Journal of Mathematical Inequalities
Volume 19, Number 2 (2025), 523–537
doi:10.7153/jmi-2025-19-33

CONSTRUCTION CONDITIONS OF BOUNDED OPERATORS WITH SUPER–HOMOGENEOUS KERNELS IN HIGH–DIMENSIONAL SPACE AND THE ESTIMATION OF OPERATOR NORM

QIAN ZHAO, BING HE* AND YONG HONG

(*Communicated by L. Mihoković*)

Abstract. The concept of super-homogeneous functions is introduced, and the construction conditions of bounded operators with super-homogeneous kernels between high-dimensional weighted Lebesgue function spaces and weighted Hilbert-type sequence spaces are discussed. The sufficient and necessary conditions for operator boundedness and the estimation formula for operator norm are obtained, and the exact calculation formula for operator norm is obtained under specific conditions. Finally, several special cases of applications are presented.

1. Introduction

Assuming $\frac{1}{p}+\frac{1}{q}=1$ $(p>1)$, $\tilde{a}=\{a_n\}\in l_p^{p-2}$, $f\in L_q(0,+\infty)$, $K(u)$ monotonically decreases on $(0,+\infty)$. In 1934, Hardy [4] gave the following Hilbert-type inequality with a non-homogeneous kernel $K(nx)$:

$$\int_0^{+\infty}\sum_{n=1}^{\infty}K(nx)a_nf(x)\mathrm{d}x\leqslant\phi\left(\frac{1}{p}\right)\left(\sum_{n=1}^{\infty}n^{p-2}|a_n|^p\right)^{1/p}\left(\int_0^{+\infty}|f(x)|^q\mathrm{d}x\right)^{1/q}$$
$$=\phi\left(\frac{1}{p}\right)\|\tilde{a}\|_{p,p-2}\|f\|_q. \tag{1}$$

Consider a discrete operator T_1 and an integral operator T_2 with the kernel $K(nx)$:

$$T_1(\tilde{a})=\sum_{n=1}^{\infty}K(nx)a_n,\quad T_2(f)_n=\int_0^{+\infty}K(nx)f(x)\mathrm{d}x.$$

Then (1) is equivalent to the following inequalities

$$\|T_1(\tilde{a})\|_p\leqslant\phi\left(\frac{1}{p}\right)\|\tilde{a}\|_{p,p-2},\quad \|T_2(f)\|_{q,q-2}\leqslant\phi\left(\frac{1}{p}\right)\|f\|_q. \tag{2}$$

Mathematics subject classification (2020): 26D15.
Keywords and phrases: Super-homogeneous kernel, weighted Lebesgue space, weighted Hilbert-type space, construction of bounded operators, estimation of operator norm.

* Corresponding author.

Paper JMI-19-33

(2) indicates that T_1 is a bounded operator from l_p^{p-2} to $L_p(0,+\infty)$ and T_2 is a bounded operator from $L_q(0,+\infty)$ to l_q^{q-2}. T_1 implements a bounded mapping from sequence space l_p^{p-2} to function space $L_p(0,+\infty)$, and T_2 implements a bounded mapping from function space $L_q(0,+\infty)$ to sequence space l_q^{q-2}.

Due to its important applications in operator theory and analytical disciplines, Hilbert-type inequalities have received widespread attention. [3,11,14–16,19,20,23,24] introduced independent parameters to generalize and improve Hilbert-type inequalities, while [1,5,6,12,13,22] discussed some high-dimensional Hilbert-type inequalities. In order to further explore the Hilbert-type inequality, [7,10,18,21] considered the optimal parameter matching problem of Hilbert-type inequality and sought the rule of optimal parameters, while [2,8,17] discussed the conditions for constructing Hilbert-type inequalities and bounded operators.

In this article, we will discuss the construction of bounded operators with super-homogeneous kernels in high-dimensional spaces. To this end, we first generalize the Lebesgue space to a weighted high-dimensional Lebesgue space: Let $r>1$, $\alpha\in\mathbb{R}$, $m\in\mathbb{N}_+:=\{1,2,3,\cdots\}$, $x=(x_1,x_2,\cdots,x_m)\in\mathbb{R}_+^m$, $\rho>0$, $\|x\|_{\rho,m}=\left(x_1^\rho+x_2^\rho+\cdots+x_m^\rho\right)^{1/\rho}$. Define

$$L_r^\alpha\left(\mathbb{R}_+^m\right)=\left\{f(x):\|f\|_{r,\alpha}=\left(\int_{\mathbb{R}_+^m}\|x\|_{\rho,m}^\alpha|f(x)|^r\mathrm{d}x\right)^{1/r}<+\infty\right\}.$$

When $m=1,\alpha=0$, $L_r^\alpha\left(\mathbb{R}_+^m\right)$ is the usual Lebesgue space $L_r(0,+\infty)$. We call

$$l_r^\alpha=\left\{\tilde{a}=\{a_n\}:\|\tilde{a}\|_{r,\alpha}=\left(\sum_{n=1}^\infty n^\alpha|a_n|^r\right)^{1/r}<+\infty\right\}$$

a weighted Hilbert-type space.

Let $\sigma_1,\sigma_2,\tau_1,\tau_2\in\mathbb{R}$, and function $K(u,v)$ satisfy the following condition for any $t>0$:

$$K(tu,v)=t^{\sigma_1}K\left(u,t^{\tau_1}v\right),\quad K(u,tv)=t^{\sigma_2}K\left(t^{\tau_2}u,v\right).$$

Then $K(u,v)$ is called a super-homogeneous function with parameters $\{\sigma_1,\sigma_2,\ \tau_1,\tau_2\}$. When $\sigma_1=\sigma_2=\lambda$, $\tau_1=\tau_2=-1$, $K(u,v)$ is a typical λ-order homogeneous function. Super-homogeneous functions are a natural extension of homogeneous functions, which include most common kernel functions. Specifically, there holds $K(t,1)=t^{\sigma_1}K\left(1,t^{\tau_1}\right)$, $K(1,t)=t^{\sigma_2}K\left(t^{\tau_2},1\right)$. Since

$$K(tu,v)=t^{\sigma_1}K\left(u,t^{t_1}v\right)=t^{\sigma_1+\tau_1\sigma_2}K\left(t^{\tau_1\tau_2}u,v\right),$$

it can be seen that in the general case, there hold $\tau_1\tau_2=1$ and $\sigma_1+\tau_1\sigma_2=0$. Therefore, in our discussions, we always assume that $\tau_1\tau_2=1$, $\sigma_1+\tau_1\sigma_2=0$.

For the sake of simplicity, denote

$$W_1(s)=\int_0^{+\infty}K(1,t)t^s\mathrm{d}t,\ W_2(s)=\int_0^{+\infty}K(t,1)t^s\mathrm{d}t,$$

$$\widetilde{A}(K,\tilde{a},f)=\int_{\mathbb{R}_+^m}\sum_{n=1}^\infty K\left(n,\|x\|_{\rho,m}\right)a_nf(x)\mathrm{d}x.$$

2. Preliminary lemmas

LEMMA 1. *Let* $\tau_1\tau_2 = 1$. *Then* $\sigma_1 + \tau_1\sigma_2 = 0$ *is equivalent to* $\sigma_2 + \tau_2\sigma_1 = 0$.

Proof. If $\sigma_1 + \tau_1\sigma_2 = 0$, then $\tau_2\sigma_1 + \tau_1\tau_2\sigma_2 = 0$. Hence $\sigma_2 + \tau_2\sigma_1 = 0$. Conversely, if $\sigma_2 + \tau_2\sigma_1 = 0$, then $\tau_1\sigma_2 + \tau_1\tau_2\sigma_1 = 0$. Thus $\sigma_1 + \tau_1\sigma_2 = 0$. □

LEMMA 2. *Let* $\tau_1\tau_2 = 1$, $\sigma_1 + \tau_1\sigma_2 = 0$. *Then* $\tau_1\left(\frac{m}{p} - \frac{\beta}{q}\right) - \left(\frac{1}{q} - \frac{\alpha}{p}\right) = \sigma_1$ *is equivalent to* $\tau_2\left(\frac{1}{q} - \frac{\alpha}{p}\right) - \left(\frac{m}{p} - \frac{\beta}{q}\right) = \sigma_2$.

