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SHARP RAMANUJAN TYPE INEQUALITIES WITH $1/(x+c)$

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Abstract. We establish the new Ramanujan type inequalities with $\frac{1}{x+c}$ as follows: for $x>0$, we have

$$\frac{1}{x+\alpha}<\sum_{k=1}^{\infty}\frac{k^{k-2}}{(x+k)^k}<\frac{1}{x+\beta},$$

where the constants $\alpha=\frac{6}{\pi^2}\cong 0.607927$ and $\beta=0$ are the best possible.

1. Introduction

In 1914, Ramanujan [6] proposed the following question.

QUESTION 1. *If x is positive, show that*

$$\sum_{k=1}^{\infty}\frac{k^{k-2}}{(x+k)^k}<\frac{1}{x}.$$

Many mathematicians [1, 2, 3, 4, 7, 8] have proven this inequality and extended it. Especially, Karamata [3] proved the following inequality.

THEOREM 2. *The inequality*

$$\frac{2e^{-x}}{x\left(x+\frac{8}{3}\right)}\leqslant\frac{1}{x}-\sum_{k=1}^{\infty}\frac{k^{k-2}}{(x+k)^k}\leqslant\frac{2e^{-x}}{x(x+2)}$$

holds for $x>0$.

In the proof of our main theorems, Theorem 2 plays an important role. In this paper, we prove the following double inequality.

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THEOREM 3. *We have*

$$\frac{1}{x+\alpha} < \sum_{k=1}^{\infty} \frac{k^{k-2}}{(x+k)^k} < \frac{1}{x+\beta}$$

for $x > 0$*, where the constants* $\alpha = \frac{6}{\pi^2} \cong 0.607927$ *and* $\beta = 0$ *are the best possible.*

It has been proven that the inequality $\sum_{k=1}^{\infty} \frac{k^{k-2}}{(x+k)^k} < \frac{1}{x+\beta}$ holds for $\beta = 0$, which is Ramanujan's problem. We show that $\beta = 0$ is the best possible constant. Moreover, no linear fractional function that evaluates $\sum_{k=1}^{\infty} \frac{k^{k-2}}{(x+k)^k}$ from below for $x > 0$ is known. We show that $\alpha = \frac{6}{\pi^2} \cong 0.607927$ is the best possible constant.

THEOREM 4. *We have*

$$\frac{1}{x} - \frac{2e^{-x}}{x(x+2)} < \frac{1}{x+\frac{6}{\pi^2}}$$

for $0 < x \leqslant \frac{13}{100}$.

From Theorem 4, we can see that $\frac{1}{x+\frac{6}{\pi^2}}$ approximates $\sum_{k=1}^{\infty} \frac{k^{k-2}}{(x+k)^k}$ better than $\frac{1}{x} - \frac{2e^{-x}}{x(x+2)}$ when sufficiently close to $x = 0$. Using Theorem 2, we further obtain the following result.

THEOREM 5. *If* a *is a positive real number, then we have*

$$\frac{1}{x+\alpha} < \sum_{k=1}^{\infty} \frac{k^{k-2}}{(x+k)^k}$$

for $x > a$*, where* $\alpha = \frac{2a}{-2+2e^a+ae^a}$*. Also, if* b *is a positive real number with* $b > \frac{1}{2}$*, then we have*

$$\sum_{k=1}^{\infty} \frac{k^{k-2}}{(x+k)^k} < \frac{1}{x+\beta}$$

for $\frac{1}{2} < x < b$*, where* $\beta = \frac{6b}{-6+8e^b+3be^b}$.

2. Proof of Theorems

Proof of Theorem 3. We consider the equation

$$\frac{1}{x+f_1(x)} = \sum_{k=1}^{\infty} \frac{k^{k-2}}{(x+k)^k}$$

and we have

$$f_1(x) = \left(\sum_{k=1}^{\infty} \frac{k^{k-2}}{(x+k)^k}\right)^{-1} - x.$$

Since it is clear that $f_1(x) > 0$ holds for $x > 0$ and $f_1(0) = \frac{6}{\pi^2}$, we will show that $\lim_{x\to+\infty} f_1(x) = 0$ and $f_1(x) < \frac{6}{\pi^2}$ for $x > 0$. First, we may show $\lim_{x\to+\infty} f_1(x) = 0$. From Theorem 2, we have

$$f_1(x) \leqslant \left(\frac{1}{x} - \frac{2e^{-x}}{x(x+2)}\right)^{-1} - x = \frac{2x}{-2+2e^x+e^x x} = f_2(x).$$

The derivative of $f_2(x)$ is

$$f_2'(x) = \frac{2(-2+2e^x-2e^x x-e^x x^2)}{(-2+2e^x+e^x x)^2} < \frac{2(-2+2e^x-2e^x x)}{(-2+2e^x+e^x x)^2} = \frac{2f_3(x)}{(-2+2e^x+e^x x)^2}.$$

