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ON NEWTON–LIKE INEQUALITIES FOR COMPLEX NUMBERS

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Abstract. I. Newton famously stated that the sequence of normalized elementary symmetric polynomials has the following property: the square of each polynomial is greater than or equal to the product of its two adjacent polynomials when evaluated at any real numbers. We introduce several novel generalizations of this property for evaluations on multisets of self-conjugate complex numbers in the angular sector $|\arg z| \leqslant \pi/4$.

1. Introduction

Let Z be a multiset of complex numbers such that $|Z| = n$. The *elementary symmetric polynomials in n variables*, $e_0, \ldots, e_n$, can be defined by the following equation in the ring of polynomials with complex coefficients and indeterminate x:

$$\prod_{z\in Z}(x+z) = \sum_{k=0}^{n} e_{n-k}(Z)x^k. \tag{1}$$

Newton's inequalities [6, 11, 15] state that

$$e_k^2(Z) \geqslant \left(1+\frac{1}{k}\right)\left(1+\frac{1}{n-k}\right) e_{k-1}(Z)e_{k+1}(Z) \tag{2}$$

for $1 \leqslant k \leqslant n-1$, where Z is an arbitrary multiset of real numbers with cardinality n. A proof of (2) for positive numbers can be found in [6, p. 53] (using induction). Pages later, in [6, §4.3, p. 104], a proof of (2) is given for arbitrary real numbers (based on Rolle's Theorem). In [13], Monov proved a generalization, known as λ-*Newton inequalities*, for multisets of self-conjugate complex numbers. Subsequently, Xu [20] introduced another generalization also applicable to multisets of self-conjugate complex numbers. Ellard and Šmigoc [4] further explored modifications of Newton's inequalities on complex numbers for specific subsequences of the elementary symmetric polynomials. Meanwhile, Ren [17] established a different type of generalization involving sums of normalized elementary symmetric polynomials, this time applicable

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to arbitrary real numbers. While more generalizations exist for arbitrary real numbers (e.g., [5, 16, 18]), this paper focuses on those valid for complex numbers. It is worth mentioning that some of these generalizations have found applications in the fields of linear algebra, particularly in the study of M-matrices (see [7, 14]), and in differential equations, specifically partial differential equations associated with curvature problems (see [10, 17]).

In this paper, we introduce an additional generalization of Newton's inequalities for multisets of self-conjugate complex numbers (see Theorem 5), utilizing primarily the concept of ultra log-concave sequences and a theorem by Liggett [9, Thm. 3], which states that the convolution of two ultra log-concave sequences is itself ultra log-concave.

The organization of this paper is as follows. In Sect. 2, we present some definitions, and some lemmas, mainly about *weighted log-concavity*, which are needed in Sect. 3, where we focus on *ultra log-concavity* as a particular case of weighted log-concavity. In Sect. 4 we use the aforementioned Liggett's theorem in order to obtain new Newton-like inequalities on self-conjugate complex numbers in Theorem 5. This theorem is our main result. We compare our results with the so-called λ-Newton inequalities in Sections 5 and 6. In Sect. 7, as a consequence of Theorem 5, we show Newton-like inequalities for some subsequences of the elementary symmetric polynomials with indices in a fixed residual class. These inequalities generalize, in a sense, the results by Ellard and Šmigoc [4]. We further compare our findings in Section 7 with theirs. Finally, in Sect. 8, again as a consequence of Theorem 5, we show Newton-like inequalities for linear combinations of the normalized elementary symmetric polynomials. This result offers an answer to a question raised by Ren [17].

2. Definitions and preliminary results

We denote the set of nonnegative integers by $\mathbb{N}$. The fields of real numbers and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. The ring of formal power series with coefficients in $\mathbb{R}$ is denoted by $\mathbb{R}[[x]]$. The semiring of formal power series with real nonnegative coefficients and indeterminate x is denoted by $\mathbb{R}_{\geqslant 0}[[x]]$. If $a, b \in \mathbb{N}$ such that $a \leqslant b$, $[a,b]$ stands for the integer interval $\{k \in \mathbb{N} : a \leqslant k \leqslant b\}$.

DEFINITION 1. Let $\Delta = (d_k)_{k\in\mathbb{N}}$ be a sequence of real numbers.

(i) The *support* of Δ, denoted $\operatorname{supp}\Delta$, is defined as

$$\operatorname{supp}\Delta = \{k \in \mathbb{N} : d_k \neq 0\}.$$

(ii) The *support* of a formal power series $p(x)$, denoted $\operatorname{supp} p(x)$, is the support of the sequence of its coefficients.

(iii) The sequence Δ *has no internal zeros* if $\operatorname{supp}\Delta$ is a nonempty interval of integer numbers.

(iv) The set of sequences of nonnegative real numbers without internal zeros is denoted by W.

(v) A formal power series $p(x) \in \mathbb{R}[[x]]$ *has the property* P if and only if the sequence of its coefficients has the same property P.

