}} \big|M_\alpha\big(b\chi_B\big)(x) - M_\alpha\big(\chi_B\big)(x) M\big(b\chi_B\big)(x)\big| \\
&\leqslant |B|^{-\frac{\alpha}{Q}} \big|M_\alpha\big(b\chi_B\big)(x) - |b(x)| M_\alpha\big(\chi_B\big)(x)\big| \\
&\quad + |B|^{-\frac{\alpha}{Q}} \big|b(x) M_\alpha\big(\chi_B\big)(x) - M_\alpha\big(\chi_B\big)(x) M\big(b\chi_B\big)(x)\big| \\
&= |B|^{-\frac{\alpha}{Q}} \big|M_\alpha\big(b\chi_B\big)(x) - b(x) M_\alpha\big(\chi_B\big)(x)\big| \\
&\quad + |B|^{-\frac{\alpha}{Q}} M_\alpha\big(\chi_B\big)(x) \big|b(x) M\big(\chi_B\big)(x) - M\big(b\chi_B\big)(x)\big| \\
&= |B|^{-\frac{\alpha}{Q}} \big|[b, M_\alpha]\big(\chi_B\big)(x)\big| + \big|[b, M]\big(\chi_B\big)(x)\big|.
\end{aligned} \tag{4.8}$$

From (4.8) we obtain, for any $x \in B$,

$$\big||B|^{-\frac{\alpha}{Q}} M_{\alpha,B}(b)(x) - M_B(b)(x)\big| \leqslant |B|^{-\frac{\alpha}{Q}} \big|[b, M_\alpha]\big(\chi_B\big)(x)\big| + \big|[b, M]\big(\chi_B\big)(x)\big|.$$

Then, it follows from Lemma 2.5 and (4.4), (4.5) that

$$\begin{aligned}
I_2 &= |B|^{-\frac{\beta}{Q}} \Psi^{-1}\big(|B|^{-1}\big) \big\| |B|^{-\frac{\alpha}{Q}} M_{\alpha,B}(b)(\cdot) - M_B(b)(\cdot) \big\|_{L^\Psi(B)} \\
&\lesssim |B|^{-\frac{\beta}{Q}} \Psi^{-1}\big(|B|^{-1}\big) |B|^{-\frac{\alpha}{Q}} \big\| [b, M_\alpha]\big(\chi_B\big) \big\|_{L^\Psi(B)} \\
&\quad + |B|^{-\frac{\beta}{Q}} \Psi^{-1}\big(|B|^{-1}\big) \big\| [b, M]\big(\chi_B\big) \big\|_{L^\Psi(B)} \\
&\lesssim \|b\|_{\dot{\Lambda}_\beta} |B|^{-\frac{\alpha+\beta}{Q}} \Psi^{-1}\big(|B|^{-1}\big)\, |B|^{\frac{\alpha+\beta}{Q}}\, \|\chi_B\|_{L^\Psi} \\
&\quad + |B|^{-\frac{\beta}{Q}}\, \|b\|_{\dot{\Lambda}_\beta} \Psi^{-1}\big(|B|^{-1}\big) |B|^{\frac{\beta}{Q}}\, \|\chi_B\|_{L^\Psi} \lesssim \|b\|_{\dot{\Lambda}_\beta}.
\end{aligned} \tag{4.9}$$

By (4.6), (4.7) and (4.9), we get

$$|B|^{-\frac{\beta}{Q}} \Psi^{-1}\big(|B|^{-1}\big)\, \big\|b(\cdot) - M_B(b)(\cdot)\big\|_{L^\Psi(B)} \lesssim \|b\|_{\dot{\Lambda}_\beta},$$

which leads us to (4.2) since B is arbitrary.

(3) $\Rightarrow$ (1): Now, let us prove $b \in \dot{\Lambda}_\beta(\mathbb{G})$ and $b \geqslant 0$. For any ball B, let $E = \{y \in B : b(y) \leqslant b_B\}$ and $F = \{y \in B : b(y) > b_B\}$. The following equality is true (see [2, page 3331]):

$$\int_E |b(y) - b_B| dy = \int_F |b(y) - b_B| dy.$$

Since $b(y) \leqslant b_B \leqslant |b_B| \leqslant |B|^{-\frac{\alpha}{Q}} M_{\alpha,B}(b)(y)$ for any $y \in E$, we obtain

$$|b(y) - b_B| \leqslant \big|b(y) - |B|^{-\frac{\alpha}{Q}} M_{\alpha,B}(b)(y)\big|,\ y \in E.$$

