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# FURTHER REFINEMENTS AND GENERALIZATIONS OF THE YOUNG AND ITS REVERSE INEQUALITIES

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*Abstract.* In this work, we mainly refine some celebrated Young-type inequalities. As an application, we present corresponding operators, determinants and matrix inequalities following from the established scalar inequalities.

## 1. Introduction

The AM-GM inequality can be stated as follows: Let $m$ be a positive integer and $x_k, p_k$ $(k = 1,2,\dots,m)$ be non-negative real numbers with $\sum_{k=1}^{m} p_k = 1$. Then

$$\prod_{k=1}^{m} x_k^{p_k} \leqslant \sum_{k=1}^{m} p_k x_k, \tag{1.1}$$

where equality holds if and only if $x_1 = x_2 = \cdots = x_m$. When $m = 2$, inequality (1.1) is just the classical Young's inequality

$$a^{1-v}b^v \leqslant (1-v)a + vb, \tag{1.2}$$

where $a,b \geqslant 0$ and $0 \leqslant v \leqslant 1$.

In [8, 9], Kittaneh and Manasrah presented a refinement of Young and its reverse inequalities:

$$a^{1-v}b^v + r\left(\sqrt{a} - \sqrt{b}\right)^2 \leqslant (1-v)a + vb \leqslant a^{1-v}b^v + R\left(\sqrt{a} - \sqrt{b}\right)^2, \tag{1.3}$$

where $a,b \geqslant 0$, $0 \leqslant v \leqslant 1$, $r = \min\{v, 1-v\}$ and $R = \max\{v, 1-v\}$.

The squared form of the refined Young and its reverse inequalities, obtained by Hirzallah etc. [4] and He etc. [3], respectively, state that

$$r^2(a-b)^2 \leqslant ((1-v)a + vb)^2 - \left(a^{1-v}b^v\right)^2 \leqslant R^2(a-b)^2, \tag{1.4}$$

where $a,b \geqslant 0$, $0 \leqslant v \leqslant 1$, $r = \min\{v, 1-v\}$ and $R = \max\{v, 1-v\}$.

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In 2015, Kittaneh and Manasrah [11], deduced a more generalization of inequalities (1.3) and (1.4) that

$$\left(a^{1-v}b^{v}\right)^{m}+r^{m}\left(a^{\frac{m}{2}}-b^{\frac{m}{2}}\right)^{2}\leqslant((1-v)a+vb)^{m}, \tag{1.5}$$

for any positive integer $m$, where $a,b\geqslant 0$, $0\leqslant v\leqslant 1$ and $r=\min\{v,1-v\}$.

The Young and its reverse inequalities, though very simple, are important in functional analysis, matrix theory, operator theory, electrical networks, etc. Many scholars had done much research in this topic. We refer the readers to [2, 10, 13–21] and references therein for other works.

The main aim of this paper is to study generalizations and refinements of the Young and its reverse inequalities. First, we give some improvements for the the Manasrah-Kittaneh's inequalities (1.3). Then, we also give some generalizations for the Hirzallah-He's inequalities (1.4), and get a result similar to (1.5). Moreover, inequalities for determinants, positive semidefinite matrices, and operators are also given.

## 2. Generalizations and refinements of the Young and its reverse inequalities for scalars

In this section, we mainly study generalizations and refinements of the Young and its reverse inequalities for scalars. Firstly, we refine the inequality (1.3).

THEOREM 2.1. *Let $a,b\geqslant 0$, $0\leqslant v\leqslant 1$ and $m$ be a positive integer. If $v\in\left[\frac{i}{m},\frac{i+1}{m}\right]$ $(i=0,1,\cdots,m-1)$, then*

$$\begin{aligned}&a^{1-v}b^{v}+(mv-i)\left(\left(1-\frac{i+1}{m}\right)a+\frac{i+1}{m}b-a^{1-\frac{i+1}{m}}b^{\frac{i+1}{m}}\right)\\&+((i+1)-mv)\left(\left(1-\frac{i}{m}\right)a+\frac{i}{m}b-a^{1-\frac{i}{m}}b^{\frac{i}{m}}\right)\leqslant(1-v)a+vb.\end{aligned}$$

*In particular,*

*(i) If $v\in\left[0,\frac{1}{m}\right]$, we have*

$$\begin{aligned}&a^{1-v}b^{v}+v\left(\sqrt{a}-\sqrt{b}\right)^{2}+mv\left(\left(1-\frac{2}{m}\right)a+\frac{2}{m}\sqrt{ab}-a^{1-\frac{1}{m}}b^{\frac{1}{m}}\right)\\&\leqslant(1-v)a+vb.\end{aligned} \tag{2.1}$$

*(ii) If $v\in\left[1-\frac{1}{m},1\right]$, we have*

$$\begin{aligned}&a^{1-v}b^{v}+(1-v)\left(\sqrt{a}-\sqrt{b}\right)^{2}+(m-mv)\left(\left(1-\frac{2}{m}\right)b+\frac{2}{m}\sqrt{ab}-a^{\frac{1}{m}}b^{1-\frac{1}{m}}\right)\\&\leqslant(1-v)a+vb.\end{aligned} \tag{2.2}$$

*Proof.* Utilizing (1.1), we have

$$\left(1-\frac{i+1}{m}\right)a+\frac{i+1}{m}b-a^{1-\frac{i+1}{m}}b^{\frac{i+1}{m}}\geqslant 0,\quad\left(1-\frac{i}{m}\right)a+\frac{i}{m}b-a^{1-\frac{i}{m}}b^{\frac{i}{m}}\geqslant 0.$$

Then

$$\begin{aligned}
&(1-v)a+vb-(mv-i)\left(\left(1-\frac{i+1}{m}\right)a+\frac{i+1}{m}b-a^{1-\frac{i+1}{m}}b^{\frac{i+1}{m}}\right)\\
&-((i+1)-mv)\left(\left(1-\frac{i}{m}\right)a+\frac{i}{m}b-a^{1-\frac{i}{m}}b^{\frac{i}{m}}\right)\\
=\;&(mv-i)a^{1-\frac{i+1}{m}}b^{\frac{i+1}{m}}+((i+1)-mv)a^{1-\frac{i}{m}}b^{\frac{i}{m}}\\
\geqslant\;&\left(a^{1-\frac{i+1}{m}}b^{\frac{i+1}{m}}\right)^{mv-i}\left(a^{1-\frac{i}{m}}b^{\frac{i}{m}}\right)^{(i+1)-mv}\\
=\;&a^{1-v}b^{v}.
\end{aligned}$$

This completes the proof. □

When $m=3$, we have the following Corollary.