(vi) If $p(x) \in \mathbb{R}_{\geqslant 0}[[x]]$, we define the symbols $\delta p(x)$ and $\partial p(x)$ by

$$\delta p(x) = \begin{cases} \min \operatorname{supp} p(x), & \text{if } p(x) \neq 0; \\ -\infty, & \text{if } p(x) = 0; \end{cases}$$

$$\partial p(x) = \begin{cases} \max \operatorname{supp} p(x), & \text{if } p(x) \neq 0; \\ -\infty, & \text{if } p(x) = 0. \end{cases}$$

(vii) If $p(x) \in \mathbb{R}[[x]]$, then $p'(x)$ stands for the formal derivative of $p(x)$.

Additionally, we need the concept of *partial ratio-dominance*, which was introduced in [1] (see also [2]).

DEFINITION 2. Let $p(x), q(x) \in \mathbb{R}_{\geqslant 0}[[x]]$, where $p(x) = \sum_k a_k x^k$ and $q(x) = \sum_k b_k x^k$. The formal power series $q(x)$ is *partially ratio-dominant over* $p(x)$, denoted $p(x) \preceq q(x)$, if $\delta p(x) \leqslant \partial q(x) + 1$ and

$$a_{k+1} b_k \leqslant a_k b_{k+1}, \quad \text{for all } k \geqslant 0. \tag{3}$$

In $[1,2]$, it is proven that the pair $(\mathrm{W}, \preceq)$ is a poset up to multiplicative constants. This means that, on W, the relationship $\preceq$ is reflexive, transitive, and antisymmetric in the following sense: $p(x) \preceq q(x)$ and $q(x) \preceq p(x)$ imply $p(x) = cq(x)$ for some positive real number c. We take advantage of these facts in the following.

DEFINITION 3. Let $\Delta = (d_k)_{k \in \mathbb{N}}$ be a sequence of nonnegative real numbers. For any $p(x) \in \mathbb{R}_{\geqslant 0}[[x]]$ such that $p(x) = \sum_k a_k x^k$, we define the operator Shf_Δ as

$$\mathrm{Shf}_\Delta[p(x)] = \sum_{k=1}^{\infty} d_k a_{k-1} x^k,$$

and we call the polynomial $p(x)$ *weighted log-concave with weights* Δ if $p(x) \preceq \mathrm{Shf}_\Delta[p(x)]$ and $p(x) \in \mathrm{W}$. In such a case, we write $p(x) \in \mathrm{WLC}(\Delta)$.

A similar weighted shifted operator was considered by Gurvits [5, p. 64], where instead of a_{k-1}, the term a_{k+1} is used. On the other hand, it is clear that if $p(x)$ has no internal zeros and neither does the sequence Δ, then $\mathrm{Shf}_\Delta[p(x)] \in \mathrm{W}$.

LEMMA 1. *Let* $p(x) \in \mathrm{WLC}(\Delta)$ *and* $d(x)$ *be the generating function of the sequence* Δ. *If* $p(x) = \sum_k a_k x^k$ *and* $d(x) = \sum_k d_k x^k$ *such that* $\operatorname{supp} p'(x) \subseteq \operatorname{supp} d'(x)$, *then*

$$d_{l+1} a_k a_l \geqslant d_k a_{k-1} a_{l+1} \tag{4}$$

for all $1 \leqslant k \leqslant l$.

Proof. Let $m \in \mathbb{N}$ with $m \geqslant 1$. Straightforwardly from Definition 3, it follows that the operator Shf_Δ is a monotone map on $(\mathrm{W}, \preceq)$, that is, preserves the partial ratio-dominant order $\preceq$. Thus,

$$p(x) \preceq \mathrm{Shf}_\Delta[p(x)] \preceq \mathrm{Shf}_\Delta[\mathrm{Shf}_\Delta[p(x)]] \preceq \cdots \preceq \mathrm{Shf}_\Delta^m[p(x)].$$

The transitivity property implies that $p(x) \preceq \mathrm{Shf}_\Delta^m[p(x)]$, which in turn implies that

$$a_{k-m}a_{k+1}d_{k-m+1}\prod_{j=0}^{m-2} d_{k-j} \leqslant a_{k+1-m}a_k d_{k+1}\prod_{j=0}^{m-2} d_{k-j} \tag{5}$$

for $k \geqslant m$, since $\mathrm{Shf}_\Delta^m[p(x)] = \sum_{k=m}^{\infty} d_k d_{k-1}\dots d_{k-m+1}a_{k-m}x^k$ and the partial ratio-domination definition. Assume first that $d_{k+1} \neq 0$. It follows that

$$d_{k-m+1}a_{k-m}a_{k+1} \leqslant d_{k+1}a_{k+1-m}a_k \tag{6}$$

because this inequality holds trivially if $d_{k-m+1} = 0$. Otherwise, if $d_{k-m+1} \neq 0$, then the common factor $\prod_{j=0}^{m-2} d_{k-j}$ in (5) is not zero, since $\Delta \in \mathrm{W}$, and we can cancel it out from both sides of (5). Now, if $d_{k+1} = 0$, then (6) also holds. Since $d_{k+1} = 0$ implies $a_{k+1} = 0$, due to $\operatorname{supp} p'(x) \subseteq \operatorname{supp} d'(x)$. Finally, note that (6) is equivalent to (4). $\square$

The following lemma shows that the shifted log-concave property is inherited by subsequences whose indices belong to a fixed residue class.