Then from Lemma 2.6 and (4.2) we have

$$\begin{aligned}
\frac{1}{|B|^{1+\frac{\beta}{Q}}}\int_B |b(y)-b_B|dy &= \frac{2}{|B|^{1+\frac{\beta}{Q}}}\int_E |b(y)-b_B|dy \\
&\leqslant \frac{2}{|B|^{1+\frac{\beta}{Q}}}\int_E \left|b(y)-|B|^{-\frac{\alpha}{Q}}M_{\alpha,B}(b)(y)\right|dy \\
&\leqslant \frac{2}{|B|^{1+\frac{\beta}{Q}}}\int_B \left|b(y)-|B|^{-\frac{\alpha}{Q}}M_{\alpha,B}(b)(y)\right|dy \\
&\leqslant 4|B|^{-\frac{\beta}{Q}}\Psi^{-1}\left(|B|^{-1}\right)\left\|b(\cdot)-|B|^{-\frac{\alpha}{Q}}M_{\alpha,B}(b)(\cdot)\right\|_{L^{\Psi}(B)} \leqslant C.
\end{aligned}$$

Thus by Lemma 3.5 we get $b\in\dot{\Lambda}_\beta(\mathbb{G})$.

In order to prove $b\geqslant 0$, it suffices to show $b^- = 0$. Observe that $0\leqslant b^+(y)\leqslant |b(y)|\leqslant M_B(b)(y)$ for $y\in B$, therefore, for any $y\in B$, there holds

$$0\leqslant b^-(y)\leqslant M_B(b)(y)-b^+(y)+b^-(y)=M_B(b)(y)-b(y).$$

Then for any ball B, we have

$$\begin{aligned}
\frac{1}{|B|}\int_B b^-(y)dy &\leqslant \frac{1}{|B|}\int_B \big(M_B(b)(y)-b(y)\big)dy \\
&= \frac{1}{|B|}\int_B \left|b(y)-M_B(b)(y)\right|dy \\
&\leqslant \frac{|B|^{\frac{\beta}{Q}}}{|B|^{1+\frac{\beta}{Q}}}\int_B \left|b(y)-M_B(b)(y)\right|dy \leqslant C|B|^{\frac{\beta}{Q}}.
\end{aligned}$$

Let $|B|\to 0$ with $x\in B$. Lebesgue's differentiation theorem assures that

$$0\leqslant b^-(x)=\lim_{|B|\to 0}\frac{1}{|B|}\int_B b^-(y)dy = 0.$$

Thus $b\in\dot{\Lambda}_\beta(\mathbb{G})$ and $b\geqslant 0$. The proof of Lemma 4.1 is completed. □

Similar to the proof of Lemma 4.1, the following lemma can be proved.

LEMMA 4.2. *Let $0<\beta<1$, $0\leqslant\alpha<Q$, $b\in L^1_{\mathrm{loc}}(\mathbb{G})$ and Φ be a Young function. Then the following statements are equivalent:*

1. *$b\in\dot{\Lambda}_\beta(\mathbb{G})$ and $b\geqslant 0$.*
2. *For all Φ we have*

$$\sup_B |B|^{-\frac{\beta}{Q}}\Phi^{-1}\left(|B|^{-1}\right)\left\|b(\cdot)-|B|^{-\frac{\alpha}{Q}}M_{\alpha,B}(b)(\cdot)\right\|_{L^{\Phi}(B)}\leqslant C. \tag{4.10}$$

3. *There exists Φ such that (4.10) is valid.*

The following theorem is valid.

THEOREM 4.3. *Let $0<\beta<1$, $0\leqslant\alpha<Q$, $0<\alpha+\beta<Q$ and b be a locally integrable function. Suppose that Φ,Ψ be Young functions, $\Phi\in\mathscr{Y}\cap\nabla_2$ and $\Psi^{-1}(t)\approx\Phi^{-1}(t)t^{-\frac{\alpha+\beta}{Q}}$. Then the following statements are equivalent:*

1. *$b\in\dot{\Lambda}_\beta(\mathbb{G})$ and $b\geqslant 0$.*
2. *$[b,M_\alpha]$ is bounded from $L^\Phi(\mathbb{G})$ to $L^\Psi(\mathbb{G})$.*
3. *There exists a constant $C>0$ such that*

$$\sup_B |B|^{-\frac{\beta}{Q}}\Psi^{-1}\big(|B|^{-1}\big)\left\|b(\cdot)-|B|^{-\frac{\alpha}{Q}}M_{\alpha,B}(b)(\cdot)\right\|_{L^\Psi(B)}\leqslant C. \tag{4.11}$$