COROLLARY 2.1. *Let $a,b\geqslant 0$ and $0\leqslant v\leqslant 1$.*
*(i) If $v\in\left[0,\frac{1}{3}\right]$, then*

$$a^{1-v}b^{v}+v\left(\sqrt{a}-\sqrt{b}\right)^{2}+3v\left(\frac{1}{3}a+\frac{2}{3}\sqrt{ab}-a^{\frac{2}{3}}b^{\frac{1}{3}}\right)\leqslant(1-v)a+vb.$$

*(ii) If $v\in\left[\frac{1}{3},\frac{2}{3}\right]$, then*

$$a^{1-v}b^{v}+(2-3v)\left(\frac{2}{3}a+\frac{1}{3}b-a^{\frac{2}{3}}b^{\frac{1}{3}}\right)+(3v-1)\left(\frac{1}{3}a+\frac{2}{3}b-a^{\frac{1}{3}}b^{\frac{2}{3}}\right)\leqslant(1-v)a+vb.$$

*(iii) If $v\in\left[\frac{2}{3},1\right]$, then*

$$a^{1-v}b^{v}+(1-v)\left(\sqrt{a}-\sqrt{b}\right)^{2}+(3-3v)\left(\frac{1}{3}b+\frac{2}{3}\sqrt{ab}-a^{\frac{1}{3}}b^{\frac{2}{3}}\right)\leqslant(1-v)a+vb.$$

REMARK 2.1. In all of inequalities appeared in Corollary 2.1, we know conclusions (i) and (iii) of Corollary 2.1 are better than the left side of (1.3). But we can calculate that (ii) of Corollary 2.1 and the left side of (1.3) are not comparable. For example, if we take $v=\frac{5}{12}$, $a=2$, $b=3$, then $v\left(\sqrt{a}-\sqrt{b}\right)^{2}\approx 0.042$ and $(2-3v)\left(\frac{2}{3}a+\frac{1}{3}b-a^{\frac{2}{3}}b^{\frac{1}{3}}\right)+(3v-1)\left(\frac{1}{3}a+\frac{2}{3}b-a^{\frac{1}{3}}b^{\frac{2}{3}}\right)\approx 0.044$; but we calculate that if $a=0.01$, $b=10000$, then $v\left(\sqrt{a}-\sqrt{b}\right)^{2}\approx 4158.34$ and

$$(2-3v)\left(\frac{2}{3}a+\frac{1}{3}b-a^{\frac{2}{3}}b^{\frac{1}{3}}\right)+(3v-1)\left(\frac{1}{3}a+\frac{2}{3}b-a^{\frac{1}{3}}b^{\frac{2}{3}}\right)\approx 4140.92.$$

We need the following Lemma 2.1, indicating the advantage of Theorem 2.1.

LEMMA 2.1. *Let $x \geqslant 0$, $m \geqslant 3$ and $m$ be a positive integer. Then*

$$(m-1)-(m+1)x^{\frac{1}{m+1}} \geqslant (m-2)-mx^{\frac{1}{m}}. \tag{2.3}$$

*Proof.* Firstly, we let $f_m(x) = mx^{\frac{1}{m}} - (m+1)x^{\frac{1}{m+1}} + 1$, where $x \geqslant 0$, $m \geqslant 3$ and $m$ be a positive integer. Then $f_m(0) = 1$, and

$$f_m'(x) = x^{\frac{1}{m}-1} - x^{\frac{1}{m+1}-1}.$$

If $x \in [0,1]$, we have $f_m'(x) \leqslant 0$, and if $x \in [1,+\infty)$, we have $f_m'(x) \geqslant 0$.

Then, we have $f_m(x) \geqslant f_m(1) = 0$. This completes the proof. □

REMARK 2.2. (1) According to (2.3) and Theorem 2.1, when $v \in \left[0, \frac{1}{m}\right] \cup \left[1-\frac{1}{m}, 1\right]$ ($m$ is a positive integer), the larger $m$, the better Theorem 2.1.

(2) When $m = 1$, Theorem 2.1 is (1.2). When $m = 2$, Theorem 2.1 is the first inequality in (1.3).

Next, we study refinements of the Young's reverse inequalities.

THEOREM 2.2. *Let $a, b \geqslant 0$, $0 \leqslant v \leqslant 1$ and $m$ be a positive integer. If $v \in \left[\frac{i}{m}, \frac{i+1}{m}\right]$ $(i = 0, 1, \cdots, m-1)$, then*

$$\begin{aligned}(1-v)a+vb \leqslant a^{1-v}b^v &+ (mv-i)\left(\left(1-\frac{i+1}{m}\right)a+\frac{i+1}{m}b+a^{\frac{i+1}{m}}b^{1-\frac{i+1}{m}}\right) \\ &+ ((i+1)-mv)\left(\left(1-\frac{i}{m}\right)a+\frac{i}{m}b+a^{\frac{i}{m}}b^{1-\frac{i}{m}}\right)-2\sqrt{ab}.\end{aligned}$$

*In particular,*

*(i) If $v \in \left[0, \frac{1}{m}\right]$, we have*

$$\begin{aligned}(1-v)a+vb \leqslant a^{1-v}b^v &+ (1-v)\left(\sqrt{a}-\sqrt{b}\right)^2 \\ &- mv\left(\left(1-\frac{2}{m}\right)b+\frac{2}{m}\sqrt{ab}-a^{\frac{1}{m}}b^{1-\frac{1}{m}}\right).\end{aligned} \tag{2.4}$$

*(ii) If $v \in \left[1-\frac{1}{m}, 1\right]$, we have*

$$\begin{aligned}(1-v)a+vb \leqslant a^{1-v}b^v &+ v\left(\sqrt{a}-\sqrt{b}\right)^2 \\ &- (m-mv)\left(\left(1-\frac{2}{m}\right)a+\frac{2}{m}\sqrt{ab}-a^{1-\frac{1}{m}}b^{\frac{1}{m}}\right).\end{aligned} \tag{2.5}$$

*Proof.* Utilizing Young's inequality and AM-GM inequality, if $v \in \left[\frac{i}{m}, \frac{i+1}{m}\right]$

$(i=0,1,\cdots,m-1)$, we have

$$\begin{aligned}
&a^{1-v}b^{v}+(mv-i)\left(\left(1-\frac{i+1}{m}\right)a+\frac{i+1}{m}b+a^{\frac{i+1}{m}}b^{1-\frac{i+1}{m}}\right)\\
&\quad+((i+1)-mv)\left(\left(1-\frac{i}{m}\right)a+\frac{i}{m}b+a^{\frac{i}{m}}b^{1-\frac{i}{m}}\right)-2\sqrt{ab}-((1-va+vb)\\
&=a^{1-v}b^{v}+(mv-i)\,a^{\frac{i+1}{m}}b^{1-\frac{i+1}{m}}+((i+1)-mv)\,a^{\frac{i}{m}}b^{1-\frac{i}{m}}-2\sqrt{ab}\\
&\geqslant a^{1-v}b^{v}+\left(a^{\frac{i+1}{m}}b^{1-\frac{i+1}{m}}\right)^{mv-i}\left(a^{\frac{i}{m}}b^{1-\frac{i}{m}}\right)^{(i+1)-mv}-2\sqrt{ab}\\
&=a^{1-v}b^{v}+a^{v}b^{1-v}-2\sqrt{ab}\geqslant 0.
\end{aligned}$$

This completes the proof. □

REMARK 2.3. (1) According to Theorem 2.2 and Lemma 2.1, when $v\in\left[0,\frac{1}{m}\right]\cup\left[1-\frac{1}{m},1\right]$ ($m$ is a positive integer), the larger $m$, the better Theorem 2.2.

(2) When $m=2$, Theorem 2.2 is the second inequality in (1.3). When $m=3$, we have the following Corollary.