LEMMA 2. *Let $p(x) = \sum_k a_k x^k$ with $p(x) \in \mathrm{WLC}(\Delta)$, and $d(x)$ the generating function of Δ such that $\operatorname{supp} p'(x) \subseteq \operatorname{supp} d'(x)$. If $m, r \in \mathbb{N}$ with $m \geqslant 1$, then*

(i) $\sum_k a_{km+r}x^k \in \mathrm{WLC}(\Delta_0)$, *where* $\Delta_0 = (\prod_{j=1}^m d_{(k-1)m+r+j})_k$.

(ii) For $1 \leqslant k \leqslant l$,

$$a_{km+r}\,a_{lm+r}\prod_{j=1}^{m} d_{lm+r+j} \geqslant a_{(k-1)m+r}a_{(l+1)m+r}\prod_{j=1}^{m} d_{(k-1)m+r+j}.$$

Proof.

(i) Assume that $\delta p(x) + m \leqslant k \leqslant \partial p(x) - m$. Then $[k-m, k] \subseteq \operatorname{supp} p(x)$. Hence, we can use (4) m times in order to get

$$\begin{aligned} d_{k+m}\dots d_{k+2}d_{k+1}\frac{a_k}{a_{k-m}} &\geqslant d_{k+m}\dots d_{k+2}d_{k+1-m}\frac{a_{k+1}}{a_{k+1-m}} \\ &\geqslant d_{k+m}\dots d_{k+3}d_{k+1-m}d_{k+2-m}\frac{a_{k+2}}{a_{k+2-m}} \geqslant \dots \geqslant d_{k+1-m}d_{k+2-m}\dots d_k\frac{a_{k+m}}{a_k}, \end{aligned}$$

which implies

$$a_k^2\prod_{j=1}^{m} d_{k+j} \geqslant a_{k-m}a_{k+m}\prod_{j=1}^{m} d_{k+j-m} \tag{7}$$

for $\delta p(x)+m \leqslant k \leqslant \partial p(x)-m$. Also, (7) trivially holds for $k<\delta p(x)+m$ and $k>\partial p(x)-m$. Therefore, (7) holds for any k. Now, by substituting k with $mk+r$ in (7), we get

$$a_{km+r}^2 \prod_{j=1}^{m} d_{km+r+j} \geqslant a_{(k-1)m+r}a_{(k+1)m+r} \prod_{j=1}^{m} d_{(k-1)m+r+j},$$

which leads to the result.

(ii) This follows from (i) and Lemma 1. □

3. From weighted to ultra log-concavity

In this section, we apply our results about weighted log-concavity to ultra-log concavity. We recall from [9] that a sequence of nonnegative real numbers $(a_k)_{k\in\mathbb{N}}$ is *ultra log-concave of order* α if it has no internal zeros, $a_k=0$ for $k>\alpha$, and

$$\frac{\alpha-k}{k+1}a_k^2 \geqslant \frac{\alpha-k+1}{k}a_{k-1}a_{k+1}, \quad k\geqslant 1. \tag{8}$$

In this case, if $p(x)$ is the generating function of the sequence $(a_k)_{k\in\mathbb{N}}$, we write $p(x)\in \mathrm{ULC}(\alpha)$. Note that (8) can be written as

$$a_k^2 \geqslant \left(1+\frac{1}{k}\right)\left(1+\frac{1}{\alpha-k}\right)a_{k-1}a_{k+1}, \quad 1\leqslant k\leqslant \partial p(x)-1. \tag{9}$$

Therefore, it makes sense to call a sequence of nonnegative real numbers $(a_k)_{k\in\mathbb{N}}$ *ultra log-concave of order infinity* if it has no internal zeros and satisfies:

$$a_k^2 \geqslant \left(1+\frac{1}{k}\right)a_{k-1}a_{k+1}, \quad k\geqslant 1. \tag{10}$$

In this case, if $p(x)$ is the related generating function, we write $p(x)\in \mathrm{ULC}(\infty)$.

THEOREM 1. *Let* $p(x)=\sum_k a_k x^k$. *If* $p(x)\in \mathrm{ULC}(\alpha)$, *then*

$$a_k a_l \geqslant \frac{(l+1)(\alpha-k+1)}{k(\alpha-l)}a_{k-1}a_{l+1} \tag{11}$$

for all $1\leqslant k\leqslant l\leqslant \partial p(x)-1$.

Proof. Let $\Delta=(d_k)_{k\in\mathbb{N}}$ where

$$d_k=\begin{cases}\frac{\alpha-k+1}{k}, & \text{if } 1\leqslant k\leqslant \partial p(x);\\ 0, & \text{otherwise.}\end{cases}$$

Then, the equation $\mathrm{ULC}(\alpha)=\mathrm{WLC}(\Delta)$ holds due to (8). Thus, we can use Lemma 1, and the result follows. □

Lemma 2 applied to subsequences of ultra log-concave sequences gives the following.