Proof. It follows from Lemma 1 that $\sigma_2 + \tau_2\sigma_1 = 0$.

If $\tau_1\left(\frac{m}{p} - \frac{\beta}{q}\right) - \left(\frac{1}{q} - \frac{\alpha}{p}\right) = \sigma_1$, then $\tau_1\tau_2\left(\frac{m}{p} - \frac{\beta}{q}\right) - \tau_2\left(\frac{1}{q} - \frac{\alpha}{p}\right) = \tau_2\sigma_1 = -\sigma_2$. Hence $\tau_2\left(\frac{1}{q} - \frac{\alpha}{p}\right) - \left(\frac{m}{p} - \frac{\beta}{q}\right) = \sigma_2$. Conversely, if $\tau_2\left(\frac{1}{q} - \frac{\alpha}{p}\right) - \left(\frac{m}{p} - \frac{\beta}{q}\right) = \sigma_2$, then $\tau_1\tau_2\left(\frac{1}{q} - \frac{\alpha}{p}\right) - \tau_1\left(\frac{m}{p} - \frac{\beta}{q}\right) = \tau_1\sigma_2 = -\sigma_1$. Thus $\tau_1\left(\frac{m}{p} - \frac{\beta}{q}\right) - \left(\frac{1}{q} - \frac{\alpha}{p}\right) = \sigma_1$. □

LEMMA 3. *Suppose that* $\frac{1}{p} + \frac{1}{q} = 1$ $(p > 1)$, $\tau_1\tau_2 = 1$, $\sigma_1 + \tau_1\sigma_2 = 0$, $K(u,v)$ *is a super-homogeneous function with parameters* $\{\sigma_1, \sigma_2, \tau_1, \tau_2\}$, *and* $\tau_1\left(\frac{m}{p} - \frac{\beta}{q}\right) - \left(\frac{1}{q} - \frac{\alpha}{p}\right) - \sigma_1 = c$. *Then*

$$W_1^{\frac{1}{p}}\left(\frac{m}{p} - \frac{\beta}{q} - 1\right) W_2^{\frac{1}{q}}\left(\frac{1}{q} - \frac{\alpha}{p} - 1 + c\right) = \frac{1}{|\tau_1|^{\frac{1}{q}}} W_1\left(\frac{m}{p} - \frac{\beta}{q} - 1\right). \tag{3}$$

Proof. Since $\tau_1\left(\frac{m}{p} - \frac{\beta}{q}\right) - \left(\frac{1}{q} - \frac{\alpha}{p}\right) - \sigma_1 = c$, it follows that

$$\begin{aligned}
W_1\left(\frac{m}{p} - \frac{\beta}{q} - 1\right) &= \int_0^{+\infty} K(1,t) t^{\frac{m}{p} - \frac{\beta}{q} - 1}\, \mathrm{d}t \\
&= \int_0^{+\infty} K(t^{\tau_2}, 1) t^{\sigma_2 + \frac{m}{p} - \frac{\beta}{q} - 1}\, \mathrm{d}t \\
&= \frac{1}{|\tau_2|} \int_0^{+\infty} K(u,1) u^{\frac{1}{\tau_2}\left(\sigma_2 + \frac{m}{p} - \frac{\beta}{q} - 1\right) + \frac{1}{\tau_2} - 1}\, \mathrm{d}u \\
&= |\tau_1| \int_0^{+\infty} K(u,1) u^{\tau_1\sigma_2 + \tau_1\left(\frac{m}{p} - \frac{\beta}{q}\right) - 1}\, \mathrm{d}u \\
&= |\tau_1| \int_0^{+\infty} K(u,1) u^{-\sigma_1 + \frac{1}{q} - \frac{\alpha}{p} + \sigma_1 + c - 1}\, \mathrm{d}u \\
&= |\tau_1| \int_0^{+\infty} K(u,1) u^{\frac{1}{q} - \frac{\alpha}{p} - 1 + c}\, \mathrm{d}u \\
&= \frac{1}{|\tau_2|} W_2\left(\frac{1}{q} - \frac{\alpha}{p} - 1 + c\right).
\end{aligned}$$

Therefore, $W_2\left(\frac{1}{q} - \frac{\alpha}{p} - 1 + c\right) = \frac{1}{|\tau_1|} W_1\left(\frac{m}{p} - \frac{\beta}{q} - 1\right)$, and (3) can be obtained. □

LEMMA 4. [9] *Let* $m \in \mathbb{N}_+,\ \rho > 0,\ r > 0,\ x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}_+^m,\ \|x\|_{\rho,m} = \left(x_1^\rho + x_2^\rho + \cdots + x_m^\rho\right)^{1/p},\ \varphi(u)$ *be a measurable function. Then*

$$\int_{\|x\|_{\rho,m} \leqslant r} \varphi\left(\|x\|_{\rho,m}\right) \mathrm{d}x = \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \int_0^r \varphi(u) u^{m-1}\, \mathrm{d}u,$$
$$\int_{\|x\|_{\rho,m} \geqslant r} \varphi\left(\|x\|_{\rho,m}\right) \mathrm{d}x = \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \int_r^{+\infty} \varphi(u) u^{m-1}\, \mathrm{d}u,$$
$$\int_{\mathbb{R}_+^m} \varphi\left(\|x\|_{\rho,m}\right) \mathrm{d}x = \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \int_0^{+\infty} \varphi(u) u^{m-1}\, \mathrm{d}u,$$

where $\Gamma(t)$ *is the Gamma function.*

LEMMA 5. *Suppose that* $\frac{1}{p} + \frac{1}{q} = 1\ (p > 1),\ m \in \mathbb{N}_+,\ \rho > 0,\ x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}_+^m,\ \|x\|_{\rho,m} = \left(x_1^\rho + x_2^\rho + \cdots + x_m^\rho\right)^{1/\rho},\ K(u,v) > 0$ *is a super-homogeneous function with parameters* $\{\sigma_1, \sigma_2, \tau_1, \tau_2\},\ K(t,1) t^{\frac{1}{q} - \frac{\alpha}{p} - 1 + c}$ *monotonically decreases on* $(0, +\infty)$. *Then*

$$\begin{aligned}\widetilde{\omega}_1(n,\beta) &= \int_{\mathbb{R}_+^m} K\left(n, \|x\|_{\rho,m}\right) \|x\|_{\rho,m}^{-\frac{\beta+m}{q}} \mathrm{d}x \\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} n^{\sigma_1 + \tau_1\left(\frac{\beta}{q} - \frac{m}{p}\right)} W_1\left(\frac{m}{p} - \frac{\beta}{q} - 1\right),\end{aligned}$$

$$\begin{aligned}\widetilde{\omega}_2(x,\alpha) &= \sum_{n=1}^{\infty} K\left(n, \|x\|_{\rho,m}\right) n^{-\frac{\alpha+1}{p} + c} \\ &\leqslant \|x\|_{\rho,m}^{\sigma_2 + \tau_2\left(\frac{\alpha}{p} - \frac{1}{q} - c\right)} W_2\left(\frac{1}{q} - \frac{\alpha}{p} - 1 + c\right).\end{aligned}$$

Proof. It follows from Lemma 4 that

$$\begin{aligned}\tilde{\omega}_1(n,\beta) &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \int_0^{+\infty} K(n,u) u^{-\frac{\beta+m}{q} + m - 1} \mathrm{d}u \\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} n^{\sigma_1} \int_0^{+\infty} K\left(1, u \cdot n^{\tau_1}\right) u^{\frac{m}{p} - \frac{\beta}{q} - 1} \mathrm{d}u \\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} n^{\sigma_1 + \tau_1\left(\frac{\beta}{q} - \frac{m}{p}\right)} \int_0^{+\infty} K(1,t) t^{\frac{m}{p} - \frac{\beta}{q} - 1} \mathrm{d}t \\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} n^{\sigma_1 + \tau_1\left(\frac{\beta}{q} - \frac{m}{p}\right)} W_1\left(\frac{m}{p} - \frac{\beta}{q} - 1\right).\end{aligned}$$

