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# $P$–NUMERICAL RADIUS INEQUALITIES FOR $2\times 2$ OPERATOR MATRICES

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*Abstract.* In this paper, we present a refinement of the triangle inequality for Schatten $p$-norm, and specific example is given to compare our result with the triangle inequality for Schatten $p$-norm. As an application, a new lower bound for $p$-numerical radius is obtained. In addition, some bounds for $p$-numerical radius of $2\times 2$ operator matrices are established, which extend the results of previous studies. Moreover, Schatten $p$-norm equalities of $2\times 2$ operator matrices are also given.

## 1. Introduction and preliminaries

Let $\mathbb{B}(\mathscr{H})$ denote the $C^*$-algebra of all bounded linear operators on a complex separable Hilbert space $\mathscr{H}$. Let $P_1,\ldots,P_n$ be a family of mutually orthogonal projections in $\mathscr{H}$ such that $\oplus P_i = I$. Given $T$ in $\mathbb{B}(\mathscr{H})$, let $T_{ij} = P_iTP_j$, $i,j = 1,2,\ldots,n$. Making the usual identification we can write $T$ in a block-matrix form

$$T = [T_{ij}],\ 1 \leqslant i,j \leqslant n. \tag{1}$$

For $T\in\mathbb{B}(\mathscr{H})$, the adjoint, the real and imaginary parts of $T$ are defined by $T^*$, $\Re(T)$ and $\Im(T)$, respectively. And according to the Cartesian decomposition, $T\in\mathbb{B}(\mathscr{H})$ can be presented as $T = \Re(T)+i\Im(T)$, where $\Re(T) = \frac{1}{2}(T+T^*)$ and $\Im(T) = \frac{1}{2i}(T-T^*)$.

Let $N(.)$ be an arbitrary norm on $\mathbb{B}(\mathscr{H})$. For every $T\in\mathbb{B}(\mathscr{H})$ and unitary operators $U,V\in\mathbb{B}(\mathscr{H})$, the norm $N(.)$ is self-adjoint if $N(T) = N(T^*)$, unitarily invariant if $N(T) = N(U^*TV)$, and weakly unitarily invariant if $N(T) = N(U^*TU)$.

A compact operator $T\in\mathbb{B}(\mathscr{H})$ belongs to the Schatten $p$-class $C_p$ for $0<p\leqslant\infty$ if $\|T\|_p = (tr|T|^p)^{\frac{1}{p}} = (\sum\limits_{j=1}^{\infty} s_j^p(T))^{\frac{1}{p}} < \infty$, where $s_1(T)\geqslant s_2(T)\geqslant\ldots$ are the eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$, and $tr(.)$ is the usual trace functional. Throughout this paper, we assume $T\in\mathbb{B}(\mathscr{H})$ is compact whenever $T\in C_p$ for $0<p\leqslant\infty$. It is known that $C_p$ is a two-sided ideal in $\mathbb{B}(\mathscr{H})$, and $C_\infty$ is the ideal of compact operators in $\mathbb{B}(\mathscr{H})$. For $1\leqslant p\leqslant\infty (0<p<1)$, $\|.\|_p$ defines a norm(a quasi norm) on $C_p$. It should be mentioned here that for quasi norm does not satisfy the triangle inequality. For the theory of the Schatten $p$-class, we refer to [8, 14, 20].

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For $0 < p \leqslant \infty$, define the $p$-numerical radius $w_p(.)$ by $w_p(T) = \sup_{\theta \in \mathbb{R}} \left\| \Re(e^{i\theta}T) \right\|_p = \sup_{\theta \in \mathbb{R}} \left\| \Im(e^{i\theta}T) \right\|_p$. It can be seen that

$$\max\left\{ \|\Re(T)\|_p, \|\Im(T)\|_p \right\} \leqslant w_p(T) \leqslant \|T\|_p.$$

For $0 < p \leqslant \infty, s > 0$,

$$\|A\|_{sp}^s = \| |A|^s \|_p = \| |A^*|^s \|_p. \tag{2}$$

Besides, if $T$ is self-adjoint, $w_p(T) = \|T\|_p$.

For $p = 2$, Hilbert-Schmidt numerical radius $w_2(.)$ has been given in [1]. In particular, it has been shown that $\frac{1}{\sqrt{2}} \|T\|_2 \leqslant w_2(T) \leqslant \|T\|_2$, where $T \in C_2$. For more basic information about $w_2(.)$, it is recommended that readers can see [2, 3, 5, 15, 19].