THEOREM 2. *If* $p(x) \in \mathrm{ULC}(\alpha)$ *such that* $p(x) = \sum_k a_k x^k$, *then*

$$a_{km+r}a_{lm+r} \geqslant a_{(k-1)m+r}a_{(l+1)m+r}\prod_{j=0}^{m-1}\left(1+\frac{m(l-k+1)}{km+r-j}\right)\left(1+\frac{m(l-k+1)}{\alpha-lm-r-j}\right) \tag{12}$$

for $1 \leqslant k \leqslant l \leqslant (\partial p(x)-r)/m-1$. *In particular, the sequence* $(a_{km+r})_{k\in\mathbb{N}}$ *is ultra log-concave of order* $(\alpha-2r)/m$.

Proof. From Lemma 2 (i), and the proof of Theorem 1, we get that $\sum_k a_{km+r}x^k \in \mathrm{WLC}(\Delta_0)$, where $\Delta_0 = (\prod_{j=1}^m d_{(k-1)m+r+j})_{k\in\mathbb{N}}$ and $d_k = (\alpha-k+1)/k$ for $1 \leqslant k \leqslant \partial p(x)$. Assume $1 \leqslant k \leqslant l \leqslant (\partial p(x)-r)/m-1$. From Lemma 2 (ii), we get,

$$a_{km+r}a_{lm+r}\prod_{j=1}^{m}\frac{\alpha-lm-r-j+1}{lm+r+j} \geqslant a_{(k-1)m+r}a_{(l+1)m+r}\prod_{j=1}^{m}\frac{\alpha-(k-1)m-r-j+1}{(k-1)m+r+j}. \tag{13}$$

We can see that:

$$\frac{lm+r+j}{(k-1)m+r+j} = 1+\frac{m(l-k+1)}{(k-1)m+r+j}$$

and

$$\frac{\alpha-(k-1)m-r-j+1}{\alpha-lm-r-j+1} = 1+\frac{m(l-k+1)}{\alpha-lm-r-j+1}.$$

Additionally, since $l \leqslant (\partial p(x)-r)/m-1$, then $\alpha-lm-r-j+1>0$ for any $1 \leqslant j \leqslant m$. Therefore (13) can be written as

$$a_{km+r}a_{lm+r} \geqslant a_{(k-1)m+r}a_{(l+1)m+r}\prod_{j=1}^{m}\left(1+\frac{m(l-k+1)}{(k-1)m+r+j}\right)\left(1+\frac{m(l-k+1)}{\alpha-lm-r-j+1}\right)$$

which is equivalent to (12). Note that in (12) the factors for $j=r$ and $l=k$ are

$$\left(1+\frac{1}{k}\right)\left(1+\frac{m}{\alpha-km-2r}\right) = \left(1+\frac{1}{k}\right)\left(1+\frac{1}{\frac{\alpha-2r}{m}-k}\right).$$

Consequently, the sequence $(a_{km+r})_k$ is ultra log-concave of order $(\alpha-2r)/m$ due to (9). $\square$

As Liggett noted, the concept of ultra log-concavity is equivalent to that of *log-concavity*.

DEFINITION 4. A sequence $(a_k)_{k\in\mathbb{N}}$ of nonnegative real numbers is *log-concave* if it has no internal zeros and

$$a_k^2 \geqslant a_{k-1}a_{k+1}, \quad k \geqslant 0.$$

In this case, if $p(x)$ is the generating function of $(a_k)_{k\in\mathbb{N}}$, we write $p(x)\in \mathrm{LC}$.

Thus, $(a_k)_{k\in\mathbb{N}}$ is ultra log-concave of order α if and only if $\left(\binom{\alpha}{k}^{-1}a_k\right)_{0\leqslant k\leqslant n}$ is log-concave, where n is the degree of the generating function of $(a_k)_{k\in\mathbb{N}}$.

The following general property of log-concave sequence leads to Newton-like inequalities for some linear combinations, as defined in [17]. We prove this in Corollary 3.

THEOREM 3. *Let $(a_k)_{1\leqslant k\leqslant n}$ be a log-concave sequence and $q(x)\in \mathrm{LC}$. If $q(x)=\sum_k b_k x^k$, then*

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \geqslant \left(\sum_{k=1}^{n} a_k b_{k-1}\right)\left(\sum_{k=1}^{n} a_k b_{k+1}\right). \tag{14}$$

Proof. Let $p(x)=\sum_{k=1}^{n} a_{n+1-k}x^k$. From the well-known fact that the product of two log-concave polynomials is log-concave (see [8, p. 394], [12], [19, Proposition 2]), it follows that $p(x)q(x)\in \mathrm{LC}$, which implies

$$\left(\sum_{i=0}^{k+1} a_{n+1-i}b_{k+1-i}\right)^2 \geqslant \left(\sum_{i=0}^{k} a_{n+1-i}b_{k-i}\right)\left(\sum_{i=0}^{k+2} a_{n+1-i}b_{k+2-i}\right), \quad k\geqslant 0. \tag{15}$$

Setting $k=n$ and noting that $a_{n+1}=0$, $a_0=0$, and $a_{-1}=0$ in (15), we obtain

$$\left(\sum_{i=1}^{n} a_{n+1-i}b_{n+1-i}\right)^2 \geqslant \left(\sum_{i=1}^{n} a_{n+1-i}b_{n-i}\right)\left(\sum_{i=1}^{n} a_{n+1-i}b_{n+2-i}\right),$$

which is equivalent to (14). $\square$

4. Ultra log-concavity and Newton-like inequalities

Building on the properties of ultra log-concave sequences established in the previous section, the following theorem from [9, Thm. 2] allows us to derive Newton-like inequalities for complex numbers.