Since $K(t,1)t^{\frac{1}{q}-\frac{\alpha}{p}-1+c}$ monotonically decreases on $(0,+\infty)$, it can be seen that

$$
\begin{aligned}
\tilde{\omega}_2(x,\alpha) &= \sum_{n=1}^{\infty} K\left(n,\|x\|_{\rho,m}\right) n^{\frac{1}{q}-\frac{\alpha}{p}-1+c} \\
&= \|x\|_{\rho,m}^{\sigma_2} \sum_{n=1}^{\infty} K\left(n\cdot\|x\|_{\rho,m}^{\tau_2},1\right) n^{\frac{1}{q}-\frac{\alpha}{p}-1+c} \\
&= \|x\|_{\rho,m}^{\sigma_2-\tau_2(\frac{1}{q}-\frac{\alpha}{p}-1+c)} \sum_{n=1}^{\infty} K\left(n\cdot\|x\|_{\rho,m}^{\tau_2},1\right)\left(n\cdot\|x\|_{\rho,m}^{\tau_2}\right)^{\frac{1}{q}-\frac{\alpha}{p}-1+c} \\
&\leqslant \|x\|_{\rho,m}^{\sigma_2+\tau_2\left(\frac{\alpha}{p}-\frac{1}{q}+1-c\right)} \int_0^{+\infty} K\left(u\cdot\|x\|_{\rho,m}^{\tau_2},1\right)\left(u\cdot\|x\|_{\rho,m}^{\tau_2}\right)^{\frac{1}{q}-\frac{\alpha}{p}-1+c}\mathrm{d}u \\
&= \|x\|_{\rho,m}^{\sigma_2+\tau_2\left(\frac{\alpha}{p}-\frac{1}{q}-c\right)} \int_0^{+\infty} K(t,1)t^{\frac{1}{q}-\frac{\alpha}{p}-1+c}\mathrm{d}t \\
&= \|x\|_{\rho,m}^{\sigma_2+\tau_2\left(\frac{\alpha}{p}-\frac{1}{q}-c\right)} W_2\left(\frac{1}{q}-\frac{\alpha}{p}-1+c\right). \quad \square
\end{aligned}
$$

3. Construction conditions for high-dimensional half-discrete Hilbert-type inequalities with super-homogeneous kernels

THEOREM 1. *Assuming* $\frac{1}{p}+\frac{1}{q}=1$ $(p>1)$, $m\in\mathbb{N}_+$, $\rho>0$, $\alpha,\beta\in\mathbb{R}$, $\tau_1\tau_2=1$, $\sigma_1+\tau_1\sigma_2=0$, $\tau_1\left(\frac{m}{p}-\frac{\beta}{q}\right)-\left(\frac{1}{q}-\frac{\alpha}{p}\right)-\sigma_1=c$, $x=(x_1,x_2,\cdots,x_m)\in\mathbb{R}_+^m$, $\|x\|_{\rho,m}=\left(x_1^{\rho}+x_2^{\rho}+\cdots+x_m^{\rho}\right)^{1/\rho}$, $K(u,v)>0$ *is a super-homogeneous function with parameters* $\{\sigma_1,\sigma_2,\tau_1,\tau_2\}$, $K(t,1)t^{\frac{1}{q}-\frac{\alpha}{p}-1+c}$ *monotonically decreases on* $(0,+\infty)$, *and* $W_1\left(\frac{m}{p}-\frac{\beta}{q}-1\right)<+\infty$.

(i) If and only if $c\geqslant 0$, *i.e.,* $\tau_1\left(\frac{m}{p}-\frac{\beta}{q}\right)-\left(\frac{1}{q}-\frac{\alpha}{p}\right)\geqslant\sigma_1$, *there exists a constant* $M>0$, *such that*

$$
\int_{\mathbb{R}_+^m}\sum_{n=1}^{\infty} K\left(n,\|x\|_{\rho,m}\right)a_n f(x)\mathrm{d}x \leqslant M\|\tilde{a}\|_{p,\alpha}\|f\|_{q,\beta}, \tag{4}
$$

where $\tilde{a}=\{a_n\}\in l_p^{\alpha}$, $f\in L_q^{\beta}\left(\mathbb{R}_+^m\right)$, $M\geqslant\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{1}{|\tau_1|}\right)^{1/q}W_1\left(\frac{m}{p}-\frac{\beta}{q}-1\right)$.

(ii) When $c=0$, *i.e.,* $\tau_1\left(\frac{m}{p}-\frac{\beta}{q}\right)-\left(\frac{1}{q}-\frac{\alpha}{p}\right)=\sigma_1$, *the optimal constant factor of (4) is*

$$
M_0=\inf\{M\}=\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{1}{|\tau_1|}\right)^{1/q}W_1\left(\frac{m}{p}-\frac{\beta}{q}-1\right).
$$

Proof. (i) *Sufficiency.* Let $c\geqslant 0$. It follows from the mixed Hölder's inequality, Lemma 5, and Lemma 3 that

$$
\begin{aligned}
&\widetilde{A}(K,\tilde{a},f) = \int_{\mathbb{R}^m_+}\sum_{n=1}^{\infty} K\left(n,\|x\|_{\rho,m}\right) a_n f(x)\mathrm{d}x \\
&= \int_{\mathbb{R}^m_+}\sum_{n=1}^{\infty}\left(\frac{n^{(\alpha+1-cp)/(pq)}}{\|x\|_{\rho,m}^{(\beta+m)/(pq)}}a_n\right)\left(\frac{\|x\|_{\rho,m}^{(\beta+m)/(pq)}}{n^{(\alpha+1-cp)/(pq)}}f(x)\right)K\left(n,\|x\|_{\rho,m}\right)\mathrm{d}x \\
&\leqslant \left(\int_{\mathbb{R}^m_+}\sum_{n=1}^{\infty} n^{(\alpha+1-cp)/q}\|x\|_{\rho,m}^{-(\beta+m)/q}|a_n|^p K\left(n,\|x\|_{\rho,m}\right)\mathrm{d}x\right)^{1/p} \\
&\quad\times\left(\int_{\mathbb{R}^m_+}\sum_{n=1}^{\infty}\|x\|_{p,m}^{(\beta+m)/p} n^{-(\alpha+1-cp)/p}|f(x)|^q K\left(n,\|x\|_{\rho,m}\right)\mathrm{d}x\right)^{1/q} \\
&= \left(\sum_{n=1}^{\infty} n^{(\alpha+1-cp)/q}|a_n|^p\widetilde{\omega}_1(n,\beta)\right)^{1/p}\left(\int_{\mathbb{R}^m_+}\|x\|_{\rho,m}^{(\beta+m)/p}|f(x)|^q\widetilde{\omega}_2(x,\alpha)\mathrm{d}x\right)^{1/q} \\
&\leqslant \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p} W_1^{1/p}\left(\frac{m}{p}-\frac{\beta}{q}-1\right)W_2^{1/q}\left(\frac{1}{q}-\frac{\alpha}{p}-1+c\right) \\
&\quad\times\left(\sum_{n=1}^{\infty} n^{\frac{\alpha+1-cp}{q}+\sigma_1+\tau_1\left(\frac{\beta}{q}-\frac{m}{p}\right)}|a_n|^p\right)^{1/p}\left(\int_{\mathbb{R}^m_+}\|x\|_{\rho,m}^{\frac{\beta+1}{p}+\sigma_2+\tau_2\left(\frac{\alpha}{p}-\frac{1}{q}-c\right)}|f(x)|^q\mathrm{d}x\right)^{1/q} \\
&= \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{1}{|\tau_1|}\right)^{1/q} W_1\left(\frac{m}{p}-\frac{\beta}{q}-1\right) \\
&\quad\times\left(\sum_{n=1}^{\infty} n^{\alpha-cp}|a_n|^p\right)^{1/p}\left(\int_{\mathbb{R}^m_+}\|x\|_{\rho,m}^{\beta}|f(x)|^q\mathrm{d}x\right)^{1/q} \\
&\leqslant \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{1}{|\tau_1|}\right)^{1/q} W_1\left(\frac{m}{p}-\frac{\beta}{q}-1\right) \\
&\quad\times\left(\sum_{n=1}^{\infty} n^{\alpha}|a_n|^p\right)^{1/p}\left(\int_{\mathbb{R}^m_+}\|x\|_{\rho,m}^{\beta}|f(x)|^q\mathrm{d}x\right)^{1/q} \\
&= \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{1}{|\tau_1|}\right)^{1/q} W_1\left(\frac{m}{p}-\frac{\beta}{q}-1\right)\|\tilde{a}\|_{p,\alpha}\|f\|_{q,\beta}.
\end{aligned}
$$

Arbitrarily taking $M \geqslant \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{1}{|\tau_1|}\right)^{1/q} W_1\left(\frac{m}{p}-\frac{\beta}{q}-1\right)$, (4) can be obtained.