Further, it was also shown in [1] that the $p$-numerical radius is equivalent to the Schatten $p$-norm, i.e., for every $T \in C_p$,

$$\frac{1}{2}\|T\|_p \leqslant \omega_p(T) \leqslant \|T\|_p, \text{ where } p \in [1, \infty). \tag{3}$$

The study of the numerical radius of operators has attracted a lot of interest due to their widespread applications in many branches in mathematics and physics. In mathematics, the numerical radius is often used as a more reliable indicator of the convergence rate of iterative methods, see [6, 11]. In physics, it has been successfully applied to quantum computation and quantum information theory, especially in the field of quantum error correction, see [7, 10, 18, 17]. Over the years, numerous outstanding mathematicians have acquired various generalized numerical radius inequalities such as $w_2(.)$ of operator. As the generalization of $w_2(.)$ of operator, it is necessary to investigate $w_p(.)$ of operator. As is known, computing the numerical radius is not easy task. However, the operator norm computations are much easier. This urges the need to find bounds of $w_p(.)$ in the terms of $\|.\|_p$. But, compared with $\|.\|_2$, $\|.\|_p$ is more complex. Consequently, calculation problem of $\|.\|_p$ urgently needs to be solved.

In this paper, we split our main results into two sections. In the first section, we give a refinement of the triangle inequality for Schatten $p$-norm. As an application, a new lower bound for $p$-numerical radius is obtained. In the second section, some bounds for $p$-numerical radius of $2 \times 2$ operator matrices, which extend the results of previous studies and obtain several new lower bound estimates about special $2 \times 2$ operator matrices. Two Schatten $p$-norm equalities of $2 \times 2$ operator matrices are also given.

## 2. Improvement of the triangle inequality for Schatten $p$-norm

LEMMA 2.1. ([12]) *Let $f:[a,b]\to\mathbb{R}$ be a convex function. Then*

$$\begin{aligned} f\Big(\frac{a+b}{2}\Big) &\leqslant \frac{1}{2}\Big(f\Big(\frac{a+3b}{4}\Big)+f\Big(\frac{3a+b}{4}\Big)\Big)\\ &\leqslant \frac{1}{b-a}\int_a^b f(x)dx\\ &\leqslant \frac{1}{2}\Big(f\Big(\frac{a+b}{2}\Big)+\frac{f(a)+f(b)}{2}\Big)\\ &\leqslant \frac{f(a)+f(b)}{2}. \end{aligned}$$

Now, we give our first main result of this section, which is a refinement of the triangle inequality for Schatten $p$-norm.

THEOREM 2.1. *Let $X,Y\in C_p$, where $p\geqslant 1$. Then*

$$\begin{aligned} \|X+Y\|_p &\leqslant \frac{1}{4}(\|3X+Y\|_p+\|X+3Y\|_p)\\ &\leqslant 2\int_0^1 \|tX+(1-t)Y\|_p dt\\ &\leqslant \frac{1}{2}(\|X+Y\|_p+\|X\|_p+\|Y\|_p)\\ &\leqslant \|X\|_p+\|Y\|_p. \end{aligned}$$

*Proof.* Let $f(t)=\|tX+(1-t)Y\|_p$ for $t\in\mathbb{R}$. It is easy to see that the function $f(t)$ is convex. By taking into consideration Lemma 2.1, we see that

$$\begin{aligned} f\Big(\frac{0+1}{2}\Big) &\leqslant \frac{1}{2}\Big(f\Big(\frac{3}{4}\Big)+f\Big(\frac{1}{4}\Big)\Big)\\ &\leqslant \int_0^1 f(x)dx\\ &\leqslant \frac{1}{2}\Big(f\Big(\frac{1}{2}\Big)+\frac{f(0)+f(1)}{2}\Big)\\ &\leqslant \frac{f(0)+f(1)}{2}. \end{aligned}$$

It follows that

$$\begin{aligned}\left\|\frac{1}{2}X+\frac{1}{2}Y\right\|_p &\leqslant \frac{1}{2}\left(\left\|\frac{3}{4}X+\frac{1}{4}Y\right\|_p+\left\|\frac{1}{4}X+\frac{3}{4}Y\right\|_p\right)\\ &\leqslant \int_0^1 \|tX+(1-t)Y\|_p dt\\ &\leqslant \frac{1}{2}\left(\left\|\frac{1}{2}X+\frac{1}{2}Y\right\|_p+\frac{\|X\|_p+\|Y\|_p}{2}\right)\\ &\leqslant \frac{\|X\|_p+\|Y\|_p}{2}.\end{aligned}$$