COROLLARY 2.2. *Let $a,b\geqslant 0$ and $0\leqslant v\leqslant 1$.*

*(i) If $v\in\left[0,\frac{1}{3}\right]$, then*

$$(1-v)a+vb\leqslant a^{1-v}b^{v}+(1-v)\left(\sqrt{a}-\sqrt{b}\right)^{2}-3v\left(\frac{1}{3}b+\frac{2}{3}\sqrt{ab}-a^{\frac{1}{3}}b^{\frac{2}{3}}\right).$$

*(ii) If $v\in\left[\frac{1}{3},\frac{2}{3}\right]$, then*

$$\begin{aligned}
(1-v)a+vb\leqslant a^{1-v}b^{v}&+(3v-1)\left(\frac{1}{3}a+\frac{2}{3}b+a^{\frac{2}{3}}b^{\frac{1}{3}}\right)\\
&+(2-3v)\left(\frac{2}{3}a+\frac{1}{3}b+a^{\frac{1}{3}}b^{\frac{2}{3}}\right)-2\sqrt{ab}.
\end{aligned}$$

*(iii) If $v\in\left[\frac{2}{3},1\right]$, then*

$$(1-v)a+vb\leqslant a^{1-v}b^{v}+v\left(\sqrt{a}-\sqrt{b}\right)^{2}-(3-3v)\left(\frac{1}{3}a+\frac{2}{3}\sqrt{ab}-a^{\frac{2}{3}}b^{\frac{1}{3}}\right).$$

REMARK 2.4. According to the AM-GM inequality, when $v\in\left[0,\frac{1}{3}\right]\cup\left[\frac{2}{3},1\right]$, the result of Corollary 2.2 is better than the second inequality in (1.3). But when $v\in\left[\frac{1}{3},\frac{2}{3}\right]$, they are not comparable. For example, if we take $v=\frac{5}{12}$, $a=2$, $b=3$, then

$$(1-v)\left(\sqrt{a}-\sqrt{b}\right)^{2}\approx 0.0589$$

and

$$(3v-1)\left(\frac{1}{3}a+\frac{2}{3}b+a^{\frac{2}{3}}b^{\frac{1}{3}}\right)+(2-3v)\left(\frac{2}{3}a+\frac{1}{3}b+a^{\frac{1}{3}}b^{\frac{2}{3}}\right)-2\sqrt{ab}\approx 0.0556;$$

if $a = 10000$, $b = 0.01$, then we can get the following

$$(1-v)\left(\sqrt{a}-\sqrt{b}\right)^2 = 5821.6725$$

and

$$(3v-1)\left(\frac{1}{3}a+\frac{2}{3}b+a^{\frac{2}{3}}b^{\frac{1}{3}}\right) + (2-3v)\left(\frac{2}{3}a+\frac{1}{3}b+a^{\frac{1}{3}}b^{\frac{2}{3}}\right) - 2\sqrt{ab} = 5839.0875.$$

Due to Remarks 2.1 and 2.4, we can get the better results when $v \in \left[0,\frac{1}{3}\right] \cup \left[\frac{2}{3},1\right]$. Then we discuss the situations when $v \in \left[0,\frac{1}{3}\right] \cup \left[\frac{2}{3},1\right]$. Next we refine and generalize the squared form of the Young and its reverse inequalities (1.4).

THEOREM 2.3. *Let $a,b \geqslant 0$ and $v \in \left[0,\frac{1}{3}\right] \cup \left[\frac{2}{3},1\right]$.*
*(i) If $v \in \left[0,\frac{1}{3}\right]$, then*

$$\begin{aligned} & v^2(a-b)^2 + 3va\left(\frac{1}{3}a+\frac{2}{3}b-a^{\frac{1}{3}}b^{\frac{2}{3}}\right) \\ \leqslant\ & ((1-v)a+vb)^2 - \left(a^{1-v}b^v\right)^2 \\ \leqslant\ & (1-v)^2(a-b)^2 - 3vb\left(\frac{2}{3}a+\frac{1}{3}b-a^{\frac{2}{3}}b^{\frac{1}{3}}\right). \end{aligned} \tag{2.6}$$

*(ii) If $v \in \left[\frac{2}{3},1\right]$, then*

$$\begin{aligned} & (1-v)^2(a-b)^2 + (3-3v)b\left(\frac{2}{3}a+\frac{1}{3}b-a^{\frac{2}{3}}b^{\frac{1}{3}}\right) \\ \leqslant\ & ((1-v)a+vb)^2 - \left(a^{1-v}b^v\right)^2 \\ \leqslant\ & v^2(a-b)^2 - (3-3v)a\left(\frac{1}{3}a+\frac{2}{3}b-a^{\frac{1}{3}}b^{\frac{2}{3}}\right). \end{aligned} \tag{2.7}$$

*Proof.* First of all, we prove the left-hand side of (2.6). If $v \in \left[0,\frac{1}{3}\right]$, utilizing Young's inequality, we can get

$$\begin{aligned} & ((1-v)a+vb)^2 - \left(v^2(a-b)^2 + 3va\left(\frac{1}{3}a+\frac{2}{3}b-a^{\frac{1}{3}}b^{\frac{2}{3}}\right)\right) \\ =\ & (1-3v)a^2 + 3va^{\frac{4}{3}}b^{\frac{2}{3}} \\ \geqslant\ & \left(a^2\right)^{1-3v}\left(a^{\frac{4}{3}}b^{\frac{2}{3}}\right)^{3v} = \left(a^{1-v}b^v\right)^2. \end{aligned}$$

The left-hand side of (2.6) is valid.

Then we prove the right-hand side of (2.6). If $v \in \left[0,\frac{1}{3}\right]$, we can get

$$\begin{aligned} & (1-v)^2(a-b)^2 - 3vb\left(\frac{2}{3}a+\frac{1}{3}b-a^{\frac{2}{3}}b^{\frac{1}{3}}\right) + \left(a^{1-v}b^v\right)^2 - ((1-v)a+vb)^2 \\ =\ & \left(a^{1-v}b^v\right)^2 + 3va^{\frac{2}{3}}b^{\frac{4}{3}} + (1-3v)b^2 - 2ab \\ \geqslant\ & \left(a^{1-v}b^v\right)^2 + \left(a^{\frac{2}{3}}b^{\frac{4}{3}}\right)^{3v}\left(b^2\right)^{1-3v} - 2ab \geqslant 0. \end{aligned}$$

This proves (2.6). In the same way, (2.7) can be shown. This completes the proof. □

Next, we generalize Theorem 2.3 to the results similar to Theorems 2.1 and 2.2.

COROLLARY 2.3. *Let $a,b \geqslant 0$, $v \in \left[0,\frac{1}{m}\right] \cup \left[1-\frac{1}{m},1\right]$ and $m$ be a positive integer.*

*(i) If $v \in \left[0,\frac{1}{m}\right]$, then*

$$\begin{aligned} & v^2(a-b)^2+mva\left(\left(1-\frac{2}{m}\right)a+\frac{2}{m}b-a^{1-\frac{2}{m}}b^{\frac{2}{m}}\right) \\ \leqslant & \left((1-v)a+vb\right)^2-\left(a^{1-v}b^v\right)^2 \\ \leqslant & (1-v)^2(a-b)^2-mvb\left(\left(1-\frac{2}{m}\right)b+\frac{2}{m}a-a^{\frac{2}{m}}b^{1-\frac{2}{m}}\right). \end{aligned} \tag{2.8}$$

*(ii) If $v \in \left[1-\frac{1}{m},1\right]$, then*

$$\begin{aligned} & (1-v)^2(a-b)^2+(m-mv)b\left(\left(1-\frac{2}{m}\right)b+\frac{2}{m}a-a^{\frac{2}{m}}b^{1-\frac{2}{m}}\right) \\ \leqslant & \left((1-v)a+vb\right)^2-\left(a^{1-v}b^v\right)^2 \\ \leqslant & v^2(a-b)^2-(m-mv)a\left(\left(1-\frac{2}{m}\right)a+\frac{2}{m}b-a^{1-\frac{2}{m}}b^{\frac{2}{m}}\right). \end{aligned} \tag{2.9}$$

*Proof.* The proof is very similar to that of Theorem 2.3, so we omit it. □

Next, we refine and generalize Theorems 2.1 and 2.3. Before giving the results, we need the following lemma. The lemma was given by Kittaneh and Manasrah [7].

LEMMA 2.2. *Let $\phi$ be a strictly increasing convex function defined on an interval $I$. If $x$, $y$, $z$, and $w$ are points in $I$ such that*

$$z-w \leqslant x-y,$$

*where $w \leqslant z \leqslant x$ and $y \leqslant x$, then*

$$0 \leqslant \phi(z)-\phi(w) \leqslant \phi(x)-\phi(y).$$

Next, we discuss the situations when $v \in \left[0,\frac{1}{m}\right] \cup \left[1-\frac{1}{m},1\right]$ and $m$ is a positive integer, based on Theorems 2.1 and 2.3.