4. *There exists a constant $C>0$ such that*

$$\sup_B |B|^{-1-\frac{\beta}{Q}}\left\|b(\cdot)-|B|^{-\frac{\alpha}{Q}}M_{\alpha,B}(b)(\cdot)\right\|_{L^1(B)}\leqslant C. \tag{4.12}$$

Proof. Part "$(1)\Leftrightarrow(3)$" and part "$(1)\Leftrightarrow(4)$" follows from Lemma 4.2.

$(1)\Rightarrow(2)$: It follows from (4.3) and Theorem 3.8 that $[b,M_\alpha]$ is bounded from $L^\Phi(\mathbb{G})$ to $L^\Psi(\mathbb{G})$ since $b\in\dot{\Lambda}_\beta(\mathbb{G})$ and $b\geqslant 0$.

$(2)\Rightarrow(3)$: For any fixed ball $B\subset\mathbb{G}$ and all $x\in B$, we have (see [12, pp. 13]).

$$M_\alpha(\chi_B)(x)=|B|^{\alpha/Q} \qquad \text{and} \qquad M_\alpha(b\chi_B)(x)=M_{\alpha,B}(b)(x).$$

Since $[b,M_\alpha]$ is bounded from $L^\Phi(\mathbb{G})$ to $L^\Psi(\mathbb{G})$, then

$$\begin{aligned}
&|B|^{-\frac{\beta}{Q}}\Psi^{-1}\big(|B|^{-1}\big)\|b(\cdot)-|B|^{-\frac{\alpha}{Q}}M_{\alpha,B}(b)(\cdot)\|_{L^\Psi(B)}\\
&=|B|^{-\frac{\alpha+\beta}{Q}}\Psi^{-1}\big(|B|^{-1}\big)\|b(\cdot)M_\alpha(\chi_B)(\cdot)-M_\alpha(b\chi_B)(\cdot)\|_{L^\Psi(B)}\\
&=|B|^{-\frac{\alpha+\beta}{Q}}\Psi^{-1}\big(|B|^{-1}\big)\|[b,M_\alpha](\chi_B)\|_{L^\Psi(B)}\\
&\leqslant C|B|^{-\frac{\alpha+\beta}{Q}}\Psi^{-1}\big(|B|^{-1}\big)\|\chi_B\|_{L^\Phi}\leqslant C
\end{aligned} \tag{4.13}$$

which implies (3) since the ball $B\subset\mathbb{G}$ is arbitrary.

$(3)\Rightarrow(4)$. We deduce (4.12) from (4.11). Assume (4.11) holds, then for any fixed ball B, it follows from Lemma 2.4 and (4.11) that

$$\begin{aligned}
&|B|^{-1-\frac{\beta}{Q}}\left\|b(\cdot)-|B|^{-\frac{\alpha}{Q}}M_{\alpha,B}(b)(\cdot)\right\|_{L^1(B)}\\
&\leqslant 2|B|^{-\frac{\beta}{Q}}\Psi^{-1}\big(|B|^{-1}\big)\left\|b(\cdot)-|B|^{-\frac{\alpha}{Q}}M_{\alpha,B}(b)(\cdot)\right\|_{L^\Psi(B)}\leqslant C,
\end{aligned}$$

where the constant C is independent of B. So we obtain (4.12).

The proof of Theorem 4.3 is completed. □

If we take $\alpha=0$ in Theorem 4.3, we have the following result.

COROLLARY 4.4. [11] *Let $0<\beta<1$ and b be a locally integrable function. Suppose that Φ,Ψ be Young functions, $\Phi\in\mathscr{Y}\cap\nabla_2$ and $\Psi^{-1}(t)\approx\Phi^{-1}(t)t^{-\frac{\beta}{Q}}$. Then the following statements are equivalent:*

1. *$b \in \dot{\Lambda}_\beta(\mathbb{G})$ and $b \geqslant 0$.*
2. *$[b,M]$ is bounded from $L^\Phi(\mathbb{G})$ to $L^\Psi(\mathbb{G})$.*
3. *There exists a constant $C > 0$ such that*

$$\sup_B |B|^{-\frac{\beta}{Q}} \Psi^{-1}\big(|B|^{-1}\big) \left\| b(\cdot) - M_B(b)(\cdot) \right\|_{L^\Psi(B)} \leqslant C.$$

4. *There exists a constant $C > 0$ such that*

$$\sup_B |B|^{-1-\frac{\beta}{Q}} \left\| b(\cdot) - M_B(b)(\cdot) \right\|_{L^1(B)} \leqslant C.$$

If we take $\Phi(t) = t^p$ and $\Psi(t) = t^q$ with $1 \leqslant p < \infty$ and $1 \leqslant q \leqslant \infty$ in Theorem 4.3, we have the following result.