Hence,

$$\begin{aligned}\|X+Y\|_p &\leqslant \frac{1}{4}(\|3X+Y\|_p+\|X+3Y\|_p)\\ &\leqslant 2\int_0^1 \|tX+(1-t)Y\|_p dt\\ &\leqslant \frac{1}{2}(\|X+Y\|_p+\|X\|_p+\|Y\|_p)\\ &\leqslant \|X\|_p+\|Y\|_p. \quad \square\end{aligned}$$

REMARK 2.1. Here, we remark that our inequalities in Theorem 2.1 is a nontrivial improvement of the triangle inequality for Schatten $p$-norm. Take $X=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$, and $Y=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$. Easy computations show that

$$\|X\|_p=\|Y\|_p=1, \quad \|X+Y\|_p=\sqrt{2}, \quad \|3X+Y\|_p=\|X+3Y\|_p=\sqrt{10},$$

and

$$\int_0^1 \|tX+(1-t)Y\|_p dt \simeq 0.812.$$

Hence,

$$\begin{aligned}\|X+Y\|_p \simeq 1.414 &< \frac{1}{4}(\|3X+Y\|_p+\|X+3Y\|_p) \simeq 1.581\\ &< 2\int_0^1 \|tX+(1-t)Y\|_p dt \simeq 1.624\\ &< \frac{1}{2}(\|X+Y\|_p+\|X\|_p+\|Y\|_p) \simeq 1.707\\ &< \|X\|_p+\|Y\|_p = 2.\end{aligned}$$

The following theorem provides a new lower bound for $p$-numerical radius of operator.

THEOREM 2.2. *Let $T \in C_p$, where $p \geqslant 2$. Then*

$$\begin{aligned}\frac{1}{4}\|T^*T+TT^*\|_{p/2} &\leqslant \frac{1}{8}(\|3\Re^2(T)+\Im^2(T)\|_{p/2}+\|\Re^2(T)+3\Im^2(T)\|_{p/2}) \\ &\leqslant \int_0^1 \|t\Re^2(T)+(1-t)\Im^2(T)\|_{p/2}dt \\ &\leqslant \frac{1}{4}\left(\|\Re^2(T)+\Im^2(T)\|_{p/2}+\|\Re^2(T)\|_{p/2}+\|\Im^2(T)\|_{p/2}\right) \\ &\leqslant \frac{1}{2}\|\Re(T)\|_p^2+\frac{1}{2}\|\Im(T)\|_p^2 \\ &\leqslant \omega_p^2(T).\end{aligned}$$

*Proof.* For $T \in C_p$, where $p \geqslant 2$, by calculation, we have

$$\|\Re^2(T)+\Im^2(T)\|_{p/2} = \frac{1}{2}\|T^*T+TT^*\|_{p/2}. \tag{4}$$

Put $X = \Re^2(T)$ and $Y = \Im^2(T)$ in Theorem 2.1, we obtain that

$$\begin{aligned}\|\Re^2(T)+\Im^2(T)\|_{p/2} &\leqslant \frac{1}{4}(\|3\Re^2(T)+\Im^2(T)\|_{p/2}+\|\Re^2(T)+3\Im^2(T)\|_{p/2}) \\ &\leqslant 2\int_0^1 \|t\Re^2(T)+(1-t)\Im^2(T)\|_{p/2}dt \\ &\leqslant \frac{1}{2}\left(\|\Re^2(T)+\Im^2(T)\|_{p/2}+\|\Re^2(T)\|_{p/2}+\|\Im^2(T)\|_{p/2}\right) \\ &\leqslant \|\Re^2(T)\|_{p/2}+\|\Im^2(T)\|_{p/2}.\end{aligned}$$

It follow from (4), we derive

$$\begin{aligned}\frac{1}{2}\|T^*T+TT^*\|_{p/2} &\leqslant \frac{1}{4}(\|3\Re^2(T)+\Im^2(T)\|_{p/2}+\|\Re^2(T)+3\Im^2(T)\|_{p/2}) \\ &\leqslant 2\int_0^1 \|t\Re^2(T)+(1-t)\Im^2(T)\|_{p/2}dt \\ &\leqslant \frac{1}{2}\left(\|\Re^2(T)+\Im^2(T)\|_{p/2}+\|\Re^2(T)\|_{p/2}+\|\Im^2(T)\|_{p/2}\right) \\ &\leqslant \|\Re(T)\|_p^2+\|\Im(T)\|_p^2 \\ &\leqslant 2\omega_p^2(T).\end{aligned}$$