THEOREM 4. (Liggett) *If $p(x)\in \mathrm{ULC}(\alpha)$ and $q(x)\in \mathrm{ULC}(\beta)$ with $\alpha,\beta\in\mathbb{N}\cup\{\infty\}$, then $p(x)q(x)\in \mathrm{ULC}(\alpha+\beta)$.*

This theorem serves as a foundation for deriving Newton-like inequalities when applied to elementary symmetric polynomials. We show this in the following theorem and its corollaries.

DEFINITION 5. A multiset Z of complex numbers is *self-conjugate* if for every element z in Z, the complex conjugate of z (denoted by z^*) is also in Z.

THEOREM 5. *Let Z be a finite multiset of self-conjugate complex numbers in the angular sector $\Omega = \{z \in \mathbb{C} : |\arg z| \leqslant \pi/4\}$. If $n = |Z|$ and*

$$\beta = \sum_{z \in Z} \left\lceil \sum_{j=0}^{\infty} \tan^{2j}(\arg z) \right\rceil,$$

then:

(i) The sequence $(e_{n-k}(Z))_{0 \leqslant k \leqslant n}$ is ultra log-concave of order β.

(ii) The sequence $(e_k(Z))_{0 \leqslant k \leqslant n}$ is ultra log-concave of order β. That is,

$$e_k^2(Z) \geqslant \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{\beta - k}\right) e_{k-1}(Z) e_{k+1}(Z) \tag{16}$$

for $1 \leqslant k \leqslant n-1$.

Proof. We can write $Z = [r_1, \ldots, r_m, z_1, \ldots, z_\ell, z_1^*, \ldots, z_\ell^*]$, where $2\ell + m = n$, $r_1, \ldots, r_m \in \mathbb{R}_{\geqslant 0}$ and $z_1, \ldots, z_\ell \in \mathbb{C} - \mathbb{R}$. Let $p(x) = \prod_{z \in Z}(x+z)$.

(i) We have the following decomposition:

$$p(x) = (x + r_1) \cdots (x + r_m)(x^2 + 2\operatorname{Re} z_1 x + |z_1|^2) \ldots (x^2 + 2\operatorname{Re} z_\ell x + |z_\ell|^2), \tag{17}$$

where each linear factor $x + z_j$ is ultra log-concave of order 1, $1 \leqslant j \leqslant m$; while each quadratic factor $x^2 + 2\operatorname{Re} z_\ell x + |z_\ell|^2$ is ultra log-concave of order α_j, defined as:

$$\alpha_j = \begin{cases} \frac{2}{1 - \operatorname{Im}^2 z_j / \operatorname{Re}^2 z_j}, & \text{if } |\arg z_j| < \pi/4; \\ \infty, & \text{if } |\arg z_j| = \pi/4, \end{cases}$$

since the equation $4\operatorname{Re}^2 z_j = 2(1 + 1/(\alpha_j - 1))|z_j|^2$ holds and (9), $1 \leqslant j \leqslant \ell$. Additionally, each quadratic factor on the right side of (17) has order $\lceil \alpha_j \rceil$ (the ceiling of α_j) for $1 \leqslant j \leqslant \ell$. This is because if a sequence is ultra log-concave of order α_j, it is also ultra log-concave of any higher order, including $\lceil \alpha_j \rceil$. Liggett's theorem implies that $p(x) \in \operatorname{ULC}(m + \lceil \alpha_1 \rceil + \cdots + \lceil \alpha_\ell \rceil)$. Now, the result follows from the fact that the elementary symmetric polynomials are defined by Equation (1).

(ii) The generating function of the sequence $(s_k(Z))_{0 \leqslant k \leqslant n}$ is the reflected polynomial $p^R(x)$ of the polynomial $p(x)$. Since $p^R(x) = x^n p(1/x)$, we get

$$p^R(x) = r_1 \ldots r_m |z_1|^2 \ldots |z_\ell|^2 q(x),$$

where $q(x) = \prod_{z \in Z}(x + z^{-1})$. This implies that $p^R(x)$ is equal to $q(x)$ in the projective space W. Applying (i) to $q(x)$ we get that $q(x) \in \operatorname{ULC}(\beta)$, because the angular sector Ω is closed under the inversion $z \mapsto z^{-1}$ and

$$\frac{\operatorname{Im}^2(z^{-1})}{\operatorname{Re}^2(z^{-1})} = \frac{\operatorname{Im}^2 z}{\operatorname{Re}^2 z}.$$

