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SOME NEW INEQUALITIES FOR (A–A) AND (G–A) m–CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. In this paper, we introduce several new inequalities for (A-A) and (G-A) m-convex functions. Additionally, we examine multi-term refinements of the classical Jensen-type inequality for these classes of functions. Our results are further extended by applying the weak sub-majorization theory. As an application, we derive new refinements of the classical Hermite-Hadamard inequality for m-convex functions. Our findings extend and generalize recent publications in this field, including [5, 6, 8, 9, 16].

1. Introduction and preliminaries

Convex functions are essential in numerous fields of mathematics, including analysis, optimization, mathematical physics, functional analysis, and operator theory. A convex function $f : I \to \mathbb{R}$ is distinguished by the property that it adheres to

$$f(\kappa u+(1-\kappa)v) \leqslant \kappa f(u)+(1-\kappa)f(v) \tag{1.1}$$

for every $u, v \in I$ and $\kappa \in (0,1)$. A function f is called log-convex if it is positive and $\log f$ is convex. A recent research direction in mathematical inequalities focuses on minimizing the gap between the two sides of (1.1) by introducing specific terms. This inequality has been improved in the literature and has many applications in both scalar and matrix settings. Inspired by the numerous classes of convexity, we propose a new type that unifies several of them.

A subset $I \subset \mathbb{R}$ is called m-convex for $m \in [0,1]$ if for all $u, v \in I$ and $\kappa \in (0,1)$, the element $\kappa u + m(1-\kappa)v \in I$. It is clear that the interval $I = [0,b]$ is an m-convex subset of $\mathbb{R}$.

The notion of an (A-A) m-convex function was initially introduced in [17]. Recall that $f : [0,b] \to \mathbb{R}$ is called (A-A) m-convex, if for every $u, v \in [0,b]$ and $\kappa \in (0,1)$, the following inequality holds

$$f(\kappa u+m(1-\kappa)v) \leqslant \kappa f(u)+m(1-\kappa)f(v), \tag{1.2}$$

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using the previously defined notations, this is equivalent to the following

$$f(m\kappa u+(1-\kappa)v)\leqslant m\kappa f(u)+(1-\kappa)f(v). \tag{1.3}$$

We recall that a function f is called an (A-G) m-convex function if f is positive and $\log f$ is an (A-A) m-convex function, characterized by the following inequality.

$$f(\kappa u+m(1-\kappa)v)\leqslant f^{\kappa}(u)f^{m(1-\kappa)}(v). \tag{1.4}$$

Recall that f is called (G-A) m-convex, if for every $u,v\in[0,b]$ and $\kappa\in(0,1)$, the following inequality holds

$$f\left(u^{\kappa}v^{m(1-\kappa)}\right)\leqslant \kappa f(u)+m(1-\kappa)f(v), \tag{1.5}$$

using the previously defined notations, this is equivalent to the following

$$f\left(u^{m\kappa}v^{(1-\kappa)}\right)\leqslant m\kappa f(u)+(1-\kappa)f(v). \tag{1.6}$$

It is evident that if $m=1$ in (1.2), we recover the classical definition of convexity. Moreover, if we take $m=1$ in inequality (1.4), we recover the classical definition of (A-G) convex functions or log-convexity which can be expressed in the following inequality

$$f(\kappa u+(1-\kappa)v)\leqslant f^{\kappa}(u)f^{1-\kappa}(v). \tag{1.7}$$

For additional properties, examples, and inequalities related to the concept of m-convexity, the reader is encouraged to consult the following papers [3, 4, 10, 17].

The inequality (1.1) has been refined and reversed in the literature in various ways. Among the most significant refinements is given in the following theorem, proven by M. Sababheh [14], stated as follows

THEOREM 1.1. *Let $f:[u,v]\to[0,\infty)$ be convex. Then*

$$\left(\frac{\kappa}{\eta}\right)^{\lambda}\leqslant\frac{(\kappa f(v)+(1-\kappa)f(v))^{\lambda}-f^{\lambda}(\kappa u+(1-\kappa)v)}{(\eta f(u)+(1-\eta)f(v))^{\lambda}-f^{\lambda}(\eta u+(1-\eta)v)}\leqslant\left(\frac{1-\kappa}{1-\eta}\right)^{\lambda} \tag{1.8}$$

for $\lambda\geqslant 1$ and $0\leqslant\kappa\leqslant\eta<1$, $\eta\neq 0$.

Another noteworthy refinement of inequality (1.1), introduced by D. Choi et al. [1], incorporates multiple refining terms and is expressed as follows

THEOREM 1.2. *Let $f:[u,v]\to\mathbb{R}$ be a a convex function and $\kappa\in[0,1]$. If N is a positive integer, then the following inequality*

$$\kappa f(u)+(1-\kappa)f(v)\geqslant f(\kappa u+(1-\kappa)v)+\sum_{n=0}^{N-1}2r_n(\kappa)\sum_{k=1}^{2^n}\Delta_f^{(u,v)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\kappa) \tag{1.9}$$

holds, for certain positive summands.