Hence,

$$
\begin{aligned}
\frac{1}{4}\|T^*T+TT^*\|_{p/2} &\leqslant \frac{1}{8}(\|3\Re^2(T)+\Im^2(T)\|_{p/2}+\|\Re^2(T)+3\Im^2(T)\|_{p/2})\\
&\leqslant \int_0^1 \|t\Re^2(T)+(1-t)\Im^2(T)\|_{p/2}dt\\
&\leqslant \frac{1}{4}\left(\|\Re^2(T)+\Im^2(T)\|_{p/2}+\|\Re^2(T)\|_{p/2}+\|\Im^2(T)\|_{p/2}\right)\\
&\leqslant \frac{1}{2}\|\Re(T)\|_p^2+\frac{1}{2}\|\Im(T)\|_p^2\\
&\leqslant \omega_p^2(T). \quad \square
\end{aligned}
$$

## 3. The $p$-numerical radius of $2\times 2$ operator matrices

To obtain the desired results of this paper, we first introduce some well-known lemmas. The first lemma is the convexity(concavity) inequalities.

LEMMA 3.1. ([4]) *Let $a,b\in[0,\infty)$. Then*

(1) $2^{p-1}(a^p+b^p)\leqslant (a+b)^p\leqslant a^p+b^p$ *for* $0\leqslant p\leqslant 1$;

(2) $a^p+b^p\leqslant (a+b)^p\leqslant 2^{p-1}(a^p+b^p)$ *for* $p\geqslant 1$.

The second lemma is the basic properties of Schatten $p$-norm.

LEMMA 3.2. ([4]) *Let $A,B\in C_p$, and $p\in(0,\infty)$. Then*

$$
\left\|\begin{pmatrix}0 & A\\ B & 0\end{pmatrix}\right\|_p=\left\|\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}\right\|_p=\sqrt[p]{\|A\|_p^p+\|B\|_p^p}\ \text{and}\ \|A\|_p=\|A^*\|_p.
$$

R. Bhatia and F. Kittaneh [9] studied the relation between the norm of operator matrix $T$ and the norm of its block matrix entries $T_{ij}$. They acquired the following result:

LEMMA 3.3. ([9]) *Let $T=[T_{ij}]$, where $T_{ij}\in C_p$, $i,j=1,2\dots,n$. Then*

(1) $\displaystyle\sum_{i,j=1}^n \|T_{ij}\|_p^2\leqslant \|T\|_p^2\leqslant n^{\frac{4}{p}-2}\sum_{i,j=1}^n \|T_{ij}\|_p^2$ *for* $1\leqslant p\leqslant 2$;

(2) $\displaystyle n^{\frac{4}{p}-2}\sum_{i,j=1}^n \|T_{ij}\|_p^2\leqslant \|T\|_p^2\leqslant \sum_{i,j=1}^n \|T_{ij}\|_p^2$ *for* $2\leqslant p\leqslant\infty$.

LEMMA 3.4. ([13]) *Let $A\in C_p$, then for $0<p\leqslant\infty$, we have*

$$
2^{\frac{2}{p}}w_p^2(A)\geqslant w_{p/2}(A^2).
$$

LEMMA 3.5. ([16]) *Let $B,C \in C_p$, then for $0<p<\infty$,*

$$w_p\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right] \geqslant 2^{\frac{1}{p}-1} \max\left\{w_p(B+C), w_p(B-C)\right\}.$$

LEMMA 3.6. *Let $A,B \in C_p$, then for $0<p\leqslant\infty$,*

$$w_p\left[\begin{pmatrix} A & B \\ B & A \end{pmatrix}\right] \leqslant \left(w_p^p(A+B)+w_p^p(A-B)\right)^{\frac{1}{p}}.$$

*In particular, if $A,B$ are self-adjoint, then*

$$w_p\left[\begin{pmatrix} A & B \\ B & A \end{pmatrix}\right] = \left(w_p^p(A+B)+w_p^p(A-B)\right)^{\frac{1}{p}}.$$