COROLLARY 4.5. *Let $0 < \beta < 1$, $0 \leqslant \alpha < Q$, $0 < \alpha + \beta < Q$, $b \in L^1_{\mathrm{loc}}(\mathbb{G})$, b be a locally integrable function, $1 < p < q \leqslant \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha+\beta}{Q}$. Then the following statements are equivalent:*

1. *$b \in \dot{\Lambda}_\beta(\mathbb{G})$ and $b \geqslant 0$.*
2. *$[b,M_\alpha]$ is bounded from $L^p(\mathbb{G})$ to $L^q(\mathbb{G})$.*
3. *There exists a constant $C > 0$ such that*

$$\sup_B \frac{1}{|B|^{\frac{\beta}{Q}}} \left(\frac{1}{|B|} \int_B \left| b(x) - |B|^{-\frac{\alpha}{Q}} M_{\alpha,B}(b)(x) \right|^q dx \right)^{1/q} \leqslant C.$$

4. *There exists a constant $C > 0$ such that*

$$\sup_B \frac{1}{|B|^{1+\frac{\beta}{Q}}} \int_B \left| b(x) - |B|^{-\frac{\alpha}{Q}} M_{\alpha,B}(b)(x) \right| dx \leqslant C.$$

REMARK 4.6. Note that Theorem in the case $\mathbb{G} = \mathbb{R}^n$, 4.3 was proved in [14].

The following theorem is valid.

THEOREM 4.7. *Let $0 < \beta < 1$, $0 \leqslant \alpha < Q$, $0 < \alpha + \beta < Q$ and b be a locally integrable function. Suppose that Φ, Ψ be Young functions, $\Phi \in \mathscr{Y} \cap \nabla_2$ and $\Psi^{-1}(t) \approx \Phi^{-1}(t) t^{-\frac{\alpha+\beta}{Q}}$. Then the following statements are equivalent:*

1. *$b \in \dot{\Lambda}_\beta(\mathbb{G})$ and $b \geqslant 0$.*
2. *$[b,M_\alpha]$ is bounded from $L^\Phi(\mathbb{G})$ to $L^\Psi(\mathbb{G})$.*
3. *There exists a constant $C > 0$ such that*

$$\sup_B |B|^{-\frac{\beta}{Q}} \Psi^{-1}\big(|B|^{-1}\big) \left\| b(\cdot) - M_B(b)(\cdot) \right\|_{L^\Psi(B)} \leqslant C. \tag{4.14}$$

4. *There exists a constant $C > 0$ such that*

$$\sup_B |B|^{-1-\frac{\beta}{Q}} \left\| b(\cdot) - M_B(b)(\cdot) \right\|_{L^1(B)} \leqslant C. \tag{4.15}$$

Proof. Part "(1) $\Leftrightarrow$ (2)" follows from Theorem 4.3 and part "(1) $\Leftrightarrow$ (4)" follows from Lemma 4.1.

(2) $\Rightarrow$ (3): We divide the proof into two cases according to the range of α.

Case 1. Assume $\alpha = 0$. For any fixed ball B and $x \in B$, we have

$$b(x) - M_B(b)(x) = b(x)M\big(\chi_B\big)(x) - M\big(b\chi_B\big)(x) = [b,M]\big(\chi_B\big)(x).$$

Since in this case we assume $\Psi^{-1}\big(r^{-Q}\big) \approx r^{\beta}\,\Phi^{-1}\big(r^{-Q}\big)$ and $[b,M]$ is bounded from $L^{\Phi}(\mathbb{G})$ to $L^{\Psi}(\mathbb{G})$, then by Lemma 2.5, we have

$$\begin{aligned}
&|B|^{-\frac{\beta}{Q}}\,\Psi^{-1}\big(|B|^{-1}\big)\big\|b(\cdot) - M_B(b)(\cdot)\big\|_{L^{\Psi}(B)} \\
&= |B|^{-\frac{\beta}{Q}}\,\Psi^{-1}\big(|B|^{-1}\big)\big\|[b,M]\big(\chi_B\big)(\cdot)\big\|_{L^{\Psi}(B)} \\
&\lesssim |B|^{-\frac{\beta}{Q}}\,\Psi^{-1}\big(|B|^{-1}\big)\big\|\chi_B\big\|_{L^{\Phi}(\mathbb{G})} \approx 1,
\end{aligned}$$

which implies (4.14).