Necessity. Supposing (4) holds. If $c<0$, let's first discuss the case where $\tau_1<0$. At this point, take $0<\varepsilon<\frac{c}{\tau_1}$. Set $a_n = n^{-(\alpha+1-\tau_1\varepsilon)/p}$ $(n=1,2,\cdots)$, and

$$
f(x) = \begin{cases} \|x\|_{\rho,m}^{-(\beta+m+\varepsilon)/q}, & \|x\|_{\rho,m}\geqslant 1, \\ 0, & \|x\|_{\rho,m}<1. \end{cases}
$$

It follows from Lemma 4 that

$$\begin{aligned}\|\tilde{a}\|_{p,\alpha}\|f\|_{q,\beta} &= \left(\sum_{n=1}^{\infty} n^{-1+\tau_1\varepsilon}\right)^{1/p} \left(\int_{\|x\|\rho,m\geqslant 1} \|x\|_{\rho,m}^{-m-\varepsilon}\mathrm{d}x\right)^{1/q} \\ &\leqslant \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q} \left(1+\int_1^{+\infty} t^{-1+\tau_1\varepsilon}\mathrm{d}t\right)^{1/p} \left(\int_1^{+\infty} u^{-1-\varepsilon}\mathrm{d}u\right)^{1/q} \\ &= \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p} \frac{1}{\varepsilon}\left(1+\frac{1}{|\tau_1|}\right)^{1/p}.\end{aligned}$$

Again by Lemma 4, we obtain

$$\begin{aligned}&\widetilde{A}(K,\tilde{a},f) = \sum_{n=1}^{\infty} n^{-\frac{\alpha+1}{p}+\frac{\tau_1\varepsilon}{p}} \left(\int_{\|x\|_{\rho,m}\geqslant 1} K\left(n,\|x\|_{\rho,m}\right)\|x\|_{\rho,m}^{-\frac{\beta+m}{q}-\frac{\varepsilon}{q}}\mathrm{d}x\right) \\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \sum_{n=1}^{\infty} n^{\frac{1}{q}-\frac{\alpha}{p}-1+\frac{\tau_1\varepsilon}{p}} \left(\int_1^{+\infty} K(n,u)u^{-\frac{\beta+m}{q}-\frac{\varepsilon}{q}+m-1}\mathrm{d}u\right) \\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \sum_{n=1}^{\infty} n^{\sigma_1+\frac{1}{q}-\frac{\alpha}{p}-1+\frac{\tau_1\varepsilon}{p}} \left(\int_1^{+\infty} K(1,u\cdot n^{\tau_1})u^{\frac{m}{p}-\frac{\beta}{q}-1-\frac{\varepsilon}{q}}\mathrm{d}u\right) \\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \sum_{n=1}^{\infty} n^{\sigma_1+\frac{1}{q}-\frac{\alpha}{p}-1+\frac{\tau_1\varepsilon}{p}-\tau_1\left(\frac{m}{p}-\frac{\beta}{q}\right)+\frac{\tau_1\varepsilon}{q}} \left(\int_{n^{\tau_1}}^{+\infty} K(1,t)t^{\frac{m}{p}-\frac{\beta}{q}-1-\frac{\varepsilon}{q}}\mathrm{d}t\right) \\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \sum_{n=1}^{\infty} n^{-1-c+\tau_1\varepsilon} \left(\int_{n^{\tau_1}}^{+\infty} K(1,t)t^{\frac{m}{p}-\frac{\beta}{q}-1-\frac{\varepsilon}{q}}\mathrm{d}t\right) \\ &\geqslant \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \sum_{n=1}^{\infty} \frac{1}{n^{1+c-\tau_1\varepsilon}} \int_1^{+\infty} K(1,t)t^{\frac{m}{p}-\frac{\beta}{q}-1-\frac{\varepsilon}{q}}\mathrm{d}t.\end{aligned}$$

It follows that

$$\begin{aligned}&\sum_{n=1}^{\infty} \frac{1}{n^{1+c-\tau_1\varepsilon}} \int_1^{+\infty} K(1,t)t^{\frac{m}{p}-\frac{\beta}{q}-1-\frac{\varepsilon}{q}}\mathrm{d}t \\ &\leqslant \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{-1/p} \frac{M}{\varepsilon}\left(1+\frac{1}{|\tau_1|}\right)^{1/p} < +\infty. \qquad (5)\end{aligned}$$

Since $c<0,\ \tau_1<0,\ 0<\varepsilon<\frac{c}{\tau_1}$, we get $1+c-\tau_1\varepsilon<1$, thus $\sum_{n=1}^{\infty}\frac{1}{n^{1+c-\tau_1\varepsilon}}=+\infty$, which contradicts (5).

Further discuss the case where $\tau_1>0$. Take $0<\varepsilon<\frac{-c}{\tau_1}$. Let $a_n = n^{-(\alpha+1+\tau_1\varepsilon)/p}$ $(n=1,2,\cdots)$, and

$$f(x)=\begin{cases}\|x\|_{\rho,m}^{-(\beta+m-\varepsilon)/q}, & \|x\|_{\rho,m}\leqslant 1,\\ 0, & \|x\|_{\rho,m}>1.\end{cases}$$

Then

$$\begin{aligned}\|\tilde{a}\|_{p,\alpha}\|f\|_{q,\beta} &= \left(\sum_{n=1}^{\infty} n^{-1-\tau_1\varepsilon}\right)^{1/p}\left(\int_{\|x\|\rho,m\leqslant 1}\|x\|_{\rho,m}^{-m+\varepsilon}\mathrm{d}x\right)^{1/q}\\ &\leqslant \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q}\left(1+\int_1^{+\infty} t^{-1-\tau_1\varepsilon}\mathrm{d}t\right)^{1/p}\left(\int_0^1 u^{-1+\varepsilon}\mathrm{d}u\right)^{1/q}\\ &= \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q}\frac{1}{\varepsilon}\left(\varepsilon+\frac{1}{\tau_1}\right)^{1/p},\end{aligned}$$

$$\begin{aligned}\widetilde{A}(K,\tilde{a},f) &= \sum_{n=1}^{\infty} n^{-\frac{\alpha+1}{p}-\frac{\tau_1\varepsilon}{p}}\left(\int_{\|x\|\rho,m\leqslant 1} K\left(n,\|x\|_{\rho,m}\right)\|x\|_{\rho,m}^{-\frac{\beta+m}{q}+\frac{\varepsilon}{q}}\mathrm{d}x\right)\\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\sum_{n=1}^{\infty} n^{\frac{1}{q}-\frac{\alpha}{p}-1-\frac{\tau_1\varepsilon}{p}}\left(\int_0^1 K(n,u)u^{-\frac{\beta+m}{q}+m-1+\frac{\varepsilon}{q}}\mathrm{d}u\right)\\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\sum_{n=1}^{\infty} n^{\sigma_1+\frac{1}{q}-\frac{\alpha}{p}-1-\frac{\tau_1\varepsilon}{p}}\left(\int_0^1 K\left(1,u\cdot n^{\tau_1}\right)u^{\frac{m}{p}-\frac{\beta}{q}-1+\frac{\varepsilon}{q}}\mathrm{d}u\right)\\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\sum_{n=1}^{\infty} n^{-1-c-\tau_1\varepsilon}\left(\int_0^{n^{\tau_2}} K(1,t)t^{\frac{m}{p}-\frac{\beta}{q}-1+\frac{\varepsilon}{q}}\mathrm{d}t\right)\\ &\geqslant \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\sum_{n=1}^{\infty}\frac{1}{n^{1+c+\tau_1\varepsilon}}\int_0^1 K(1,t)t^{\frac{m}{p}-\frac{\beta}{q}-1+\frac{\varepsilon}{q}}\mathrm{d}t.\end{aligned}$$