*Proof.* Let $U=\frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ -I & I \end{pmatrix}$, then $U$ is a unitary operator. So, by using weakly unitary invariance of $w_p(.)$ and [16], we have

$$\begin{aligned} w_p\left[\begin{pmatrix} A & B \\ B & A \end{pmatrix}\right] &= w_p\left[U\begin{pmatrix} A & B \\ B & A \end{pmatrix}U^*\right] \\ &= w_p\left[\begin{pmatrix} A+B & 0 \\ 0 & A-B \end{pmatrix}\right] \\ &\leqslant \left(w_p^p(A+B)+w_p^p(A-B)\right)^{\frac{1}{p}} \quad \text{for } 0<p\leqslant\infty. \end{aligned}$$

In particular, if $A,B$ are self-adjoint, then by using Lemma 3.2, we get

$$w_p\left[\begin{pmatrix} A & B \\ B & A \end{pmatrix}\right] = \left(w_p^p(A+B)+w_p^p(A-B)\right)^{\frac{1}{p}}. \qquad \square$$

LEMMA 3.7. *Let $A,B,C,D \in C_p$, then for $1\leqslant p\leqslant\infty$, we have*

(1) $w_p\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] \geqslant w_p\left[\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\right]$;

(2) $w_p\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] \geqslant w_p\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right]$.

*Proof.* The proof is similar to the technique used in [19]. $\square$

Now, we give upper and lower bound estimates for $p$-numerical radius of $2\times 2$ operator matrices.

THEOREM 3.1. *Let $T=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A,B,C,D \in C_p$ and $\alpha\in[0,1]$. Then*

(1) $w_p(T) \leqslant 2^{\frac{2}{p}-1}\sqrt{2\left(\alpha^2+(1-\alpha)^2\right)\left(w_p^2(A)+w_p^2(D)\right)+\|B\|_p^2+\|C\|_p^2}$ *for* $1\leqslant p\leqslant 2$;

(2) $w_p(T) \leqslant \sqrt{2\left(\alpha^2+(1-\alpha)^2\right)\left(w_p^2(A)+w_p^2(D)\right)+\|B\|_p^2+\|C\|_p^2}$ *for* $2\leqslant p\leqslant\infty$.

*Proof.* For $1 \leqslant p \leqslant 2$,

$$\left\|\Re(e^{i\theta}T)\right\|_p^2 = \frac{1}{4}\left\|\begin{pmatrix} 2\Re\left(e^{i\theta}A\right) & e^{i\theta}B + e^{-i\theta}C^* \\ e^{i\theta}C + e^{-i\theta}B^* & 2\Re\left(e^{i\theta}D\right)\end{pmatrix}\right\|_p^2$$

$$\leqslant \frac{1}{4}\left(\left\|\begin{pmatrix} 2\alpha\Re\left(e^{i\theta}A\right) & e^{i\theta}B \\ e^{-i\theta}B^* & 2\alpha\Re\left(e^{i\theta}D\right)\end{pmatrix}\right\|_p + \left\|\begin{pmatrix} 2(1-\alpha)\Re\left(e^{i\theta}A\right) & e^{-i\theta}C^* \\ e^{i\theta}C & 2(1-\alpha)\Re\left(e^{i\theta}D\right)\end{pmatrix}\right\|_p\right)^2$$

$$\leqslant \frac{1}{2}\left(\left\|\begin{pmatrix} 2\alpha\Re\left(e^{i\theta}A\right) & e^{i\theta}B \\ e^{-i\theta}B^* & 2\alpha\Re\left(e^{i\theta}D\right)\end{pmatrix}\right\|_p^2 + \left\|\begin{pmatrix} 2(1-\alpha)\Re\left(e^{i\theta}A\right) & e^{-i\theta}C^* \\ e^{i\theta}C & 2(1-\alpha)\Re\left(e^{i\theta}D\right)\end{pmatrix}\right\|_p^2\right)$$

(by using the convexity of the function $f(t) = t^2$ on $[0,+\infty)$)

$$\leqslant 2^{\frac{4}{p}-3}\left((2\alpha)^2\left(\left\|\Re(e^{i\theta}A)\right\|_p^2 + \left\|\Re(e^{i\theta}D)\right\|_p^2\right) + 2\|B\|_p^2\right)$$

$$+ 2^{\frac{4}{p}-3}\left((2(1-\alpha))^2\left(\left\|\Re(e^{i\theta}A)\right\|_p^2 + \left\|\Re(e^{i\theta}D)\right\|_p^2\right) + 2\|C\|_p^2\right) \quad \text{(by Lemma 3.3)}$$

$$= 2^{\frac{4}{p}-2}\left[2\left(\alpha^2 + (1-\alpha)^2\right)\left(\left\|\Re(e^{i\theta}A)\right\|_p^2 + \left\|\Re(e^{i\theta}D)\right\|_p^2\right) + \|B\|_p^2 + \|C\|_p^2\right].$$