The derivative of $f_3(x)$ is $f_3'(x) = -2e^x x < 0$ for $x > 0$ and $f_3(x)$ is monotonically decreasing for $x > 0$. By $f_3(x) < f_3(0) = 0$ for $x > 0$, we have $f_2'(x) < 0$ for $x > 0$ and $f_2(x)$ is monotonically decreasing for $x > 0$. From $f_1(x) > 0$ for $x > 0$ and $\lim_{x\to+\infty} f_2(x) = 0$, we can get $\lim_{x\to+\infty} f_1(x) = 0$. We next show $f_1(x) < \frac{6}{\pi^2}$ for $x > 0$. From $e^x > 1 + x + \frac{x^2}{2}$ for $x > 0$, the following inequality holds for $x > \frac{15}{100}$.

$$f_1(x) < f_2(x) < f_2\left(\frac{15}{100}\right) = \frac{3}{10\left(\frac{43}{20}e^{\frac{3}{20}} - 2\right)} < \frac{3}{10\left(\frac{43}{20}\left(1 + \left(\frac{3}{20}\right) + \frac{1}{2}\left(\frac{3}{20}\right)^2\right) - 2\right)}$$
$$= \frac{1600}{2649} < \frac{800}{1323} = \frac{6}{\left(\frac{315}{100}\right)^2} < \frac{6}{\pi^2}.$$

Thus, we obtain $f_1(x) < \frac{6}{\pi^2}$ for $x > \frac{15}{100}$. The derivative of $f_1(x)$ is

$$f_1'(x) = \frac{\sum_{k=1}^{\infty} \frac{k^{k-1}}{(x+k)^{k+1}}}{\left(\sum_{k=1}^{\infty} \frac{k^{k-2}}{(x+k)^k}\right)^2} - 1$$

and we consider the sign of the following function $f_4(x)$.

$$\left(\sum_{k=1}^{\infty} \frac{k^{k-2}}{(x+k)^k}\right)^2 f_1'(x) = \sum_{k=1}^{\infty} \frac{k^{k-1}}{(x+k)^{k+1}} - \left(\sum_{k=1}^{\infty} \frac{k^{k-2}}{(x+k)^k}\right)^2 = f_4(x).$$

Here, we have

$$f_4(x) < \sum_{k=1}^{\infty} \frac{k^{k-1}}{(0+k)^{k+1}} - \left(\frac{1}{x} - \frac{2e^{-x}}{x(2+x)}\right)^2 = \frac{\pi^2}{6} - \left(\frac{1}{x} - \frac{2e^{-x}}{x(2+x)}\right)^2 = f_5(x)$$

and the derivative of $f_5(x)$ is

$$f_5'(x) = \frac{2e^{-2x}(-2+2e^x+e^x x)(-4+4e^x-8x+4e^x x-2x^2+e^x x^2)}{x^3(2+x)^3}.$$

Since we have

$$\begin{aligned} x^3(2+x)^3 f_5'(x) &= 2e^{-2x}(-2+2e^x+e^x x)(-4+4e^x-8x+4e^x x-2x^2+e^x x^2) \\ &> 2e^{-2x}(-2+2e^0+e^x x)\left(-4+4\left(1+x+\frac{x^2}{2}\right)-8x+4e^0 x-2x^2+e^0 x^2\right) \\ &= 2e^{-x}x^3 > 0, \end{aligned}$$

$f_5(x)$ is monotonically increasing for $0 < x < \frac{15}{100}$. From $e^x > 1+x+\frac{x^2}{2}$ for $x > 0$, we have

$$\begin{aligned} &\frac{\pi}{\sqrt{6}} - \left(\frac{20}{3} - \frac{800}{129e^{\frac{3}{20}}}\right) < \frac{\pi}{\sqrt{6}} - \frac{20}{3} + \frac{800}{129\left(1+\left(\frac{3}{20}\right)+\frac{1}{2}\left(\frac{3}{20}\right)^2\right)} \\ &= \frac{\pi}{\sqrt{6}} - \frac{52980}{39947} < \frac{\frac{315}{100}}{\frac{141}{100}\cdot\frac{173}{100}} - \frac{52980}{39947} = -\frac{11336880}{324809057} < 0. \end{aligned}$$

Hence, we have $f_5\left(\frac{15}{100}\right) = \frac{\pi^2}{6} - \left(\frac{20}{3} - \frac{800}{129e^{3/20}}\right)^2 < 0$ for $0 < x < \frac{15}{100}$. Therefore, $f_4(x) < 0$ for $0 < x < \frac{15}{100}$ and $f_1(x)$ is monotonically decreasing for $0 < x < \frac{15}{100}$. Thus, we can get $f_1(x) < f_1(0) = \frac{6}{\pi^2}$ for $0 < x < \frac{15}{100}$. This completes the proof of Theorem 3. □