THEOREM 2.4. *Let $a,b \geqslant 0$, $k$ be a real number such that $k \geqslant 1$, $m$ be a positive integer and $v \in \left[0,\frac{1}{m}\right] \cup \left[1-\frac{1}{m},1\right]$.*

*(i) If $v \in \left[0,\frac{1}{m}\right]$, then*

$$\begin{aligned} & v^k\left((m-1)a+b\right)^k-(mv)^k\left(a^{1-\frac{1}{m}}b^{\frac{1}{m}}\right)^k \\ \leqslant & \left((1-v)a+vb\right)^k-\left(a^{1-v}b^v\right)^k \\ \leqslant & \left((1-v)a+(1-(m-1)v)b+mva^{\frac{1}{m}}b^{1-\frac{1}{m}}\right)^k-2^k(ab)^{\frac{k}{2}}. \end{aligned} \tag{2.10}$$

*(ii) If* $v \in \left[1-\frac{1}{m}, 1\right]$, *then*

$$
\begin{aligned}
&(1-v)^k (a+(m-1)\,b)^k - (m-mv)^k \left(a^{\frac{1}{m}} b^{1-\frac{1}{m}}\right)^k \\
&\leqslant ((1-v)\,a+vb)^k - \left(a^{1-v} b^v\right)^k \\
&\leqslant \left((1-(m-1)\,(1-v))\,a+vb+(m-mv)\,a^{1-\frac{1}{m}} b^{\frac{1}{m}}\right)^k - 2^k (ab)^{\frac{k}{2}}. \qquad (2.11)
\end{aligned}
$$

*Proof.* First of all, we prove the left-hand side of (2.10).

We take $z=v\,((m-1)\,a+b)$, $w=mva^{1-\frac{1}{m}} b^{\frac{1}{m}}$, $x=(1-v)\,a+vb$ and $y=a^{1-v}b^v$, where $m$ is a positive integer, $v\in\left[0,\frac{1}{m}\right]\cup\left[1-\frac{1}{m},1\right]$ and $a,b\geqslant 0$.

Duo to Young's inequality and AM-GM inequality, we get $w\leqslant z$ and $y\leqslant x$.

If $v\in\left[0,\frac{1}{m}\right]$ and $a,b\geqslant 0$, we have $1-mv\geqslant 0$, observe that $z\leqslant x$.

We take $\phi(x)=x^k$ ($k$ is a real number such that $k\geqslant 1$), which is a strictly increasing convex function defined on an interval $[0,+\infty)$. The left-hand side of (2.10) is valid.

Then we prove the right-hand side of (2.10).

We take $w'=a^{1-v}b^v$, $z'=(1-v)\,a+vb$, $x'=(1-v)\,a+(1-(m-1)\,v)\,b+mva^{\frac{1}{m}}b^{1-\frac{1}{m}}$ and $y'=2\sqrt{ab}$ where $v\in\left[1-\frac{1}{m},1\right]$ and $a,b\geqslant 0$. We observe that $w'\leqslant z'\leqslant x'$.

Utilizing Young's inequality and AM-GM inequality, we obtain

$$
\begin{aligned}
&x'-y' \\
&=(1-v)\,a+(1-(m-1)\,v)\,b+mva^{\frac{1}{m}}b^{1-\frac{1}{m}}-2\sqrt{ab} \\
&=(1-v)\,a+vb+(1-mv)\,b+mva^{\frac{1}{m}}b^{1-\frac{1}{m}}-2\sqrt{ab} \\
&\geqslant a^{1-v}b^v+b^{1-mv}\left(a^{\frac{1}{m}}b^{1-\frac{1}{m}}\right)^{mv}-2\sqrt{ab} \\
&=a^{1-v}b^v+a^vb^{1-v}-2\sqrt{ab}\geqslant 0.
\end{aligned}
$$

We take $\phi(x)=x^k$ ($k$ is a real number such that $k\geqslant 1$), which is a strictly increasing convex function defined on an interval $[0,+\infty)$. The right-hand side of (2.10) is valid. Therefore, inequalities (2.10) is valid. In the same way, (2.13) is also valid. This completes the proof. □

## 3. Generalizations and refinements of the Young and its reverse inequalities for operators, matrices and determinants

Let $B(\mathbb{H})$ be the $C^*$-algebra of all bounded linear operators on a complex Hilbert space $\mathbb{H}$ and $I$ be the identity operator. For two self-adjoint operators $A$ and $B$, the symbol $B\leqslant A$ means that $A-B$ is a positive operator. Let $A,B\in B(\mathbb{H})$ be positive operators and $0\leqslant v\leqslant 1$. The $v$-weighted arithmetic operator mean of $A$ and $B$, denoted by $A\nabla_v B$, is defined by

$$A\nabla_v B=(1-v)A+vB.$$

Moreover, if $A$ is an invertible positive operator, the $v$-weighted geometric operator mean of $A$ and $B$, denoted by $A\#_v B$, is defined by

$$A\#_v B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^v A^{\frac{1}{2}}.$$

For $v > 1$, the definition of $A\#_v B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^v A^{\frac{1}{2}}$ is still well defined. In the following, we use $A\#_v B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^v A^{\frac{1}{2}}$ for any $v \geqslant 0$. When $v = \frac{1}{2}$, the operators $A\nabla_{\frac{1}{2}} B$ and $A\#_{\frac{1}{2}} B$ are called the arithmetic operator mean and geometric operator mean, respectively. Usually, we write $A\nabla B$ and $A\# B$ for brevity, respectively.

Let $M_n(\mathbb{C})$ denote the space of all $n\times n$ complex matrices and $|||.|||$ denote any unitarily invariant (or symmetric) norm on $M_n(\mathbb{C})$. So, $|||UAV||| = |||A|||$ for all $A \in M_n(\mathbb{C})$ and for all unitary matrices $U$, $V \in M_n(\mathbb{C})$. For $A \in M_n(\mathbb{C})$, the Hilbert-Schmidt norm, and the trace norm of $A$ are defined respectively by

$$\|A\|_2 = \left(\sum_{i=1}^{n} s_i^2(A)\right)^{\frac{1}{2}} = \left(\mathrm{tr}|A|^2\right)^{\frac{1}{2}} = \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{\frac{1}{2}}, \quad \|A\|_1 = \mathrm{tr}(|A|) = \sum_{i=1}^{n} s_i(A),$$

where $s_1(A) \geqslant s_2(A) \geqslant \cdots \geqslant s_n(A)$ are the singular values of $A$, that is, the eigenvalues of the positive matrix $|A| = (A^*A)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity and $\mathrm{tr}(\cdot)$ is the usual trace function. It is evident that these norms are unitarily invariant, and it is known that each unitarily invariant norm is a symmetric guage function of singular values [1].

In this section, we mainly study generalizations and refinements of the Young and its reverse inequalities for operators, matrices and determinants. Before giving the main results, we need the following lemma, which is the monotonicity property for operator functions [12].

LEMMA 3.1. *Let $X \in B(\mathbb{H})$ be a self-adjoint operator and $f$ and $g$ be continuous functions such that $f(t) \geqslant g(t)$ for all $t \in Sp(X)$ (the spectrum of $X$). Then $f(X) \leqslant g(X)$.*

Based on Theorems 2.1 and 2.2, we have the following operator inequalities.