Therefore $p^R(x) \in \operatorname{ULC}(\beta)$. $\square$

5. On λ –Newton inequalities

Several Newton-like inequalities valid on the complex semiplane $\operatorname{Re} z \geqslant 0$ have been found. In contrast, our proposed Newton-like inequalities (16) holds just in the angular sector $|\arg z| \leqslant \pi/4$. Thus, to facilitate comparisons, throughout the rest of this paper, we shall assume that all the symmetric elementary polynomials and their normalizations are evaluated over multisets Z in said angular sector. For instance, Newton-type inequalities appeared in [13, Thm. 2.1]. For a multiset Z satisfying the conditions of Theorem 5, these inequalities take the form:

$$e_k^2(Z) \geqslant \lambda \left(1+\frac{1}{k}\right)\left(1+\frac{1}{n-k}\right) e_{k-1}(Z) e_{k+1}(Z). \tag{18}$$

Here, $1 \leqslant k \leqslant n-1$ and $\lambda = \cos^2(\max\{\arg z : z \in Z\})$. As we prove in the following proposition, the inequalities (16) are stronger than (18) for some values of k but weaker for others. This implies that neither set of inequalities universally implies the other.

PROPOSITION 1. *Let $\beta \geqslant n > k \geqslant 1$, $0 \leqslant \lambda < 1$, and $M_k = 1 + \frac{1}{\beta-k} - \lambda\left(1+\frac{1}{n-k}\right)$ for $1 \leqslant k \leqslant n-1$. There exists an integer m_0 such that $1 \leqslant m_0 \leqslant n-1$, which holds $M_k > 0$ for $1 \leqslant k < m_0$, while $M_k \leqslant 0$ for $m_0 \leqslant k \leqslant n-1$.*

Proof. We have

$$M_k = \frac{q_{\beta,\lambda,n}(k)}{(n-k)(\beta-k)},$$

where

$$q_{\beta,\lambda,n}(k) = (1-\lambda)\,k^2 + (\lambda-1)(\beta+n+1)\,k - (n+1)\,\beta\lambda + n\,(\beta+1).$$

The sign of M_k depends on $q_{\beta,\lambda,n}(k)$. The discriminant of this quadratic polynomial in k is equal to $(1-\lambda)d_{\lambda,n}(\beta)$, where

$$d_{\lambda,n}(\beta) = (1-\lambda)\,\beta^2 + 2\left((n+1)\lambda - n + 1\right)\beta + (n-1)^2 - (n+1)^2\lambda. \tag{19}$$

This is another quadratic convex polynomial in β, which has its own larger root equal to

$$\frac{(n+1)\lambda - 2\sqrt{\lambda} - n + 1}{\lambda - 1}$$

which is lower or equal to n. Thus, for any number greater or equal than n, such as β, this quadratic polynomial in β is nonnegative. This implies that the roots of $q_{\beta,\lambda,n}(k)$ are real. They are given by

$$r^{\pm}(\beta,\lambda,n) = \frac{(1-\lambda)(\beta+n+1) \pm \sqrt{1-\lambda}\sqrt{d_{\lambda,n}(\beta)}}{2(1-\lambda)}.$$

The middle point between $r^{\pm}(\beta,\lambda,n)$ is $(\beta+n+1)/2$, which is strictly greater than n. Additionally, the inequality $n-k < r^{-}(\beta,\lambda,n)$ is equivalent to $0 < (1-\lambda)(\beta - n +$

$2k+1)^2 - d_{\lambda,n}(\beta)$. But the right side of this inequality equals to $-4((\lambda-1)k^2 + (\lambda-1)(\beta-n+1) + (\beta-n)\lambda)$. Thus, $n-k < r^-(\beta,\lambda,n)$ is equivalent to $\lambda < g_{\beta,n}(k)$, where

$$g_{\beta,n}(k) = \frac{1+(\beta-n)/(k+1)}{1+(\beta-n)/k}.$$

Since $g_{\beta,n}(n-1) = \frac{1-1/n}{1-1/\beta} \leqslant 1$ and $g_{\beta,n}(k)$ is a strictly increasing function in k, there exists an integer k_0 such that $1 \leqslant k_0 \leqslant n-1$ and

$$0 < g_{\beta,n}(1) < \cdots < g_{\beta,n}(k_0) \leqslant \lambda < g_{\beta,n}(k_0+1) < \cdots < g_{\beta,n}(n-1) \leqslant 1,$$

which implies $k < r^-(\beta,\lambda,n)$ for $1 \leqslant k \leqslant n-k_0-1$ and $k \geqslant r^-(\beta,\lambda,n)$ for $n-k_0 \leqslant k \leqslant n-1$. This leads to $g_{\beta,n}(k) > 0$ for $1 \leqslant k \leqslant n-k_0-1$ and $g_{\beta,n}(k) \leqslant 0$ for $n-k_0 \leqslant k \leqslant n-1$, as $q_{\beta,\lambda,n}(k)$ is a convex quadratic polynomial in k. The result follows. □

6. On generalized λ–Newton inequalities

COROLLARY 1. *Under the conditions of Theorem 5,*

$$e_k(Z)e_l(Z) \geqslant \frac{(l+1)(\beta-k+1)}{k(\beta-l)} e_{k-1}(Z)e_{l+1}(Z) \tag{20}$$

for all $1 \leqslant k \leqslant l \leqslant n-1$.