In this paper, we focused on (A-A) and (G-A) m-convex functions. Specifically, we extended the inequalities presented in the introduction to this class, thereby generalizing a wide range of inequalities from the literature.

This paper is organized as follows: In Section 2, we begin by proving the inequalities presented in Theorem 1.1 for the class of (A-A) and (G-A) m-convex functions in the case $\lambda = 1$, and we introduce additional additive refinement terms for inequalities (1.2)–(1.6). Additionally, we address the case when $\kappa \notin [0,1]$. In Section 3, we extend some of the results obtained in Section 2 using the so-called weak sub-majorization theory. In Section 4, we establish new refinements of Jensen-type inequalities for (A-A) and (G-A) m-convex functions. Section 5 is dedicated to applying our results to derive new refinements of the classical Hermite-Hadamard-type inequality for (A-A) m-convex functions.

2. Preliminaries and auxiliary results

This section is divided into two parts, thus: In the first part, we investigate several important inequalities for (A-A) and (G-A) m-convex functions; In the second part, we consider the case where $\kappa \notin (0,1)$.

In the remainder of this paper, we will use the following notations. For $u, v \in [0,b]$ and $m, \kappa \in (0,1)$, we denote by $u\sharp_{(\kappa,m)}v = u^{\kappa}v^{m(1-\kappa)}$, $u\nabla_{(\kappa,m)}v = \kappa u + m(1-\kappa)v$, $u\sharp_{(m,\kappa)}v = u^{m\kappa}v^{1-\kappa}$ and $u\nabla_{(m,\kappa)}v = m\kappa u + (1-\kappa)v$.

2.1. The case $\kappa \in [0,1]$

In the following theorem, we present a significant refinement and an elegant reversal of inequality (1.2), offering a substantial improvement to its formulation. This results in a refined version of inequalities (1.2) and (1.3) by incorporating two weights.

THEOREM 2.1. *Let $f : [0,b] \to \mathbb{R}$ be a (A-A) m-convex function, and $0 \leqslant \kappa \leqslant \eta < 1$, $\eta \neq 0$. Then for any $u, v \in [0,b]$ we have*

$$f\left(u\nabla_{(\kappa,m)}v\right) + \frac{\kappa}{\eta}\left(f(u)\nabla_{(\eta,m)}f(v) - f\left(u\nabla_{(\eta,m)}v\right)\right) \leqslant f(u)\nabla_{(\kappa,m)}f(v),$$

and

$$f(u)\nabla_{(m,\kappa)}f(v) \leqslant f\left(u\nabla_{(m,\kappa)}v\right) + \frac{1-\kappa}{1-\eta}\left(f(u)\nabla_{(m,\eta)}f(v) - f\left(u\nabla_{(m,\eta)}v\right)\right).$$

Proof. By applying the property that f is an m-convex function, we obtain

$$\begin{aligned}
&f(u)\nabla_{(\kappa,m)}f(v) - \frac{\kappa}{\eta}\left(f(u)\nabla_{(\eta,m)}f(v) - f\left(u\nabla_{(\eta,m)}v\right)\right)\\
&= \kappa f(u) + m(1-\kappa)f(v) - \frac{\kappa}{\eta}\left(\eta f(u) + m(1-\eta)f(v) - f\left(u\nabla_{(\eta,m)}v\right)\right)\\
&= \frac{\kappa}{\eta}f\left(u\nabla_{(\eta,m)}v\right) + m\left(1 - \frac{\kappa}{\eta}\right)f(v)
\end{aligned}$$

$$\begin{aligned}
&\geqslant f\left(\frac{\kappa}{\eta}\left(u\nabla_{(\eta,m)}v\right)+m\left(1-\frac{\kappa}{\eta}\right)v\right)\\
&=f\left(u\nabla_{(\kappa,m)}v\right).
\end{aligned}$$

The second inequality is satisfied directly by using the fact that if $0\leqslant\kappa\leqslant\eta\leqslant 1$, then $0\leqslant 1-\eta\leqslant 1-\kappa\leqslant 1$, and by substituting κ, η, u, and v with $1-\eta$, $1-\kappa$, v, and u, respectively. $\square$

If we replace f with $\log f$ in the previous theorem, we directly obtain the following multiplicative refinement and reversed of inequality (1.4).