It follows that

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{1}{n^{1+c+\tau_1\varepsilon}}\int_0^1 K(1,t)t^{\frac{m}{p}-\frac{\beta}{q}-1+\frac{\varepsilon}{q}}\mathrm{d}t\\ &\leqslant \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{-1/p}\frac{M}{\varepsilon}\left(\varepsilon+\frac{1}{\tau_1}\right)^{1/p} < +\infty. \qquad (6)\end{aligned}$$

In view of $c<0$, $\tau_1>0$, and $0<\varepsilon<\frac{-c}{\tau_1}$, we have $1+c+\tau_1\varepsilon<1$. Therefore, $\sum_{n=1}^{\infty}\frac{1}{n^{1+c+\tau_1\varepsilon}}=+\infty$, which contradicts (6).

From the above, it can be concluded that $c\geqslant 0$.

(ii) Let $c=0$. If the optimal constant factor of (4) is not M_0, then there exists a constant $\overline{M}>0$, such that

$$\tilde{A}(K,\tilde{a},f)\leqslant \overline{M}\|\tilde{a}\|_{p,\alpha}\|f\|_{q,\beta},$$

$$\overline{M}<M_0=\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{1}{|\tau_1|}\right)^{1/q}W_1\left(\frac{n}{p}-\frac{\beta}{q}-1\right). \qquad (7)$$

If $\tau_1 < 0$, taking sufficiently small $\varepsilon > 0$ and $\delta > 0$, setting $a_n = n^{-(\alpha+1-\tau_1\varepsilon)/p}$ $(n = 1, 2, \cdots)$, and

$$f(x) = \begin{cases} \|x\|_{\rho,m}^{-(\beta+m+\varepsilon)/q}, & \|x\|_{\rho,m} \geqslant \delta, \\ 0, & 0 < \|x\|_{\rho,m} < \delta. \end{cases}$$

A similar calculation yields

$$\begin{aligned} \|\tilde{a}\|_{p,\alpha}\|f\|_{q,\beta} &= \left(\sum_{n=1}^{\infty} n^{-1+\tau_1\varepsilon}\right)^{1/p} \left(\int_{\|x\|_{\rho,m}\geqslant\delta} \|x\|_{\rho,m}^{-m-\varepsilon} \mathrm{d}x\right)^{1/q} \\ &\leqslant \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q} \left(1 + \int_1^{+\infty} u^{-1+\tau_1\varepsilon} \mathrm{d}u\right)^{1/p} \left(\int_\delta^{+\infty} u^{-1-\varepsilon} \mathrm{d}u\right)^{1/q} \\ &= \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q} \frac{1}{\varepsilon} \left(\varepsilon + \frac{1}{|\tau_1|}\right)^{1/p} \delta^{-\varepsilon/q}, \end{aligned}$$

$$\begin{aligned} \widetilde{A}(K,\tilde{a},f) &= \sum_{n=1}^{\infty} n^{-\frac{\alpha+1}{p}+\frac{\tau_1\varepsilon}{p}} \left(\int_{\|x\|_{\rho,m}\geqslant\delta} K\left(n, \|x\|_{\rho,m}\right) \|x\|_{\rho,m}^{-\frac{\beta+m}{q}-\frac{\varepsilon}{q}} \mathrm{d}x\right) \\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \sum_{n=1}^{\infty} n^{\sigma_1+\frac{1}{q}-\frac{\alpha}{p}-1+\frac{\tau_1\varepsilon}{p}} \left(\int_\delta^{+\infty} K\left(1, u\cdot n^{\tau_1}\right) u^{\frac{m}{p}-\frac{\beta}{q}-1-\frac{\varepsilon}{q}} \mathrm{d}u\right) \\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \sum_{n=1}^{\infty} n^{-1+\tau_1\varepsilon} \left(\int_{\delta n^{\tau_1}}^{+\infty} K(1,t) t^{\frac{m}{p}-\frac{\beta}{q}-1-\frac{\varepsilon}{q}} \mathrm{d}t\right) \\ &\geqslant \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \sum_{n=1}^{\infty} n^{-1+\tau_1\varepsilon} \int_\delta^{+\infty} K(1,t) t^{\frac{m}{p}-\frac{\beta}{q}-1-\frac{\varepsilon}{q}} \mathrm{d}t \\ &\geqslant \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \int_1^{+\infty} t^{-1+\tau_1\varepsilon} \mathrm{d}t \int_\delta^{+\infty} K(1,t) t^{\frac{m}{p}-\frac{\beta}{q}-1-\frac{\varepsilon}{q}} \mathrm{d}t \\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \frac{1}{|\tau_1|\varepsilon} \int_\delta^{+\infty} K(1,t) t^{\frac{m}{p}-\frac{\beta}{q}-1-\frac{\varepsilon}{q}} \mathrm{d}t. \end{aligned}$$

Consequently,

$$\begin{aligned} &\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p} \frac{1}{|\tau_1|} \int_\delta^{+\infty} K(1,t) t^{\frac{m}{p}-\frac{\beta}{q}-1-\frac{\varepsilon}{q}} \mathrm{d}t \\ &\leqslant \overline{M} \left(\varepsilon + \frac{1}{|\tau_1|}\right)^{1/p} \delta^{-\varepsilon/q}. \end{aligned} \tag{8}$$

Without losing scientific nature, consider ε as a decreasing sequence $\{c_k\}$ that converges to 0. It follows from Fatou's lemma that

$$\begin{aligned} \int_\delta^{+\infty} K(1,t) t^{\frac{m}{p}-\frac{\beta}{q}-1} \mathrm{d}t &= \int_\delta^{+\infty} \liminf_{k\to+\infty} K(1,t) t^{\frac{m}{p}-\frac{\beta}{q}-1-\frac{c_k}{q}} \mathrm{d}t \\ &\leqslant \liminf_{k\to+\infty} \int_\delta^{+\infty} K(1,t) t^{\frac{m}{p}-\frac{\beta}{q}-1-\frac{c_k}{q}} \mathrm{d}t. \end{aligned} \tag{9}$$

According to (9), letting $\varepsilon \to 0^+$ in (8), that is, taking the lower limit when $k \to +\infty$, we can obtain

$$\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p} \frac{1}{|\tau_1|} \int_\delta^{+\infty} K(1,t) t^{\frac{m}{p}-\frac{\beta}{q}-1} \mathrm{d}t \leqslant \overline{M} \left(\frac{1}{|\tau_1|}\right)^{1/p}.$$

It follows that

$$\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p} \left(\frac{1}{|\tau_1|}\right)^{1/q} \int_\delta^{+\infty} K(1,t) t^{\frac{m}{p}-\frac{\beta}{q}-1} \mathrm{d}t \leqslant \overline{M}.$$

Then let $\delta \to 0^+$, we have

$$\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p} \left(\frac{1}{|\tau_1|}\right)^{1/q} W_1\left(\frac{m}{p}-\frac{\beta}{q}-1\right) \leqslant \overline{M},$$

which contradicts (7).