Therefore, by taking supremum to both sides of the above inequality over all real numbers $\theta$, we have

$$w_p(T) \leqslant 2^{\frac{2}{p}-1}\sqrt{2\left(\alpha^2 + (1-\alpha)^2\right)\left(w_p^2(A) + w_p^2(D)\right) + \|B\|_p^2 + \|C\|_p^2}.$$

Similarly, for $2 \leqslant p \leqslant \infty$, we have

$$w_p(T) \leqslant \sqrt{2\left(\alpha^2 + (1-\alpha)^2\right)\left(w_p^2(A) + w_p^2(D)\right) + \|B\|_p^2 + \|C\|_p^2}. \quad \square$$

REMARK 3.1. By taking $\alpha = \frac{1}{2}$ in Theorem 3.1, we obtain

(1) $w_p\left[\begin{pmatrix} A & B \\ C & D\end{pmatrix}\right] \leqslant 2^{\frac{2}{p}-1}\sqrt{w_p^2(A) + w_p^2(D) + \|B\|_p^2 + \|C\|_p^2}$ *for* $1 \leqslant p \leqslant 2$;

(2) $w_p\left[\begin{pmatrix} A & B \\ C & D\end{pmatrix}\right] \leqslant \sqrt{w_p^2(A) + w_p^2(D) + \|B\|_p^2 + \|C\|_p^2}$ *for* $2 \leqslant p \leqslant \infty$.

Taking $\alpha = \frac{1}{2}$ and $p = 2$ in Theorem 3.1, we derive

$$w_2\left[\begin{pmatrix} A & B \\ C & D\end{pmatrix}\right] \leqslant \sqrt{w_2^2(A) + w_2^2(D) + \|B\|_2^2 + \|C\|_2^2},$$

which has been given in [3].

THEOREM 3.2. *Let* $T=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *where* $A,B,C,D\in C_p$ *and* $\alpha\in[0,1]$. *Then*

(1) $w_p(T)\leqslant 2^{\frac{2}{p}-\frac{3}{2}}\Big[\sqrt{2\alpha^2\big(w_p^2(A)+w_p^2(D)\big)+\|B\|_p^2}$
$+\sqrt{2(1-\alpha)^2\big(w_p^2(A)+w_p^2(D)\big)+\|C\|_p^2}\Big]$ *for* $1\leqslant p\leqslant 2$;

(2) $w_p(T)\leqslant 2^{-\frac{1}{2}}\Big[\sqrt{2\alpha^2\big(w_p^2(A)+w_p^2(D)\big)+\|B\|_p^2}$
$+\sqrt{2(1-\alpha)^2\big(w_p^2(A)+w_p^2(D)\big)+\|C\|_p^2}\Big]$ *for* $2\leqslant p\leqslant\infty$.

*Proof.* For $1\leqslant p\leqslant 2$,

$$\begin{aligned}
w_p(T)&=\frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|\begin{pmatrix} 2\Re\left(e^{i\theta}A\right) & e^{i\theta}B+e^{-i\theta}C^* \\ e^{i\theta}C+e^{-i\theta}B^* & 2\Re\left(e^{i\theta}D\right)\end{pmatrix}\right\|_p\\
&\leqslant\frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|\begin{pmatrix} 2\alpha\Re\left(e^{i\theta}A\right) & e^{i\theta}B \\ e^{-i\theta}B^* & 2\alpha\Re\left(e^{i\theta}D\right)\end{pmatrix}\right\|_p\\
&\quad+\frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|\begin{pmatrix} 2(1-\alpha)\Re\left(e^{i\theta}A\right) & e^{-i\theta}C^* \\ e^{i\theta}C & 2(1-\alpha)\Re\left(e^{i\theta}D\right)\end{pmatrix}\right\|_p\\
&\leqslant 2^{\frac{2}{p}-\frac{3}{2}}\Big[\sqrt{2\alpha^2\big(w_p^2(A)+w_p^2(D)\big)+\|B\|_p^2}\\
&\quad+\sqrt{2(1-\alpha)^2\big(w_p^2(A)+w_p^2(D)\big)+\|C\|_p^2}\Big]\\
&\quad(\text{by Lemma 3.3}).
\end{aligned}$$