Case 2. Assume $0 < \alpha < Q$. For any fixed ball B,

$$\begin{aligned}
&|B|^{-\frac{\beta}{Q}}\,\Psi^{-1}\big(|B|^{-1}\big)\,\|b(\cdot) - M_B(b)(\cdot)\|_{L^{\Psi}(B)} \\
&\leqslant |B|^{-\frac{\beta}{Q}}\,\Psi^{-1}\big(|B|^{-1}\big)\,\Big\|b(\cdot) - |B|^{-\frac{\alpha}{Q}}M_{\alpha,B}(b)(\cdot)\Big\|_{L^{\Psi}(B)} \\
&\quad + |B|^{-\frac{\beta}{Q}}\,\Psi^{-1}\big(|B|^{-1}\big)\,\Big\|M_B(b)(\cdot) - |B|^{-\frac{\alpha}{Q}}M_{\alpha,B}(b)(\cdot)\Big\|_{L^{\Psi}(B)} \\
&:= I_1 + I_2.
\end{aligned} \tag{4.16}$$

For I_1. From the definition of $M_{\alpha,B}$, it is not difficult to check that $M_{\alpha,B}\big(\chi_B\big)(x) = |B|^{\frac{\alpha}{Q}}$ for all $x \in B$.

Note that, for any $x \in B$, $M_{\alpha}\big(b\chi_B\big)(x) = M_{\alpha,B}(b)(x)$ (see, for example, [12, pp. 13]) and then $M_{\alpha}\big(\chi_B\big)(x) = M_{\alpha,B}\big(\chi_B\big)(x) = |B|^{\frac{\alpha}{Q}}$.

Then, for any $x \in B$,

$$\begin{aligned}
&b(x) - |B|^{-\frac{\alpha}{Q}}M_{\alpha,B}(b)(x) = |B|^{-\frac{\alpha}{Q}}\,\Big(b(x)|B|^{\frac{\alpha}{Q}} - M_{\alpha,B}(b)(x)\Big) \\
&= |B|^{-\frac{\alpha}{Q}}\,\Big(b(x)M_{\alpha,B}\big(\chi_B\big)(x) - M_{\alpha}\big(b\chi_B\big)(x)\Big) = |B|^{-\frac{\alpha}{Q}}\,[b,M_{\alpha}]\big(\chi_B\big)(x).
\end{aligned}$$

Since $[b,M_{\alpha}]$ is bounded from $L^{\Phi}(\mathbb{G})$ to $L^{\Psi}(\mathbb{G})$, then

$$\begin{aligned}
I_1 &= |B|^{-\frac{\beta}{Q}}\,\Psi^{-1}\big(|B|^{-1}\big)\,\Big\|b(\cdot) - |B|^{-\frac{\alpha}{Q}}M_{\alpha,B}(b)(\cdot)\Big\|_{L^{\Psi}(B)} \\
&= |B|^{-\frac{\beta}{Q}}\,\Psi^{-1}\big(|B|^{-1}\big)\,|B|^{-\frac{\alpha}{Q}}\,\big\|[b,M_{\alpha}]\big(\chi_B\big)\big\|_{L^{\Psi}(B)} \\
&\lesssim |B|^{-\frac{\alpha+\beta}{Q}}\,\Psi^{-1}\big(|B|^{-1}\big)\,\|\chi_B\|_{L^{\Phi}(B)} \approx 1,
\end{aligned} \tag{4.17}$$

where in the last step we have applied Lemma 2.5 and the hypothesis $\Psi^{-1}(r^{-Q}) \approx r^{\alpha+\beta}\Phi^{-1}(r^{-Q})$.

Next, we estimate I_2.

Note that $b \in \dot{\Lambda}_\beta(\mathbb{G})$ implies $|b| \in \dot{\Lambda}_\beta(\mathbb{G})$.