If $\tau_1 > 0$, taking a sufficiently small $\varepsilon > 0$ and a sufficiently large $N > 0$, setting $a_n = n^{-(\alpha+1+\tau_1\varepsilon)/p}$ $(n = 1,2,\cdots)$, and

$$f(x) = \begin{cases} \|x\|_{\rho,m}^{-(\beta+m-\varepsilon)/q}, & \|x\|_{\rho,m} \leqslant N, \\ 0, & \|x\|_{\rho,m} > N. \end{cases}$$

Then a similar discussion leads to

$$\|\tilde{a}\|_{p,\alpha}\|f\|_{q,\beta} \leqslant \left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q} \frac{1}{\varepsilon}\left(\varepsilon+\frac{1}{|\tau_1|}\right)^{1/p} N^{\varepsilon/q}.$$

$$\begin{aligned}
\widetilde{A}(K,\tilde{a},f) &= \sum_{n=1}^{\infty} n^{-\frac{\alpha+1}{p}-\frac{\tau_1\varepsilon}{p}} \left(\int_{\|x\|\rho,m\leqslant N} K\left(n,\|x\|_{\rho,m}\right) \|x\|_{\rho,m}^{-\frac{\beta+1}{q}+\frac{\varepsilon}{q}} \,\mathrm{d}x\right) \\
&= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \sum_{n=1}^{\infty} n^{\sigma_1+\frac{1}{q}-\frac{\alpha}{p}-1-\frac{\tau_1\varepsilon}{p}} \left(\int_0^N K\left(1,u\cdot n^{\tau_1}\right) u^{\frac{m}{p}-\frac{\beta}{q}-1+\frac{\varepsilon}{q}} \,\mathrm{d}u\right) \\
&= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \sum_{n=1}^{\infty} n^{-1-\tau_1\varepsilon} \left(\int_0^{Nn^{\tau_1}} K(1,t) t^{\frac{m}{p}-\frac{\beta}{q}-1+\frac{\varepsilon}{q}} \,\mathrm{d}t\right) \\
&\geqslant \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \int_1^{+\infty} t^{-1-\tau_1\varepsilon} \mathrm{d}t \int_0^N K(1,t) t^{\frac{m}{p}-\frac{\beta}{q}-1+\frac{\varepsilon}{q}} \,\mathrm{d}t \\
&= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \frac{1}{|\tau_1|\,\varepsilon} \int_0^N K(1,t) t^{\frac{m}{p}-\frac{\beta}{q}-1+\frac{\varepsilon}{q}} \,\mathrm{d}t.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p} \frac{1}{|\tau_1|} \int_0^N K(1,t) t^{\frac{m}{p}-\frac{\beta}{q}-1+\frac{\varepsilon}{q}} \,\mathrm{d}t \\
\leqslant &\overline{M}\left(\varepsilon+\frac{1}{|\tau_1|}\right)^{1/p} N^{\varepsilon/q}.
\end{aligned}$$

Let $\varepsilon \to 0^+$. It follows from Fatou's lemma that

$$\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{1}{|\tau_1|}\right)^{1/q}\int_0^N K(1,t)t^{\frac{m}{p}-\frac{\beta}{q}-1}\,\mathrm{d}t \leqslant \overline{M}.$$

Then let $N \to +\infty$, we have

$$\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{1}{|\tau_1|}\right)^{1/q} W_1\left(\frac{m}{p}-\frac{\beta}{q}-1\right) \leqslant \overline{M},$$

which contradicts (7).

To sum up, M_0 is the best factor of (4). □

4. Parameter conditions and norms for bounded operators with super-homogeneous kernels

THEOREM 2. *Suppose that* $\frac{1}{p}+\frac{1}{q}=1$ $(p>1)$, $m \in \mathbb{N}_+$, $\rho>0$, $\alpha,\beta \in \mathbb{R}$, $\tau_1\tau_2=1$, $\sigma_1+\tau_1\sigma_2=0$, $\tau_1\left(\frac{m}{p}-\frac{\beta}{q}\right)-\left(\frac{1}{q}-\frac{\alpha}{p}\right)-\sigma_1=c$, $x=(x_1,x_2,\cdots,x_m)\in\mathbb{R}_+^m$, $\|x\|_{\rho,m}=\left(x_1^\rho+x_2^\rho+\cdots+x_m^\rho\right)^{1/\rho}$, $K(u,v)>0$ *is a super-homogeneous function with parameters* $\{\sigma_1,\sigma_2,\tau_1,\tau_2\}$, $K(t,1)^{\frac{1}{q}-\frac{\alpha}{p}-1+c}$ *monotonically decreases on* $(0,+\infty)$, *and* $W_1\left(\frac{m}{p}-\frac{\beta}{q}-1\right)<+\infty$. *The operators* T_1 *and* T_2 *are defined by*

$$T_1(\tilde{a})(x)=\sum_{n=1}^{\infty}K\left(n,\|x\|_{\rho,m}\right)a_n,\ \tilde{a}=\{a_n\}\in l_p^\alpha,$$

$$T_2(f)_n=\int_{\mathbb{R}_+^m}K\left(n,\|x\|_{\rho,m}\right)f(x)\mathrm{d}x,\ f\in L_q^\beta\left(\mathbb{R}_+^m\right).$$

(i) T_1 *is a bounded operator from* l_p^α *to* $L_p^{\beta(1-p)}\left(\mathbb{R}_+^m\right)$ *and* T_2 *is a bounded operator from* $L_q^\beta\left(\mathbb{R}_+^m\right)$ *to* $l_q^{\alpha(1-q)}$ *if and only if* $c \geqslant 0$, *i.e.,* $\tau_1\left(\frac{m}{p}-\frac{\beta}{q}\right)-\left(\frac{1}{q}-\frac{\alpha}{p}\right)\geqslant\sigma_1$. *At this point, the operator norms of* T_1 *and* T_2 *satisfy*

$$\|T_1\|=\|T_2\|\leqslant\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{1}{|\tau_1|}\right)^{1/q}W_1\left(\frac{m}{p}-\frac{\beta}{q}-1\right).$$

(ii) When $c=0$, *i.e.,* $\tau_1\left(\frac{m}{p}-\frac{\beta}{q}\right)-\left(\frac{1}{q}-\frac{\alpha}{p}\right)=\sigma_1$, *the operator norms of* $T_1: l_p^\alpha \to L_p^{\beta(1-p)}\left(\mathbb{R}_+^m\right)$ *and* $T_2: L_q^\beta\left(\mathbb{R}_+^m\right)\to l_q^{\alpha(1-q)}$ *are*

$$\|T_1\|=\|T_2\|=\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{1}{|\tau_1|}\right)^{1/q}W_1\left(\frac{m}{p}-\frac{\beta}{q}-1\right).$$

Proof. According to the basic theory of Hilbert-type inequality [9], (4) is equivalent to the following operator inequalities

$$\|T_1(\tilde{a})\|_{p,\beta(1-p)}\leqslant M\|\tilde{a}\|_{p,\alpha},\ \|T_2(f)\|_{q,\alpha(1-q)}\leqslant M\|f\|_{q,\beta}.$$

Hence by Theorem 1, Theorem 2 holds. □

COROLLARY 1. *Let* $\frac{1}{p}+\frac{1}{q}=1\ (p>1),\ m\in\mathbb{N}_+,\ \rho>0,\ \lambda_1>0,\ \lambda_2>0,$ $-1\leqslant\alpha<p-1,\ \frac{1}{\lambda_1}\left(\frac{1}{q}-\frac{\alpha}{p}\right)=\frac{1}{\lambda_2}\left(\frac{m}{p}-\frac{\beta}{q}\right),\ x=(x_1,x_2,\cdots,x_m)\in\mathbb{R}_+^m,\ \|x\|_{\rho,m}=$ $\left(x_1^\rho+x_2^\rho+\cdots+x_m^\rho\right)^{1/\rho}$*, operators* T_1 *and* T_2 *be defined by*

$$T_1(\tilde{a})(x)=\sum_{n=1}^{\infty}\mathrm{e}^{-n^{\lambda_1}\|x\|_{\rho,m}^{\lambda_2}}a_n,\ \ \tilde{a}=\{a_n\}\in l_p^\alpha,$$
$$T_2(f)_n=\int_{\mathbb{R}_+^m}\mathrm{e}^{-n^{\lambda_1}\|x\|_{\rho,m}^{\lambda_2}}f(x)\mathrm{d}x,\ \ f\in L_q^\beta\left(\mathbb{R}_+^m\right),$$

respectively. Then T_1 *is a bounded operator from* l_p^α *to* $L_p^{\beta(1-p)}\left(\mathbb{R}_+^m\right)$ *and* T_2 *is a bounded operator from* $L_q^\beta\left(\mathbb{R}_+^m\right)$ *to* $l_q^{\alpha(1-q)}$*, and the operator norms of* T_1 *and* T_2 *are*

$$\|T_1\|=\|T_2\|=\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\frac{1}{\lambda_1^{1/q}\lambda_2^{1/p}}\Gamma\left(\frac{1}{\lambda_1}\left(\frac{1}{q}-\frac{\alpha}{p}\right)\right).$$