Similarly, for $2\leqslant p\leqslant\infty$, we have

$$\begin{aligned}
w_p(T)\leqslant 2^{-\frac{1}{2}}\Big[&\sqrt{2\alpha^2\big(w_p^2(A)+w_p^2(D)\big)+\|B\|_p^2}\\
&+\sqrt{2(1-\alpha)^2\big(w_p^2(A)+w_p^2(D)\big)+\|C\|_p^2}\Big].\qquad\square
\end{aligned}$$

REMARK 3.2. By taking $\alpha=\frac{1}{2}$ in Theorem 3.2, we have

(1) $w_p\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right]\leqslant 2^{\frac{2}{p}-2}\left[\sqrt{w_p^2(A)+w_p^2(D)+2\|B\|_p^2}+\sqrt{w_p^2(A)+w_p^2(D)+2\|C\|_p^2}\right]$
for $1\leqslant p\leqslant 2$;

(2) $w_p\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right]\leqslant \frac{1}{2}\left[\sqrt{w_p^2(A)+w_p^2(D)+2\|B\|_p^2}+\sqrt{w_p^2(A)+w_p^2(D)+2\|C\|_p^2}\right]$
for $2\leqslant p\leqslant\infty$.

REMARK 3.3. Using the concavity of the function $f(t)=t^{\frac{1}{2}}$ on $[0,\infty)$, it follows that Remark 3.2 is a refinement of Remark 3.1.

REMARK 3.4. By taking $p=2$ in Theorem 3.2, we can derive

$$w_2\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] \leqslant \sqrt{\alpha^2(w_2^2(A)+w_2^2(D))+\frac{1}{2}\|B\|_2^2} + \sqrt{(1-\alpha)^2(w_2^2(A)+w_2^2(D))+\frac{1}{2}\|C\|_2^2},$$

which has been given in [2].

THEOREM 3.3. *Let $A,B,C,D \in C_p$, then for $1 \leqslant p < \infty$, we have*

$$w_p\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] \geqslant 2^{\frac{1}{p}-1}\max\left\{w_p(A+D), w_p(A-D), w_p(B+C), w_p(B-C)\right\}.$$

*Proof.*

$$\begin{aligned} w_p\left[\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\right] &= \sup_{\theta\in\mathbb{R}}\left\|\begin{pmatrix} \Re(e^{i\theta}A) & 0 \\ 0 & \Re(e^{i\theta}D) \end{pmatrix}\right\|_p \\ &= \sup_{\theta\in\mathbb{R}} \sqrt[p]{\|\Re(e^{i\theta}A)\|_p^p + \|\Re(e^{i\theta}D)\|_p^p} \\ &\geqslant 2^{\frac{1}{p}-1}\sup_{\theta\in\mathbb{R}}\left(\left\|\Re(e^{i\theta}A)\right\|_p + \left\|\Re(e^{i\theta}D)\right\|_p\right) \quad \text{(by Lemma 3.1(a))} \\ &\geqslant 2^{\frac{1}{p}-1} w_p(A+D). \end{aligned}$$

Similarly, we get

$$w_p\left[\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\right] \geqslant 2^{\frac{1}{p}-1} w_p(A-D).$$

Hence,

$$w_p\left[\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\right] \geqslant 2^{\frac{1}{p}-1}\max\left\{w_p(A+D), w_p(A-D)\right\}.$$

Finally, by using Lemma 3.5, Lemma 3.7 and the above inequality,

$$\begin{aligned} w_p\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] &\geqslant \max\left\{w_p\left[\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\right], w_p\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right]\right\} \\ &\geqslant 2^{\frac{1}{p}-1}\max\left\{w_p(A+D), w_p(A-D), w_p(B+C), w_p(B-C)\right\}. \quad \square \end{aligned}$$

REMARK 3.5. By taking $p=2$ in Theorem 3.3, we can obtain

$$w_2\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] \geqslant 2^{-\frac{1}{2}}\max\{w_2(A+D), w_2(A-D), w_2(B+C), w_2(B-C)\},$$

which has been given in [2].

In what follows, we obtain several new lower bound estimates for special $2\times 2$ operator matrices.