Proof of Theorem 4. We consider the function

$$g_1(x) = \frac{1}{x+\frac{6}{\pi^2}} - \left(\frac{1}{x} - \frac{2e^{-x}}{x(x+2)}\right)$$

and the derivative of $g_1(x)$ is

$$g_1'(x) = \frac{12(3+\pi^2 x)}{x^2(6+\pi^2 x)^2} - \frac{2(2+4x+x^2)}{e^x x^2(2+x)^2}.$$

Since the inequality $e^x < \frac{2+x}{2-x}$ holds for $0 < x < 2$ (see [5] in pp 269), we have

$$\begin{aligned} x^2 g_1'(x) &= \frac{12(3+\pi^2 x)}{(6+\pi^2 x)^2} - \frac{2(2+4x+x^2)}{e^x(2+x)^2} \leqslant \frac{12(3+\pi^2 x)}{(6+\pi^2 x)^2} - \frac{2(2+4x+x^2)}{\left(\frac{2+x}{2-x}\right)(2+x)^2} \\ &= \frac{2x^2 g_2(x)}{(2+x)^3(6+\pi^2 x)^2}, \end{aligned}$$

where $g_2(x) = 180 - 4\pi^4 + 54x + 60\pi^2 x - 6\pi^4 x + 18\pi^2 x^2 + 2\pi^4 x^2 + \pi^4 x^3$. Since we have

$$\begin{aligned} g_2(x) <& 180 - 4\left(\frac{314}{100}\right)^4 + 54x + 60\left(\frac{315}{100}\right)^2 x - 6\left(\frac{314}{100}\right)^4 x \\ &+ 18\left(\frac{315}{100}\right)^2 x + 2\left(\frac{315}{100}\right)^4 x + \left(\frac{315}{100}\right)^4 x \\ =& -\frac{326323201}{1562500} + \frac{54005274579}{100000000} x \\ <& -\frac{326323201}{1562500} + \frac{54005274579}{100000000}\left(\frac{3}{10}\right) = -\frac{46831024903}{1000000000} < 0 \end{aligned}$$

for $0 < x < \frac{3}{10}$, $g_2(x) < 0$ for $0 < x < \frac{3}{10}$ and $g_1(x)$ is monotonically decreasing for $0 < x < \frac{3}{10}$. From $e^x < \frac{2+x}{2-x}$ for $0 < x < 2$, we have

$$\begin{aligned} g_1(x) \geqslant g_1\left(\frac{13}{100}\right) &= \frac{1}{\frac{13}{100} + \frac{6}{\pi^2}} - \frac{100}{13} + \frac{20000}{2769 e^{\frac{13}{100}}} \\ &> \frac{1}{\frac{13}{100} + \frac{6}{\left(\frac{314}{100}\right)^2}} - \frac{100}{13} + \frac{20000}{2769\left(\frac{2+\frac{13}{100}}{2-\frac{13}{100}}\right)} = \frac{237260000}{82591406253} > 0 \end{aligned}$$

for $0 < x \leqslant \frac{13}{100}$. Therefore, this completes the proof of Theorem 4. □

Proof of Theorem 5. We consider the functions $f_1(x)$ and $f_2(x)$ in the proof of Theorem 3. Since the function $f_2(x)$ is monotonically decreasing for $x > 0$, we have $f_1(x) < f_2(a) = \frac{2a}{-2+2e^a+ae^a}$ for $x > a$. Also, from Theorem 2, we have

$$f_1(x) \geqslant \left(\frac{1}{x} - \frac{2e^{-x}}{x\left(x+\frac{8}{3}\right)}\right)^{-1} - x = \frac{2x}{-2+\frac{8}{3}e^x + e^x x} = h_1(x)$$

and the derivative of $h_1(x)$ is

$$h_1'(x) = \frac{6h_2(x)}{(-6+(8+3x)e^x)^2},$$

where $h_2(x) = -6 + (8 - 8x - 3x^2)e^x$. Since the derivative of $h_2(x)$ is $h_2'(x) = -e^x x(14 + 3x) < 0$, $h_2(x)$ is monotonically decreasing for $x > 0$. From

$$h_2\left(\frac{1}{2}\right) = \frac{13\sqrt{e}-24}{4} < \frac{13\sqrt{3}-24}{4} < \frac{13\cdot\frac{174}{100}-24}{4} = -\frac{69}{200} < 0,$$

$h_2(x) < 0$ for $x > \frac{1}{2}$ and $h_1(x)$ is monotonically decreasing for $x > \frac{1}{2}$. Hence, we have $h_1(b) = \frac{6b}{-6+8e^b+3be^b} < h_1(x)$ for $\frac{1}{2} < x < b$. This completes the proof of Theorem 5. □

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