Proof. Let

$$K(u,v)=\mathrm{e}^{-u^{\lambda_1}v^{\lambda_2}},\ \ u>0,\ v>0.$$

Then $K(u,v)>0$ and is a super-homogeneous function with parameters $\left\{0,0,\frac{\lambda_1}{\lambda_2},\frac{\lambda_2}{\lambda_1}\right\}$, that is, $\sigma_1=\sigma_2=0,\ \tau_1=\frac{\lambda_1}{\lambda_2},\ \tau_2=\frac{\lambda_2}{\lambda_1}$. Obviously, $\tau_1\tau_2=1,\ \sigma_1+\tau_1\sigma_2=0$, and $\tau_1\left(\frac{m}{p}-\frac{\beta}{q}\right)-\left(\frac{1}{q}-\frac{\alpha}{p}\right)=\sigma_1$ can transformed into $\frac{1}{\lambda_1}\left(\frac{1}{q}-\frac{\alpha}{p}\right)=\frac{1}{\lambda_2}\left(\frac{m}{p}-\frac{\beta}{q}\right)$.

Since $-1<\alpha<p-1,\ \frac{1}{\lambda_1}\left(\frac{1}{q}-\frac{\alpha}{p}\right)=\frac{1}{\lambda_2}\left(\frac{m}{p}-\frac{\beta}{q}\right)$, we have $\frac{m}{p}-\frac{\beta}{q}>0$ and $\frac{1}{q}-\frac{\alpha}{p}-1<0$. Thus

$$K(t,1)t^{\frac{1}{q}-\frac{\alpha}{p}-1}=t^{\frac{1}{q}-\frac{\alpha}{p}-1}\mathrm{e}^{-t^{\lambda_1}}$$

monotonically decreases on $(0,+\infty)$, and

$$\begin{aligned}W_1\left(\frac{m}{p}-\frac{\beta}{q}-1\right)&=\int_0^{+\infty}K(1,t)t^{\frac{m}{p}-\frac{\beta}{q}-1}\,\mathrm{d}t\\&=\int_0^{+\infty}t^{\frac{m}{p}-\frac{\beta}{q}-1}\mathrm{e}^{-t^{\lambda_2}}\,\mathrm{d}t=\frac{1}{\lambda_2}\int_0^{+\infty}u^{\frac{1}{\lambda_2}\left(\frac{m}{p}-\frac{\beta}{q}\right)-1}\mathrm{e}^{-u}\,\mathrm{d}u\\&=\frac{1}{\lambda_2}\int_0^{+\infty}u^{\frac{1}{\lambda_1}\left(\frac{1}{q}-\frac{\alpha}{p}\right)-1}\mathrm{e}^{-u}\,\mathrm{d}u=\frac{1}{\lambda_2}\Gamma\left(\frac{1}{\lambda_1}\left(\frac{1}{q}-\frac{\alpha}{p}\right)\right)<+\infty.\end{aligned}$$

It follows from Theorem 2 that T_1 is a bounded operator from l_p^α to $L_p^{\beta(1-p)}\left(\mathbb{R}_+^m\right)$ and T_2 is a bounded operator from $L_q^\beta\left(\mathbb{R}_+^m\right)$ to $l_q^{\alpha(1-q)}$, and the operator norms of T_1 and T_2 are

$$\begin{aligned}\|T_1\|=\|T_2\|&=\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{1}{|\tau_1|}\right)^{1/q}W_1\left(\frac{m}{p}-\frac{\beta}{q}-1\right)\\&=\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\frac{1}{\lambda_1^{1/q}\lambda_2^{1/p}}\Gamma\left(\frac{1}{\lambda_1}\left(\frac{1}{q}-\frac{\alpha}{p}\right)\right).\quad\Box\end{aligned}$$

COROLLARY 2. *Assuming* $\frac{1}{p}+\frac{1}{q}=1\ (p>1),\ m\in\mathbb{N}_+,\ \rho>0,\ 0\leqslant\lambda<1,$ $\lambda_1>0,\ \lambda_2>0,\ \alpha>-1,\ \frac{1}{\lambda_1}\left(\frac{1}{q}-\frac{\alpha}{p}\right)=\frac{1}{\lambda_2}\left(\frac{m}{p}-\frac{\beta}{q}\right)=1,\ x=(x_1,x_2,\cdots,x_m)\in\mathbb{R}_+^m,$ $\|x\|_{\rho,m}=\left(x_1^\rho+x_2^\rho+\cdots+x_m^\rho\right)^{1/\rho}$, *operators* T_1 *and* T_2 *are defined by*

$$T_1(\tilde{a})(x)=\sum_{n=1}^{\infty}\frac{a_n}{\left(n^{2\lambda_1}+\|x\|_{\rho,m}^{2\lambda_2}\right)\arctan^\lambda\left(n^{\lambda_1}/\|x\|_{\rho,m}^{\lambda_2}\right)},\ \ \tilde{a}=\{a_n\}\in l_p^\alpha,$$

$$T_2(f)_n=\int_{\mathbb{R}_+^m}\frac{f(x)\mathrm{d}x}{\left(n^{2\lambda_1}+\|x\|_{\rho,m}^{2\lambda_2}\right)\arctan^\lambda\left(n^{\lambda_1}/\|x\|_{\rho,m}^{\lambda_2}\right)},\ \ f\in L_q^\beta\left(\mathbb{R}_+^m\right),$$

respectively. Then T_1 *is a bounded operator from* l_p^α *to* $L_p^{\beta(1-p)}\left(\mathbb{R}_+^m\right)$ *and* T_2 *is a bounded operator from* $L_q^\beta\left(\mathbb{R}_+^m\right)$ *to* $l_q^{\alpha(1-q)}$, *and the operator norms of* T_1 *and* T_2 *are*

$$\|T_1\|=\|T_2\|=\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\frac{1}{(1-\lambda)\lambda_1^{1/q}\lambda_2^{1/p}}\left(\frac{\pi}{2}\right)^{1-\lambda}.$$

Proof. Set

$$K(u,v)=\frac{1}{\left(u^{2\lambda_1}+v^{2\lambda_2}\right)\arctan^\lambda\left(u^{\lambda_1}/v^{\lambda_2}\right)},\ \ u>0,\ v>0.$$

Then $K(u,v)>0$ and is a super-homogeneous function with parameters $\left\{-2\lambda_1,-2\lambda_2,-\frac{\lambda_1}{\lambda_2},-\frac{\lambda_2}{\lambda_1}\right\}$, that is, $\sigma_1=-2\lambda_1,\ \sigma_2=-2\lambda_2,\ \tau_1=-\frac{\lambda_1}{\lambda_2},\ \tau_2=-\frac{\lambda_2}{\lambda_1}$. Obviously, $\tau_1\tau_2=1,\ \sigma_1+\tau_1\sigma_2=0.$