THEOREM 3.1. *Let $A, B \in B(\mathbb{H})$ be positive operators such that $A$ is invertible, $0 \leqslant v \leqslant 1$ and $m$ be a positive integer. If $v \in \left[\frac{i}{m}, \frac{i+1}{m}\right]$ $(i = 0, 1, \cdots, m-1)$, then*

$$\begin{aligned}
&A\#_v B + (mv - i)\left(\left(1 - \frac{i+1}{m}\right)A + \frac{i+1}{m}B - A\#_{\frac{i+1}{m}} B\right)\\
&\quad + ((i+1) - mv)\left(\left(1 - \frac{i}{m}\right)A + \frac{i}{m}B - A\#_{\frac{i}{m}} B\right)\\
&\leqslant A\nabla_v B \leqslant A\#_v B + (mv - i)\left(\left(1 - \frac{i+1}{m}\right)A + \frac{i+1}{m}B + A\#_{1-\frac{i+1}{m}} B\right)\\
&\quad + ((i+1) - mv)\left(\left(1 - \frac{i}{m}\right)A + \frac{i}{m}B + A\#_{1-\frac{i}{m}} B\right) - 2A\# B.
\end{aligned}$$

*In particular,*

*(i) If* $v \in \left[0, \frac{1}{m}\right]$*, then*

$$A\#_v B + v\,(A + B - 2A\#B) + mv\left(\left(1 - \frac{2}{m}\right)A + \frac{2}{m}A\#B - A\#_{\frac{1}{m}}B\right)$$
$$\leqslant A\nabla_v B \leqslant A\#_v B + (1-v)\,(A + B - 2A\#B)$$
$$- mv\left(\left(1 - \frac{2}{m}\right)B + \frac{2}{m}A\#B - A\#_{1-\frac{1}{m}}B\right). \tag{3.1}$$

*(ii) If* $v \in \left[1 - \frac{1}{m}, 1\right]$*, then*

$$A\#_v B + (1-v)\,(A + B - 2A\#B) + (m - mv)\left(\left(1 - \frac{2}{m}\right)B + \frac{2}{m}A\#B - A\#_{1-\frac{1}{m}}B\right)$$
$$\leqslant A\nabla_v B \leqslant A\#_v B + v\,(A + B - 2A\#B)$$
$$-(m - mv)\left(\left(1 - \frac{2}{m}\right)A + \frac{2}{m}A\#B - A\#_{\frac{1}{m}}B\right). \tag{3.2}$$

*Proof.* By Theorems 2.1 and 2.2, we have

$$a^v + (mv - i)\left(\left(1 - \frac{i+1}{m}\right) + \frac{i+1}{m}a - a^{\frac{i+1}{m}}\right)$$
$$+ ((i+1) - mv)\left(\left(1 - \frac{i}{m}\right) + \frac{i}{m}a - a^{\frac{i}{m}}\right)$$
$$\leqslant (1-v) + va$$
$$\leqslant a^v + (mv - i)\left(\left(1 - \frac{i+1}{m}\right) + \frac{i+1}{m}a + a^{1-\frac{i+1}{m}}\right)$$
$$+ ((i+1) - mv)\left(\left(1 - \frac{i}{m}\right) + \frac{i}{m}a + a^{1-\frac{i}{m}}\right) - 2\sqrt{a}.$$

for $a > 0$.

Since $A$ and $B$ are positive operators, then so is the operator $T = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. Therefore, by Lemma 3.1, we obtain

$$T^v + (mv - i)\left(\left(1 - \frac{i+1}{m}\right)I + \frac{i+1}{m}T - T^{\frac{i+1}{m}}\right)$$
$$+ (i + 1 - mv)\left(\left(1 - \frac{i}{m}\right)I + \frac{i}{m}T - T^{\frac{i}{m}}\right)$$
$$\leqslant (1-v)I + vT \leqslant T^v + (mv - i)\left(\left(1 - \frac{i+1}{m}\right)I + \frac{i+1}{m}T + T^{1-\frac{i+1}{m}}\right)$$
$$+ (i + 1 - mv)\left(\left(1 - \frac{i}{m}\right)I + \frac{i}{m}T + T^{1-\frac{i}{m}}\right) - 2T^{\frac{1}{2}}.$$

Multiplying both sides of the above inequality by $A^{\frac{1}{2}}$, we have

$$\begin{aligned}
&A\#_v B + (mv-i)\left(\left(1-\frac{i+1}{m}\right)A + \frac{i+1}{m}B - A\#_{\frac{i+1}{m}}B\right)\\
&+(i+1-mv)\left(\left(1-\frac{i}{m}\right)A + \frac{i}{m}B - A\#_{\frac{i}{m}}B\right)\\
\leqslant\ & A\nabla_v B \leqslant A\#_v B + (mv-i)\left(\left(1-\frac{i+1}{m}\right)A + \frac{i+1}{m}B + A\#_{1-\frac{i+1}{m}}B\right)\\
&+(i+1-mv)\left(\left(1-\frac{i}{m}\right)A + \frac{i}{m}B + A\#_{1-\frac{i}{m}}B\right) - 2A\#B.
\end{aligned}$$

This completes the proof. □

In the same way, based on Corollary 2.3, we get the following operator inequalities.

THEOREM 3.2. *Let $A,B \in B(\mathbb{H})$ be positive operators such that $A$ is invertible, $v \in \left[0, \frac{1}{m}\right] \cup \left[1-\frac{1}{m}, 1\right]$ and $m$ be a positive integer.*
*(i) If $v \in \left[0, \frac{1}{m}\right]$, then*

$$\begin{aligned}
&v^2(A + A\#_2 B - 2B) + mv\left(\left(1-\frac{2}{m}\right)A + \frac{2}{m}B - A\#_{\frac{2}{m}}B\right)\\
\leqslant\ & A\#_2(A\nabla_v B) - A\#_{2v}B\\
\leqslant\ & (1-v)^2(A + A\#_2 B - 2B) - mv\left(\left(1-\frac{2}{m}\right)A\#_2 B + \frac{2}{m}B - A\#_{2-\frac{2}{m}}B\right).
\end{aligned}\tag{3.3}$$

*(ii) If $v \in \left[1-\frac{1}{m}, 1\right]$, then*

$$\begin{aligned}
&(1-v)^2(A + A\#_2 B - 2B) + (m-mv)\left(\left(1-\frac{2}{m}\right)A\#_2 B + \frac{2}{m}B - A\#_{2-\frac{2}{m}}B\right)\\
\leqslant\ & A\#_2(A\nabla_v B) - A\#_{2v}B\\
\leqslant\ & v^2(A + A\#_2 B - 2B) - (m-mv)\left(\left(1-\frac{2}{m}\right)A + \frac{2}{m}B - A\#_{\frac{2}{m}}B\right).
\end{aligned}\tag{3.4}$$

*Proof.* The proof is very similar to that of Theorem 3.1, so we omit it here. □

In the same way, based on Theorem 2.4, we get

THEOREM 3.3. *Let $A,B \in B(\mathbb{H})$ be positive operators such that $A$ is invertible, $k \geqslant 1$ and $k$ be real number, $m$ be a positive integer and $v \in \left[0, \frac{1}{m}\right] \cup \left[1-\frac{1}{m}, 1\right]$.*
*(i) If $v \in \left[0, \frac{1}{m}\right]$, then*

$$\begin{aligned}
&v^k A\#_k((m-1)A + B) - (mv)^k A\#_{\frac{k}{m}}B \leqslant A\#_k(A\nabla_v B) - A\#_{kv}B\\
\leqslant\ & A\#_k\left((1-v)A + (1-(m-1)v)B + mvA\#_{1-\frac{1}{m}}B\right) - 2^k A\#_{\frac{k}{2}}B.
\end{aligned}\tag{3.5}$$

*(ii) If* $v \in \left[1-\frac{1}{m}, 1\right]$, *then*

$$(1-v)^k A\#_k (A+(m-1)B) - (m-mv)^k A\#_{\frac{(m-1)k}{m}} B \leqslant A\#_k (A\nabla_v B) - A\#_{kv} B$$

$$\leqslant A\#_k \left((1-(m-1)(1-v))A + vB + (m-mv)A\#_{\frac{1}{m}} B\right) - 2^k A\#_{\frac{k}{2}} B. \tag{3.6}$$

*Proof.* The proof is very similar to that of Theorems 3.1 and 3.2, so we omit it here. □

Next, we study generalizations and refinements of the Young's inequality for unitarily invariant norms and the trace norm. Before giving the main results, we need the following lemma [6], which is a Heinz-Kato type inequality for unitarily invariant norms.