Proof. Use Theorem 5 and Theorem 1. □

Additional Newton-like inequalities appear in [20, Thm. 2.13]. These are

$$e_k(Z)e_l(Z) \geqslant \lambda \frac{(l+1)(n-k+1)}{k(n-l)} e_{k-1}(Z)e_{l+1}(Z) \tag{21}$$

for all $1 \leqslant k \leqslant l \leqslant n-1$. These are quite similar to those in (20). The following proposition shows that, in general, neither of these Newton-like inequalities implies the other directly.

PROPOSITION 2. *Let* $\beta \geqslant n > l \geqslant k \geqslant 1$, $z \in \mathbb{C} - \{0\}$, *and*

$$M_{l,k} = \frac{\beta-k+1}{\beta-l} - \lambda \frac{n-k+1}{n-l}.$$

where $\lambda = \cos^2(\arg z)$.

(i) If $\arg z \geqslant \pi/6$ *and* $l \leqslant n/4$, *then* $M_{l,k} > 0$.

(ii) If $\beta = m + \lceil 2\lambda/(2\lambda-1) \rceil$, $2\lambda/(2\lambda-1) \notin \mathbb{N}$, $l = n-1$, *and* $n = m+2$, *then* $M_{l,k} < 0$.

Proof.

(i) The condition $l \leqslant n/4$ implies that $3(l-k)+l \leqslant n-3$, which in turn implies

$$\tan^2 \arccos \sqrt{\frac{n-l}{n-k+1}} = \frac{l-k+1}{n-l} \leqslant \frac{1}{3}.$$

Thus,

$$\arccos \sqrt{\frac{n-l}{n-k+1}} \leqslant \arctan \sqrt{\frac{1}{3}} \leqslant \arg z.$$

So, by transitivity,

$$\sqrt{\frac{n-l}{n-k+1}} \geqslant \cos \arg z = \sqrt{\lambda},$$

which implies $1 \geqslant \lambda(n-k+1)/(n-l)$. Therefore,

$$\frac{\beta-k+1}{\beta-l} > 1 \geqslant \lambda \frac{n-k+1}{n-l}.$$

(ii) Let us recall that a Moebius transformation of the form $M(x) = (x-a)/(x-b)$, with $a,b \in \mathbb{R}$ such that $a < b$, is decreasing when restricted to $\mathbb{R}$. Thus

$$\frac{\beta-k+1}{\beta-n+1} - \lambda(n-k+1) < \frac{m+\frac{2\lambda}{2\lambda-1}-k+1}{m+\frac{2\lambda}{2\lambda-1}-n+1} - \lambda(n-k+1) = (m-k+1)(\lambda-1).$$

This is, $M_{l,k} < (m-k+1)(\lambda-1) \leqslant 0$, because $l \geqslant k$ implies $m+1 > k$. □

Therefore, from Proposition 2 (i), we get that, if we take a multiset Z of self-conjugate complex numbers in the slices $\pi/6 \leqslant |\arg z| \leqslant \pi/4$ of the complex plane, then the Newton-like inequalities (20) are stronger than those in (21) for $l \leqslant n/4$. However, it is the other way around if Z has m real numbers and two mutually conjugated nonreal complex numbers z, z^* such that $2/(1-\tan^2(\arg z)) \notin \mathbb{N}$, according to Proposition 2 (ii).

7. Subsequences

COROLLARY 2. *Under the conditions of Theorem 5,*

$$e_{km+r}(Z)e_{lm+r}(Z) \geqslant$$
$$e_{(k-1)m+r}(Z)e_{(l+1)m+r}(Z) \prod_{j=0}^{m-1} \left(1+\frac{m(l-k+1)}{km+r-j}\right)\left(1+\frac{m(l-k+1)}{\beta-lm-r-j}\right) \qquad (22)$$

for $1 \leqslant k \leqslant l \leqslant (n-r)/m-1$. In particular, the subsequence of $(e_k(Z))_k$ made of terms with indices in a residual class r modulo m is ultra log-concave of order $(\beta-2r)/m$.

Proof. Use Theorem 5 and Theorem 2. □

7.1. Examples

Let Z be a self-conjugate multiset of complex numbers in the angular sector $|\arg z| \leqslant \pi/4$ such that $|Z| = n$. For $m = 2$ and $r = 0$ in Corollary 2, we get

$$\begin{aligned} e_{2k}(Z)e_{2l}(Z) \geqslant & \left(1+\frac{l-k+1}{k}\right)\left(1+\frac{l-k+1}{k-1/2}\right) \\ & \left(1+\frac{l-k+1}{\beta/2-l}\right)\left(1+\frac{l-k+1}{\beta/2-l-1/2}\right)e_{2k-2}(Z)e_{2l+2}(Z) \end{aligned} \tag{23}$$

for $1 \leqslant k \leqslant l \leqslant n/2-1$; and for $m = 2$ and $r = 1$, we get

$$\begin{aligned} e_{2k+1}(Z)e_{2l+1}(Z) \geqslant & \left(1+\frac{l-k+1}{k}\right)\left(1+\frac{l-k+1}{k+1/2}\right) \\ & \left(1+\frac{l-k+1}{\beta/2-l-1}\right)\left(1+\frac{l-k+1}{\beta/2-l-1/2}\right)e_{2k-1}(Z)e_{2l+3}(Z) \end{aligned} \tag{24}$$