In view of $\lambda\geqslant0,\ \lambda_1>0,\ \lambda_2>0,\ \alpha>-1$, we obtain $\frac{1}{q}-\frac{\alpha}{p}-1\leqslant0$. Hence

$$K(t,1)t^{\frac{1}{q}-\frac{\alpha}{p}-1}=\frac{1}{\left(t^{2\lambda_1}+1\right)\arctan^\lambda\left(t^{\lambda_1}\right)}t^{\frac{1}{q}-\frac{\alpha}{p}-1}$$

monotonically decreases on $(0,+\infty)$. Since $0\leqslant\lambda<1,\ \frac{1}{\lambda_2}\left(\frac{m}{p}-\frac{\beta}{q}\right)=1$, it follows that

$$\begin{aligned}
W_1\left(\frac{m}{p}-\frac{\beta}{q}-1\right)&=\int_0^{+\infty}K(1,t)t^{\frac{m}{p}-\frac{\beta}{q}-1}\,\mathrm{d}t\\
&=\int_0^{+\infty}\frac{1}{\left(1+t^{2\lambda_2}\right)\arctan^\lambda\left(1/t^{\lambda_2}\right)}t^{\frac{m}{p}-\frac{\beta}{q}-1}\,\mathrm{d}t\\
&=\frac{1}{\lambda_2}\int_0^{+\infty}\frac{1}{(1+u^2)\arctan^\lambda u}u^{1-\frac{1}{\lambda_2}\left(\frac{m}{p}-\frac{\beta}{q}\right)}\,\mathrm{d}u\\
&=\frac{1}{\lambda_2}\int_0^{+\infty}\frac{1}{(1+u^2)\arctan^\lambda u}\,\mathrm{d}u=\frac{1}{\lambda_2}\int_0^{+\infty}\arctan^{-\lambda}u\,\mathrm{d}(\arctan u)\\
&=\frac{1}{\lambda_2}\frac{1}{1-\lambda}\arctan^{1-\lambda}u\Big|_0^{+\infty}=\frac{1}{\lambda_2(1-\lambda)}\left(\frac{\pi}{2}\right)^{1-\lambda}<+\infty.
\end{aligned}$$

From $\frac{1}{\lambda_1}\left(\frac{1}{q}-\frac{\alpha}{p}\right)=\frac{1}{\lambda_2}\left(\frac{m}{p}-\frac{\beta}{q}\right)=1$, we have $\frac{1}{\lambda_1}\left(\frac{1}{q}-\frac{\alpha}{p}\right)+\frac{1}{\lambda_2}\left(\frac{m}{p}-\frac{\beta}{q}\right)=2$, and then $\tau_1\left(\frac{m}{p}-\frac{1}{q}\right)-\left(\frac{1}{q}-\frac{\alpha}{p}\right)=\sigma_1$.

In summary and by Theorem 2, T_1 is a bounded operator from l_p^{α} to $L_p^{\beta(1-p)}\left(\mathbb{R}_+^m\right)$ and T_2 is a bounded operator from $L_q^{\beta}\left(\mathbb{R}_+^m\right)$ to $l_q^{\alpha(1-q)}$, and

$$\|T_1\|=\|T_2\|=\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\frac{1}{|\tau_1|}\right)^{1/q}W_1\left(\frac{m}{p}-\frac{\beta}{q}-1\right)$$
$$=\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\frac{1}{(1-\lambda)\lambda_1^{1/q}\lambda_2^{1/p}}\left(\frac{\pi}{2}\right)^{1-\lambda}. \quad \square$$

Funding. Supported by Guangzhou Huashang College Daoshi Project (No. 2024HSDS10), NNSF of China (No. 12471176), Key Construction Discipline Scientific Research Ability Promotion Project of Guangdong Province (No. 2021ZDJS055), Science and Technology Plan Project of Guangzhou Haizhu District (No. HKGSXJ2022-37).

REFERENCES

[1] I. BRNETIĆ, M. KRNIĆ, J. PEČARIĆ, *Multiple Hilbert and Hardy-Hilbert inequalities with non-conjugate parameters*, Bull. Austral. Math. Soc. **71** (2005): 447–457.
[2] J. F. CAO, B. HE, Y. HONG et al., *Equivalent conditions and applications of a class of Hilbert-type integral inequalities involving multiple functions with quasi-homogeneous kernels*, J. Inequal. Appl. **2018** (2018): 1–12.
[3] M. Z. GAO, *On the Hilbert inequality*, J. for Anal. Appl. **18** (4) (1999): 1117–1122.
[4] G. H. HARDY, J. E. LITTLEWOOD, G. PÓLYA, *Inequalities*, London: Gambridge University Press, 1952.
[5] B. HE, Y. HONG AND Z. LI, *Conditions for the validity of a class of optimal Hilbert type multiple integral inequalities with nonhomogeneous kernels*, J. Inequal. Appl. **2021** (2021): 1–12.
[6] Y. HONG, *On multiple Hardy-Hilbert integral inequalities with some parameters*, J. Inequal. Appl. **2006** (2006): 1–11.
[7] Y. HONG, *Structural characteristics and applications of Hilbert's type integral inequalities with homogenous kernel* (in Chinese), J. Jilin Univ. (Science Edition), **55** (2017): 189–194.
[8] Y. HONG, Q. CHEN, *Equivalence condition for the best matching parameters of multiple integral operator with generalized homogeneous kernel and applications* (in Chinese), Sci. Sin. Math. **53** (2023): 717–728.
[9] Y. HONG, B. HE, *Theory and applications of Hilbert-type inequalities* (in Chinese), Beijing: Science Press, 2023: 15–17.
[10] Y. HONG, Y. M. WEN, *A necessary and sufficient condition of that Hilbert type series inequality with homogeneous kernel and the best constant factor*, Chinese Annals of Mathematics **37A** (2016): 243–260.
[11] Q. L. HUANG, *A new extension of a Hardy-Hilbert-type inequality*, J. Inequal. Appl. **2015** (2015): 1–12.
[12] M. KRNIĆ, *On a strengthened multidimensional Hilbert-type inequality*, Math. Slovaca **64** (5) (2014): 1165–1182.
[13] M. KRNIĆ AND P. VUKOVIĆ, *Multidimensional Hilbert-type inequalities obtained via local fractional calculus*, Acta Appl. Math. **169** (2020): 667–680.
[14] J. C. KUANG, *Note on new extensions of Hilbert's integral inequality*, J. Math. Anal. Appl. **235** (1999): 608–614.

[15] Z. X. LU, *Some new inverse type Hilbert-Pachpatte inequalities*, Tamkang Journal of Mathematics **34** (2003): 155–162.
[16] T. M. RASSIAS, B. C. YANG, *A Hilbert-type integral inequality in the whole plane related to the hypergeometric function and the beta function*, J. Math. Anal. Appl. **428** (2) (2015): 1286–1308.
[17] T. M. RASSIAS, B. C. YANG, *Equivalent properties of a Hilbert-type integral inequality with the best constant factor related to the Hurwitz zeta function*, **9** (2) (2018): 282–295.
[18] T. M. RASSIAS, B. C. YANG, *On a few equivalent statements of a Hilbert-type integral inequality in the whole plane with the Hurwitz zeta function*, Analysis and Operator Theory **146** (2019): 319–352.
[19] S. R. SALEM, *Some new Hilbert type inequalities*, Kyungpook Math. J. **46** (2006): 19–29.
[20] W. T. SULAIMAN, *New ideas on Hardy-Hilbert's integral inequality*, Pan. American Mathematical Journal **15** (2005): 95–100.
[21] W. L. WU, B. C. YANG, *A few equivalent statements of a Hilbert-type integral inequality with the Reimann-zeta function*, J. Appl. Anal. Comput. **10** (6) (2020): 2400–2417.
[22] B. C. YANG, *A more accurate multidimensional Hardy-Hilbert's inequality*, J. Appl. Anal. Comput. **8** (2) (2018): 558–572.
[23] B. C. YANG, *On Hilbert's integral inequality*, J. Math. Anal. Appl. **220** (1998): 778–785.
[24] M. H. YOU, *On a class of Hilbert-type inequalities in the whole plane related to exponent function*, J. Inequal. Appl. **2021** (2021): 1–13.

(Received September 26, 2024)

Qian Zhao
Artificial Intelligence College
Guangzhou Huashang College
Guangzhou, Guangdong 511300, P. R. China
e-mail: eunicezhao_777@163.com

Bing He
Department of Mathematics
Guangdong University of Education
Guangzhou, Guangdong 510303, P. R. China
e-mail: hzs314@163.com

Yong Hong
Artificial Intelligence College
Guangzhou Huashang College
Guangzhou, Guangdong 511300, P. R. China
e-mail: hongyonggdcc@yeah.net