LEMMA 3.2. *Let* $A, B, X \in M_n(\mathbb{C})$ *be such that* $A$ *and* $B$ *are positive semidefinite. Then*

$$|||A^{1-v} X B^v||| \leqslant |||AX|||^{1-v} |||XB|||^v.$$

*In particular,*

$$||A^{1-v} B^v||_1 \leqslant ||A||_1^{1-v} ||B||_1^v.$$

Based on Theorem 2.1, we get

THEOREM 3.4. *Let* $A, B, X \in M_n(\mathbb{C})$ *be such that* $A$ *and* $B$ *are positive semidefinite,* $0 \leqslant v \leqslant 1$ *and* $m$ *be a positive integer. If* $v \in \left[\frac{i}{m}, \frac{i+1}{m}\right]$ $(i = 0, 1, \cdots, m-1)$, *then*

$$|||A^{1-v} X B^v||| + (mv-i)\left(\left(1-\frac{i+1}{m}\right)|||AX||| + \frac{i+1}{m}|||XB||| - |||AX|||^{1-\frac{i+1}{m}} |||XB|||^{\frac{i+1}{m}}\right)$$

$$+ ((i+1)-mv)\left(\left(1-\frac{i}{m}\right)|||AX||| + \frac{i}{m}|||XB||| - |||AX|||^{1-\frac{i}{m}} |||XB|||^{\frac{i}{m}}\right)$$

$$\leqslant (1-v)|||AX||| + v|||XB|||. \tag{3.7}$$

*Furthermore,*

$$||A^{1-v} B^v||_1 + (mv-i)\left(\left(1-\frac{i+1}{m}\right)||A||_1 + \frac{i+1}{m}||B||_1 - ||A||_1^{1-\frac{i+1}{m}} ||B||_1^{\frac{i+1}{m}}\right)$$

$$+ ((i+1)-mv)\left(\left(1-\frac{i}{m}\right)||A||_1 + \frac{i}{m}||B||_1 - ||A||_1^{1-\frac{i}{m}} ||B||_1^{\frac{i}{m}}\right)$$

$$\leqslant ||(1-v)A + vB||_1. \tag{3.8}$$

*In particular,*

*(i) If* $v \in \left[0, \frac{1}{m}\right]$, *we have*

$$|||A^{1-v} X B^v||| + mv\left(\left(1-\frac{2}{m}\right)|||AX||| + \frac{2}{m}|||A^{\frac{1}{2}} X B^{\frac{1}{2}}||| - |||AX|||^{1-\frac{1}{m}} |||XB|||^{\frac{1}{m}}\right)$$

$$+ v\left(|||AX|||^{\frac{1}{2}} - |||XB|||^{\frac{1}{2}}\right)^2$$

$$\leqslant (1-v)|||AX||| + v|||XB|||. \tag{3.9}$$

*and*

$$||A^{1-v}B^{v}||_1 + mv\left(\left(1-\frac{2}{m}\right)||A||_1 + \frac{2}{m}||A^{\frac{1}{2}}B^{\frac{1}{2}}||_1 - ||A||_1^{1-\frac{1}{m}}||B||_1^{\frac{1}{m}}\right)$$
$$+v\left(||A||_1^{\frac{1}{2}} - ||B||_1^{\frac{1}{2}}\right)^2 \leqslant ||(1-v)A+vB||_1. \tag{3.10}$$

*(ii) If* $v \in \left[1-\frac{1}{m}, 1\right]$, *we have*

$$|||A^{1-v}XB^{v}||| + (m-mv)\left(\left(1-\frac{2}{m}\right)|||XB||| + \frac{2}{m}|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| - |||AX|||^{\frac{1}{m}}|||XB|||^{1-\frac{1}{m}}\right)$$
$$+(1-v)\left(|||AX|||^{\frac{1}{2}} - |||XB|||^{\frac{1}{2}}\right)^2 \leqslant (1-v)|||AX||| + v|||XB|||. \tag{3.11}$$

*and*

$$||A^{1-v}B^{v}||_1 + (m-mv)\left(\left(1-\frac{2}{m}\right)||B||_1 + \frac{2}{m}||A^{\frac{1}{2}}B^{\frac{1}{2}}||_1 - ||A||_1^{\frac{1}{m}}||B||_1^{1-\frac{1}{m}}\right)$$
$$+(1-v)\left(||A||_1^{\frac{1}{2}} - ||B||_1^{\frac{1}{2}}\right)^2 \leqslant ||(1-v)A+vB||_1. \tag{3.12}$$

*Proof.* First of all, we prove the inequalities (3.7). Utilizing Theorem 2.1 and Lemma 3.2, we get

$$\begin{aligned}
&|||A^{1-v}XB^{v}||| + (mv-i)\left(\left(1-\frac{i+1}{m}\right)|||AX||| + \frac{i+1}{m}|||XB||| - |||AX|||^{1-\frac{i+1}{m}}|||XB|||^{\frac{i+1}{m}}\right)\\
&+((i+1)-mv)\left(\left(1-\frac{i}{m}\right)|||AX||| + \frac{i}{m}|||XB||| - |||AX|||^{1-\frac{i}{m}}|||XB|||^{\frac{i}{m}}\right)\\
\leqslant\ & |||AX|||^{1-v}|||XB|||^{v}\\
&+(mv-i)\left(\left(1-\frac{i+1}{m}\right)|||AX||| + \frac{i+1}{m}|||XB||| - |||AX|||^{1-\frac{i+1}{m}}|||XB|||^{\frac{i+1}{m}}\right)\\
&+((i+1)-mv)\left(\left(1-\frac{i}{m}\right)|||AX||| + \frac{i}{m}|||XB||| - |||AX|||^{1-\frac{i}{m}}|||XB|||^{\frac{i}{m}}\right)\\
\leqslant\ & (1-v)|||AX||| + v|||XB|||.
\end{aligned}$$

Therefore, (3.7) is valid. In the same way, (3.8) can be shown.

Due to Lemma 3.2, we have

$$|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| \leqslant |||AX|||^{\frac{1}{2}}|||XB|||^{\frac{1}{2}}, \quad ||A^{\frac{1}{2}}B^{\frac{1}{2}}||_1 \leqslant ||A||_1^{\frac{1}{2}}||B||_1^{\frac{1}{2}}.$$

Through the proof of (3.7), (3.9)-(3.12) are also valid. □

Next, we give some inequalities involving the trace norm and the Hilbert-Schmidt norm. To do this, we need the following lemma [14].