for $1 \leqslant k \leqslant l \leqslant (n-1)/2-1$. However, Ellard and Šmigoc [4, Theorem 2.9] proved the following Newton-like inequalities:

$$e_{2k}(Z)e_{2l}(Z) \geqslant \left(1+\frac{l-k+1}{k}\right)\left(1+\frac{l-k+1}{\lfloor n/2\rfloor-l}\right)e_{2k-2}(Z)e_{2l+2}(Z) \tag{25}$$

for $1 \leqslant k \leqslant l \leqslant n/2-1$; and

$$e_{2k+1}(Z)e_{2l+1}(Z) \geqslant \left(1+\frac{l-k+1}{k}\right)\left(1+\frac{l-k+1}{\lceil n/2\rceil-l-1}\right)e_{2k-1}(Z)e_{2l+3}(Z) \tag{26}$$

for $1 \leqslant k \leqslant l \leqslant (n-1)/2-1$. Note that, according to Theorem 1, these inequalities are equivalent to the fact that the sequences $(e_{2k}(Z))_k$ and $(e_{2k+1}(Z))_k$ are ultra log-concave with orders $\lfloor n/2\rfloor$ and $\lceil n/2\rceil-1$, respectively.

In the next proposition, we make comparisons between all these Newton-like inequalities.

PROPOSITION 3.

(i) *If* $1 \leqslant l < \lfloor n/2\rfloor/2+1/4$, *then the inequalities* (23) *are stronger than those in* (25).

(ii) *If* $1 \leqslant l < \lceil n/2\rceil/2-3/4$, *then the inequalities* (24) *are stronger than those in* (26).

(iii) *For* $k > \lfloor n/2\rfloor/2+1/4$ *and sufficiently large* β *the inequalities* (25) *are stronger than those in* (23).

(iv) *For* $k > \lceil n/2\rceil/2-3/4$ *and sufficiently large* β *the inequalities* (26) *are stronger than those in* (24).

Proof.

(i) If $1 \leqslant l < \lfloor n/2 \rfloor/2 + 1/4$ and $k \leqslant l$, then $l + k < \lfloor n/2 \rfloor + 1/2$, which implies that the factor
$$\left(1 + \frac{l-k+1}{k-1/2}\right)$$
in (23) is strictly larger than the factor
$$\left(1 + \frac{l-k+1}{\lfloor n/2 \rfloor - l}\right)$$
in (25).

(ii) Similarly, the condition $1 \leqslant l < \lceil n/2 \rceil/2 - 3/4$ implies that
$$\left(1 + \frac{l-k+1}{k+1/2}\right) > \left(1 + \frac{l-k+1}{\lceil n/2 \rceil - 1 - l}\right).$$

(iii) Let
$$D_{k,l,n}(\beta) = \left(1 + \frac{l-k+1}{k-1/2}\right)\left(1 + \frac{l-k+1}{\beta/2 - l}\right)\left(1 + \frac{l-k+1}{\beta/2 - l - 1/2}\right) - \left(1 + \frac{l-k+1}{\lfloor n/2 \rfloor - l}\right).$$
Then,
$$\lim_{\beta \to \infty} D_{k,l,n}(\beta) = (l-k+1)\left(\frac{1}{k-1/2} - \frac{1}{\lfloor n/2 \rfloor - l}\right)$$
being this rational function negative if $k > \lfloor n/2 \rfloor/2 + 1/4$. The condition $l \geqslant k > \lfloor n/2 \rfloor/2 + 1/4$ implies that $l + k > \lfloor n/2 \rfloor + 1/2$, i.e., $k - 1/2 > \lfloor n/2 \rfloor - l$.

(iv) Similar to (iii). □

8. Linear combinations

In [17], Ren asked for structural conditions on the coefficients $(a_k)_{1 \leqslant k \leqslant n}$ of the linear combinations in (27) under which such inequality holds. In the following corollary, we show that these structural conditions are log-concavity, together with Z being a multiset of self-conjugate complex numbers in the angular sector $|\arg z| \leqslant \pi/4$.

COROLLARY 3. *Under the conditions of Theorem 5, if* $E_k(Z) = e_k(Z)/\binom{\beta}{k}$, $0 \leqslant k \leqslant n$, *and* $(a_k)_{1 \leqslant k \leqslant n}$ *is a log-concave sequence, then*
$$\left(\sum_{k=1}^{n} a_k E_k(Z)\right)^2 \geqslant \left(\sum_{k=1}^{n} a_k E_{k-1}(Z)\right)\left(\sum_{k=1}^{n} a_k E_{k+1}(Z)\right). \tag{27}$$

Proof. From Theorem 5 (ii), we obtain that the sequence $(e_k(Z))_{0\leqslant k\leqslant n}$ is ultra log-concave of order β. Consequently, the sequence $(E_k(Z))_{0\leqslant k\leqslant n}$ is log-concave. Therefore, we can utilize Theorem 3 to derive (27). $\square$

We elaborate on inequalities of the type shown in (27) in [3].

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