From (4.8) we obtain, for any $x \in B$,

$$\left||B|^{-\frac{\alpha}{Q}} M_{\alpha,B}(b)(x) - M_B(b)(x)\right| \leqslant |B|^{-\frac{\alpha}{Q}}\left|[|b|, M_\alpha](\chi_B)(x)\right| + \left|[|b|, M](\chi_B)(x)\right|.$$

Then, it follows from Lemma 2.5 that

$$\begin{aligned} I_2 =& |B|^{-\frac{\beta}{Q}}\Psi^{-1}(|B|^{-1})\left\||B|^{-\frac{\alpha}{Q}} M_{\alpha,B}(b)(\cdot) - M_B(b)(\cdot)\right\|_{L^\Psi(B)} \\ \lesssim& |B|^{-\frac{\beta}{Q}}\Psi^{-1}(|B|^{-1})|B|^{-\frac{\alpha}{Q}}\left\|[|b|, M_\alpha](\chi_B)\right\|_{L^\Psi(B)} \\ &+ |B|^{-\frac{\beta}{Q}}\Psi^{-1}(|B|^{-1})\left\|[|b|, M](\chi_B)\right\|_{L^\Psi(B)} \\ \lesssim& \|b\|_{\dot{\Lambda}_\beta}|B|^{-\frac{\alpha+\beta}{Q}}\Psi^{-1}(|B|^{-1})\ \|\chi_B\|_{L^\Phi} \\ &+ |B|^{-\frac{\beta}{Q}}\|b\|_{\dot{\Lambda}_\beta}\Psi^{-1}(|B|^{-1})\ \|\chi_B\|_{L^\Psi} \lesssim \|b\|_{\dot{\Lambda}_\beta}. \end{aligned} \tag{4.18}$$

By (4.6), (4.7) and (4.9), we get

$$|B|^{-\frac{\beta}{Q}}\Psi^{-1}(|B|^{-1})\ \left\|b(\cdot) - M_B(b)(\cdot)\right\|_{L^\Psi(B)} \lesssim \|b\|_{\dot{\Lambda}_\beta},$$

which leads us to (4.14) since B is arbitrary.

$(3) \Rightarrow (4)$: We deduce (4.15) from (4.14). Assume (4.14) holds, then for any fixed ball B, it follows from Lemma 2.4 and (4.14) that

$$\begin{aligned} &|B|^{-1-\frac{\beta}{Q}}\ \|b(\cdot) - b_B\|_{L^1(B)} \\ &\leqslant 2\,|B|^{-\frac{\beta}{Q}}\Psi^{-1}(|B|^{-1})\left\|b(\cdot) - b_B\right\|_{L^\Psi(B)} \leqslant C, \end{aligned}$$

where the constant C is independent of B. So we obtain (4.15). $\square$

If we take $\Phi(t) = t^p$ and $\Psi(t) = t^q$ with $1 \leqslant p < \infty$ and $1 \leqslant q \leqslant \infty$ in Theorem 4.7, we have the following result.

COROLLARY 4.8. *Let $0 < \beta < 1$, $0 \leqslant \alpha < Q$, $0 < \alpha + \beta < Q$, $b \in L^1_{\mathrm{loc}}(\mathbb{G})$, b be a locally integrable function, $1 < p < q \leqslant \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha+\beta}{Q}$. Then the following statements are equivalent:*

1. *$b \in \dot{\Lambda}_\beta(\mathbb{G})$ and $b \geqslant 0$.*
2. *$[b, M_\alpha]$ is bounded from $L^p(\mathbb{G})$ to $L^q(\mathbb{G})$.*
3. *There exists a constant $C > 0$ such that*

$$\sup_B \frac{1}{|B|^{\frac{\beta}{Q}}}\left(\frac{1}{|B|}\int_B \left|b(x) - M_B(b)(x)\right|^q dx\right)^{1/q} \leqslant C.$$

4. *There exists a constant $C > 0$ such that*

$$\sup_B \frac{1}{|B|^{1+\frac{\beta}{Q}}} \int_B \big|b(x) - M_B(b)(x)\big| dx \leqslant C.$$

REMARK 4.9. Note that in the case $\mathbb{G} = \mathbb{R}^n$ theorem 4.7 is a combination of results in [14] and [29].

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Vagif S. Guliyev
Institute of Applied Mathematics
Baku State University
Baku, Azerbaijan
and
Department of Mathematics
Kirsehir Ahi Evran University
Kirsehir, Turkey
and
Peoples Friendship University of Russia (RUDN University)
6 Miklukho-Maklaya St, Moscow, 117198, Russian Federation
e-mail: vagif@guliyev.com