LEMMA 3.3. *Let $A,B,X \in M_n(\mathbb{C})$. Then*

$$\sum_{j=1}^{n} s_j(AB) \leqslant \sum_{j=1}^{n} s_j(A)\, s_j(B).$$

Based on Theorem 2.1, we get

THEOREM 3.5. *Let $A,B \in M_n(\mathbb{C})$ be positive semidefinite, $0 \leqslant v \leqslant 1$ and $m$ be a positive integer. If $v \in \left[\frac{i}{m}, \frac{i+1}{m}\right]$ $(i = 0,1,\cdots,m-1)$, then*

$$\begin{aligned}
&||A^{1-v}B^{v}||_1 + (mv-i)\left(\left(1-\frac{i+1}{m}\right)||A||_1 + \frac{i+1}{m}||B||_1 - ||A^{1-\frac{i+1}{m}}||_2||B^{\frac{i+1}{m}}||_2\right)\\
&+((i+1)-mv)\left(\left(1-\frac{i}{m}\right)||A||_1 + \frac{i}{m}||B||_1 - ||A^{1-\frac{i}{m}}||_2||B^{\frac{i}{m}}||_2\right)\\
&\leqslant ||(1-v)A+vB||_1. \qquad (3.13)
\end{aligned}$$

*In particular,*

*(i) If $v \in \left[0, \frac{1}{m}\right]$, then*

$$\begin{aligned}
&||A^{1-v}B^{v}||_1 + mv\left(\left(1-\frac{2}{m}\right)||A||_1 + \frac{2}{m}||A^{\frac{1}{2}}||_2||B^{\frac{1}{2}}||_2 - ||A^{1-\frac{1}{m}}||_2||B^{\frac{1}{m}}||_2\right)\\
&+v\left(||A^{\frac{1}{2}}||_2 - ||B^{\frac{1}{2}}||_2\right)^2 \leqslant ||(1-v)A+vB||_1. \qquad (3.14)
\end{aligned}$$

*(ii) If $v \in \left[1-\frac{1}{m}, 1\right]$, then*

$$\begin{aligned}
&||A^{1-v}B^{v}||_1 + (m-mv)\left(\left(1-\frac{2}{m}\right)||B||_1 + \frac{2}{m}||A^{\frac{1}{2}}||_2||B^{\frac{1}{2}}||_2 - ||A^{\frac{1}{m}}||_2||B^{1-\frac{1}{m}}||_2\right)\\
&+(1-v)\left(||A^{\frac{1}{2}}||_2 - ||B^{\frac{1}{2}}||_2\right)^2 \leqslant ||(1-v)A+vB||_1. \qquad (3.15)
\end{aligned}$$

*Proof.* By the Theorem 2.1, we have

$$\begin{aligned}
s_j\left(A^{1-v}\right)s_j\left(B^{v}\right) \leqslant\ & (mv-i)\, s_j\left(A^{1-\frac{i+1}{m}}\right)s_j\left(B^{\frac{i+1}{m}}\right)\\
&+((i+1)-mv)\, s_j\left(A^{1-\frac{i}{m}}\right)s_j\left(B^{\frac{i}{m}}\right).
\end{aligned}$$

where $m$ is a positive integer, $v \in \left[\frac{i}{m}, \frac{i+1}{m}\right]$ $(i = 0,1,\cdots,m-1)$ and $j = 1,2,\cdots,n$.

To prove the inequality(3.13), we just need to prove the following inequality

$$||A^{1-v}B^{v}||_1 \leqslant (mv-i)\,||A^{1-\frac{i+1}{m}}||_2||B^{\frac{i+1}{m}}||_2 + ((i+1)-mv)\,||A^{1-\frac{i}{m}}||_2||B^{\frac{i}{m}}||_2.$$

Thus, by Theorem 2.1, Lemma 3.3 and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
||A^{1-v}B^{v}||_1 &= \sum_{j=1}^{n} s_j\left(A^{1-v}B^{v}\right) \leqslant \sum_{j=1}^{n} s_j\left(A^{1-v}\right) s_j\left(B^{v}\right)\\
&\leqslant (mv-i)\sum_{j=1}^{n} s_j\left(A^{1-\frac{i+1}{m}}\right) s_j\left(B^{\frac{i+1}{m}}\right) + ((i+1)-mv)\sum_{j=1}^{n} s_j\left(A^{1-\frac{i}{m}}\right) s_j\left(B^{\frac{i}{m}}\right)\\
&\leqslant (mv-i)\left(\sum_{j=1}^{n} s_j^2\left(A^{1-\frac{i+1}{m}}\right)\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n} s_j^2\left(B^{\frac{i+1}{m}}\right)\right)^{\frac{1}{2}}\\
&\quad + ((i+1)-mv)\left(\sum_{j=1}^{n} s_j^2\left(A^{1-\frac{i}{m}}\right)\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n} s_j^2\left(B^{\frac{i}{m}}\right)\right)^{\frac{1}{2}}\\
&= (mv-i)\,||A^{1-\frac{i+1}{m}}||_2||B^{\frac{i+1}{m}}||_2 + ((i+1)-mv)\,||A^{1-\frac{i}{m}}||_2||B^{\frac{i}{m}}||_2.
\end{aligned}$$

Therefore, (3.13) is valid.

Note that

$$||A^{\frac{1}{2}}||_2 = ||A||_1^{\frac{1}{2}}, \quad ||A^{\frac{1}{2}}||_2^2 = ||A||_1.$$

To prove the (3.14), we just need to prove the following inequality

$$||A^{1-v}B^{v}||_1 \leqslant mv||A^{1-\frac{1}{m}}||_2||B^{\frac{1}{m}}||_2 + (1-mv)\,||A||_1.$$

In fact, through the proof of (3.13), we can obtain

$$\begin{aligned}
||A^{1-v}B^{v}||_1 &= \sum_{j=1}^{n} s_j\left(A^{1-v}B^{v}\right) \leqslant \sum_{j=1}^{n} s_j\left(A^{1-v}\right) s_j\left(B^{v}\right)\\
&\leqslant mv\sum_{j=1}^{n} s_j\left(A^{1-\frac{1}{m}}\right) s_j\left(B^{\frac{1}{m}}\right) + (1-mv)\sum_{j=1}^{n} s_j(A)s_j\left(I\right)\\
&\leqslant mv\left(\sum_{j=1}^{n} s_j^2\left(A^{1-\frac{1}{m}}\right)\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n} s_j^2\left(B^{\frac{1}{m}}\right)\right)^{\frac{1}{2}} + (1-mv)\,||A||_1\\
&= mv||A^{1-\frac{1}{m}}||_2||B^{\frac{1}{m}}||_2 + (1-mv)\,||A||_1.
\end{aligned}$$

In the same way, (3.15) can be shown. □

Next, we give some inequalities involving the trace norm by Corollary 2.3.

THEOREM 3.6. *Let $A,B,X \in M_n(\mathbb{C})$ be such that $A$ and $B$ are positive semidefinite, $v \in \left[0,\frac{1}{m}\right] \cup \left[1-\frac{1}{m},1\right]$ and $m$ be a positive integer.*

*(i) If* $v \in \left[0, \frac{1}{m}\right]$*, then*

$$v^2||AX-XB||_2^2 + mv\left(\left(1-\frac{2}{m}\right)||AX||_2^2 + \frac{2}{m}||A^{\frac{1}{2}}XB^{\frac{1}{2}}||_2^2 - ||A^{1-\frac{1}{m}}XB^{\frac{1}{m}}||_2^2\right)$$
$$\leqslant ||(1-v)AX+vXB||_2^2 - ||A^{1-v}XB^v||_2^2 \leqslant (1-v)^2||AX-XB||_2^2$$
$$-mv\left(\left(1-\frac{2}{m}\right)||XB||_2^2 + \frac{2}{m}||A^{\frac{1}{2}}XB^{\frac{1}{2}}||_2^2 - ||A^{\frac{1}{m}}XB^{1-\frac{1}{m}}||_2^2\right). \quad (3.16)$$

*(ii) If* $v \in \left[1-\frac{1}{m}, 1\right]$*, then*

$$(1-v)^2||AX-XB||_2^2 + (m-mv)\left(\left(1-\frac{2}{m}\right)||XB||_2^2 + \frac{2}{m}||A^{\frac{1}{2}}XB^{\frac{1}{2}}||_2^2 - ||A^{\frac{1}{m}}XB^{1-\frac{1}{m}}||_2^2\right)$$
$$\leqslant ||(1-v)AX+vXB||_2^2 - ||A^{1-v}XB^v||_2^2$$
$$\leqslant v^2||AX-XB||_2^2 - (m-mv)\left(\left(1-\frac{2}{m}\right)||AX||_2^2 + \frac{2}{m}||A^{\frac{1}{2}}XB^{\frac{1}{2}}||_2^2 - ||A^{1-\frac{1}{m}}XB^{\frac{1}{m}}||_2^2\right). \quad (3.17)$$

*Proof.* Since $A$ and $B$ are positive semidefinite, it follows by spectral theorem that there exist unitary matrices $U, V \in M_n(\mathbb{C})$, such that

$$A = U\Lambda_1 U^*, \qquad B = V\Lambda_2 V^*$$

where $\Lambda_1 = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$, $\Lambda_2 = diag(\mu_1, \mu_2, \cdots, \mu_n)$ for $\lambda_p$, $\mu_q$ are eigenvalues of $A$ and $B$ respectively, $p, q = 1, 2, \cdots, n$.