THEOREM 3.4. *Let $A,B\in C_p$ be such that $AB,BA$ are self-adjoint. Then for $0<p\leqslant\infty$, we have*

$$w_p^2\left[\begin{pmatrix} 0 & A\\ B & 0\end{pmatrix}\right]\geqslant 2^{-\frac{2}{p}}\left(w_{p/2}^{p/2}(AB)+w_{p/2}^{p/2}(BA)\right)^{\frac{2}{p}}.$$

*Proof.* Let $T=\begin{pmatrix} 0 & A\\ B & 0\end{pmatrix}$, then by using Lemma 3.2 and Lemma 3.4, we have

$$\begin{aligned}2^{\frac{2}{p}}w_p^2(T)&\geqslant w_{p/2}(T^2)\\ &=w_{p/2}\left[\begin{pmatrix} AB & 0\\ 0 & BA\end{pmatrix}\right]\\ &=\left(w_{p/2}^{p/2}(AB)+w_{p/2}^{p/2}(BA)\right)^{\frac{2}{p}},\end{aligned}$$

thus we obtain the desired result. □

THEOREM 3.5. *Let $A,B\in C_p$ be such that $A^2-B^2$, $AB-BA$ are self-adjoint. Then for $0<p\leqslant\infty$, we have*

$$w_p\left[\begin{pmatrix} A & B\\ -B & -A\end{pmatrix}\right]\geqslant 2^{-\frac{1}{p}}\left(w_{p/2}^{p/2}((A-B)(A+B))+w_{p/2}^{p/2}((A+B)(A-B))\right)^{\frac{1}{p}}.$$

*Proof.* Let $T=\begin{pmatrix} A & B\\ -B & -A\end{pmatrix}$, by using Lemma 3.4 and Lemma 3.6, we have

$$\begin{aligned}2^{\frac{2}{p}}w_p^2(T)&\geqslant w_{p/2}(T^2)\\ &=w_{p/2}\left[\begin{pmatrix} A^2-B^2 & AB-BA\\ AB-BA & A^2-B^2\end{pmatrix}\right]\\ &=\left(w_{p/2}^{p/2}((A-B)(A+B))+w_{p/2}^{p/2}((A+B)(A-B))\right)^{\frac{2}{p}},\end{aligned}$$

thus we obtain

$$w_p\left[\begin{pmatrix} A & B\\ -B & -A\end{pmatrix}\right]\geqslant 2^{-\frac{1}{p}}\left(w_{p/2}^{p/2}((A-B)(A+B))+w_{p/2}^{p/2}((A+B)(A-B))\right)^{\frac{1}{p}}.\quad\square$$

Next results are Schatten $p$-norm equalities of $2\times 2$ operator matrices.

COROLLARY 3.1. *Let $A,B\in C_p$, then*

$$(1)\quad \left\|\begin{pmatrix} A & B \\ -B & -A \end{pmatrix}\right\|_p = \left(\|A+B\|_p^p + \|A-B\|_p^p\right)^{\frac{1}{p}};$$

$$(2)\quad \left\|\begin{pmatrix} A & B \\ -B & -A \end{pmatrix}^2\right\|_p = \left(\|(A-B)(A+B)\|_p^p + \|(A+B)(A-B)\|_p^p\right)^{\frac{1}{p}}.$$

*Proof.* Let $T=\begin{pmatrix} A & B \\ -B & -A \end{pmatrix}$, by Lemma 3.6 and $(2)$, we get

$$\begin{aligned}
\|T\|_p^2 &= \|TT^*\|_{p/2} \\
&= w_{p/2}(TT^*) \\
&= w_{p/2}\left[\begin{pmatrix} AA^*+BB^* & -AB^*-BA^* \\ -AB^*-BA^* & AA^*+BB^* \end{pmatrix}\right] \\
&= \left(w_{p/2}^{p/2}((A-B)(A-B)^*) + w_{p/2}^{p/2}((A+B)(A+B)^*)\right)^{\frac{2}{p}} \\
&= \left(\|(A-B)(A-B)^*\|_{p/2}^{p/2} + \|(A+B)(A+B)^*\|_{p/2}^{p/2}\right)^{\frac{2}{p}} \\
&= \left(\|(A-B)\|_p^p + \|(A+B)\|_p^p\right)^{\frac{2}{p}}.
\end{aligned}$$

Thus,

$$\left\|\begin{pmatrix} A & B \\ -B & -A \end{pmatrix}\right\|_p = \left(\|A+B\|_p^p + \|A-B\|_p^p\right)^{\frac{1}{p}}.$$

Similarly,

$$\left\|\begin{pmatrix} A & B \\ -B & -A \end{pmatrix}^2\right\|_p = \left(\|(A-B)(A+B)\|_p^p + \|(A+B)(A-B)\|_p^p\right)^{\frac{1}{p}}. \quad \square$$

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