For our computations, let $Y = U^*XV = [y_{pq}]$. Then we have

$$(1-v)AX + vXB = U[(1-v)\Lambda_1 Y + vY\Lambda_2]V^* = U[((1-v)\lambda_p + v\mu_q)y_{pq}]V^*,$$
$$A^{1-v}XB^v = U[(\lambda_p^{1-v}\mu_q^v)y_{pq}]V^*, \quad AX - XB = U[(\lambda_p - \mu_q)y_{pq}]V^*,$$
$$AX = U[(\lambda_p)y_{pq}]V^*, \quad XB = U[(\mu_q)y_{pq}]V^*.$$

By (2.8), if $v \in \left[0, \frac{1}{m}\right]$, we have

$$||(1-v)AX+vXB||_2^2 - ||A^{1-v}XB^v||_2^2$$
$$= \sum_{p,q=1}^{n}((1-v)\lambda_p + v\mu_q)^2|y_{pq}|^2 - \sum_{p,q=1}^{n}\left(\lambda_p^{1-v}\mu_q^v\right)^2|y_{pq}|^2$$
$$\geqslant mv\left[\left(1-\frac{2}{m}\right)\sum_{p,q=1}^{n}\lambda_p^2|y_{pq}|^2 + \frac{2}{m}\sum_{p,q=1}^{n}\left(\lambda_p^{\frac{1}{2}}\mu_q^{\frac{1}{2}}\right)^2|y_{pq}|^2 - \sum_{p,q=1}^{n}\left(\lambda_p^{1-\frac{1}{m}}\mu_q^{\frac{1}{m}}\right)^2|y_{pq}|^2\right]$$
$$+v^2\sum_{p,q=1}^{n}(\lambda_p - \mu_q)^2|y_{pq}|^2$$
$$= mv\left(\left(1-\frac{2}{m}\right)||AX||_2^2 + \frac{2}{m}||A^{\frac{1}{2}}XB^{\frac{1}{2}}||_2^2 - ||A^{1-\frac{1}{m}}XB^{\frac{1}{m}}||_2^2\right) + v^2||AX-XB||_2^2.$$

Therefore, the left-hand side of (3.16) is valid. In the same way, the right-hand side of (3.16) can be shown. The proof of (3.17) is similar to the proof of (3.16). □

Finally, based on Theorem 2.4, we obtain the generalizations and refinements of the Young and its reverse inequalities for determinants. To do this, we need the following lemma (see [5]).

LEMMA 3.4. *Let $A,B \in M_n(\mathbb{C})$ be positive semidefinite matrices, then*

$$\det(A+B)^{\frac{1}{n}} \geqslant \det A^{\frac{1}{n}} + \det B^{\frac{1}{n}}.$$

Based on Theorem 2.4, we have the following determinant inequalities.

THEOREM 3.7. *Let $A,B \in M_n(\mathbb{C})$ be positive semidefinite matrices, $k \geqslant 1$ and $k$ be real number, $m$ be a positive integer such that $v \in \left[0, \frac{1}{m}\right] \cup \left[1-\frac{1}{m}, 1\right]$.*

*(i) If $v \in \left[0, \frac{1}{m}\right]$, then*

$$\det\left((1-v)A+vB\right)^k \geqslant \det\left(A^{1-v}B^v\right)^k + v^{nk}\left((m-1)\det A^{\frac{1}{n}} + \det B^{\frac{1}{n}}\right)^{nk} - (mv)^{nk}\det\left(A^{1-\frac{1}{m}}B^{\frac{1}{m}}\right)^k. \tag{3.18}$$

*and*

$$\det\left(A^{1-v}B^v\right)^k \geqslant \left[\left(\det A^{\frac{1}{n}}\right)\nabla_v\left(\det B^{\frac{1}{n}}\right)\right]^{nk} + 2^{nk}\det\left(A^{\frac{1}{2}}B^{\frac{1}{2}}\right)^k - \det\left((1-v)A+(1-(m-1)v)B+mvA^{\frac{1}{m}}B^{1-\frac{1}{m}}\right)^k. \tag{3.19}$$

*(ii) If $v \in \left[1-\frac{1}{m}, 1\right]$, then*

$$\det\left((1-v)A+vB\right)^k \geqslant \det\left(A^{1-v}B^v\right)^k + (1-v)^{nk}\left(\det A^{\frac{1}{n}} + (m-1)\det B^{\frac{1}{n}}\right)^{nk} - (m-mv)^{nk}\det\left(A^{\frac{1}{m}}B^{1-\frac{1}{m}}\right)^k. \tag{3.20}$$

*and*

$$\det\left(A^{1-v}B^v\right)^k \geqslant \left[\left(\det A^{\frac{1}{n}}\right)\nabla_v\left(\det B^{\frac{1}{n}}\right)\right]^{nk} + 2^{nk}\det\left(A^{\frac{1}{2}}B^{\frac{1}{2}}\right)^k - \det\left((1-(m-1)(1-v))A+vB+(m-mv)A^{1-\frac{1}{m}}B^{\frac{1}{m}}\right)^k. \tag{3.21}$$

*Proof.* By (2.10) and Lemma 3.4, if $v \in \left[0, \frac{1}{m}\right]$, we have

$$\det\left((1-v)A+vB\right)^{k} = \left[\det\left((1-v)A+vB\right)^{\frac{1}{n}}\right]^{nk}$$

$$\geqslant \left[\left(\det(1-v)A\right)^{\frac{1}{n}} + \left(\det(vB)\right)^{\frac{1}{n}}\right]^{nk}$$

$$= \left[(1-v)\det A^{\frac{1}{n}} + v\det B^{\frac{1}{n}}\right]^{nk}$$

$$\geqslant \left[\left(\det A^{\frac{1}{n}}\right)^{1-v}\left(\det B^{\frac{1}{n}}\right)^{v}\right]^{nk} + v^{nk}\left((m-1)\det A^{\frac{1}{n}} + \det B^{\frac{1}{n}}\right)^{nk}$$

$$-(mv)^{nk}\left[\left(\det A^{\frac{1}{n}}\right)^{1-\frac{1}{m}}\left(\det B^{\frac{1}{n}}\right)^{\frac{1}{m}}\right]^{nk}$$

$$= \det\left(A^{1-v}B^{v}\right)^{k} + v^{nk}\left((m-1)\det A^{\frac{1}{n}} + \det B^{\frac{1}{n}}\right)^{nk} - (mv)^{nk}\det\left(A^{1-\frac{1}{m}}B^{\frac{1}{m}}\right)^{k}.$$

In the same way, (3.19) can be shown.

Using the same method, we can get (3.20) and (3.21) similarly, so we omit it. □

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