COROLLARY 2.2. *Let $f:[0,b]\to(0,+\infty)$ be a (A-G) m-convex function, and $0\leqslant\kappa\leqslant\eta<1$, $\eta\neq 0$. Then for any $u,v\in[0,b]$ we have*

$$f\left(u\nabla_{(\kappa,m)}v\right)\left(\frac{f(u)\sharp_{(\eta,m)}f(v)}{f\left(u\nabla_{(\eta,m)}v\right)}\right)^{\frac{\kappa}{\eta}}\leqslant f(u)\sharp_{(\kappa,m)}f(v),$$

and

$$f(u)\sharp_{(m,\kappa)}f(v)\leqslant f\left(u\nabla_{(m,\kappa)}v\right)\left(\frac{f(u)\sharp_{(m,\eta)}f(v)}{f\left(u\nabla_{(m,\eta)}v\right)}\right)^{\frac{1-\kappa}{1-\eta}}.$$

In the following theorem, we establish an analogous refinement to that of Theorem 2.1 for the class of (G-A) m-convex functions.

THEOREM 2.3. *Let $f:[0,b]\to\mathbb{R}$ be a (G-A) m-convex function, and $0\leqslant\kappa\leqslant\eta<1$, $\eta\neq 0$. Then for any $u,v\in[0,b]$ we have*

$$f\left(u\sharp_{(\kappa,m)}v\right)+\frac{\kappa}{\eta}\left(f(u)\nabla_{(\eta,m)}f(v)-f\left(u\sharp_{(\eta,m)}v\right)\right)\leqslant f(u)\nabla_{(\kappa,m)}f(v),$$

and

$$f(u)\nabla_{(m,\kappa)}f(v)\leqslant f\left(u\sharp_{(m,\kappa)}v\right)+\frac{1-\kappa}{1-\eta}\left(f(u)\nabla_{(m,\eta)}f(v)-f\left(u\sharp_{(m,\eta)}v\right)\right).$$

Proof. By utilizing the fact that f is a (G-A) m-convex function, we derive

$$\begin{aligned}
&f(u)\nabla_{(\kappa,m)}f(v)-\frac{\kappa}{\eta}\left(f(u)\nabla_{(\eta,m)}f(v)-f\left(u\sharp_{(\eta,m)}v\right)\right)\\
&=\kappa f(u)+m(1-\kappa)f(v)-\frac{\kappa}{\eta}\left(\eta f(u)+m(1-\eta)f(v)-f\left(u\sharp_{(\eta,m)}v\right)\right)\\
&=\frac{\kappa}{\eta}f\left(u\sharp_{(\eta,m)}v\right)+m\left(1-\frac{\kappa}{\eta}\right)f(v)\\
&\geqslant f\left(\left(u\sharp_{(\eta,m)}v\right)^{\frac{\kappa}{\eta}}v^{m\left(1-\frac{\kappa}{\eta}\right)}\right)\\
&=f\left(u\sharp_{(\kappa,m)}v\right).
\end{aligned}$$

The second inequality holds directly by noting that if $0 \leqslant \kappa \leqslant \eta \leqslant 1$, then $0 \leqslant 1-\eta \leqslant 1-\kappa \leqslant 1$, and by swapping κ, η, u, and v with $1-\eta$, $1-\kappa$, v, and u, respectively. $\square$

By replacing f with $\log f$ in the previous theorem, we immediately derive the following multiplicative refinement of the corresponding inequality for (G-G) m-convex functions.

COROLLARY 2.4. *Let $f:[0,b] \to (0,+\infty)$ be a (G-G) m-convex function, and $0 \leqslant \kappa \leqslant \eta < 1$, $\eta \neq 0$. Then for any $u,v \in [0,b]$ we have*

$$f\left(u \sharp_{(\kappa,m)} v\right) \left(\frac{f(u) \sharp_{(\eta,m)} f(v)}{f\left(u \sharp_{(\eta,m)} v\right)} \right)^{\frac{\kappa}{\eta}} \leqslant f(u) \sharp_{(\kappa,m)} f(v),$$

and

$$f(u) \sharp_{(m,\kappa)} f(v) \leqslant f\left(u \sharp_{(m,\kappa)} v\right) \left(\frac{f(u) \sharp_{(m,\eta)} f(v)}{f\left(u \sharp_{(m,\eta)} v\right)} \right)^{\frac{1-\kappa}{1-\eta}}.$$

REMARK 2.5. The previous corollary establishes an extension of Theorem 6 from [16] by utilizing two weights.

In the following theorem, we demonstrate a refinement of inequality (1.2) involving multiple terms.

THEOREM 2.6. *Let $f:[0,b] \to \mathbb{R}$ be a (A-A) m-convex function. If N is a positive integer, then for all $u,v \in [0,b]$ and $\kappa \in [0,\frac{1}{2^N}]$ we have*

$$\begin{aligned} & f(\kappa u + m(1-\kappa)v) \\ & + \sum_{k=1}^{N} 2^k \kappa \left[\frac{m f(v) + f\left(\frac{m(2^{k-1}-1)v+u}{2^{k-1}}\right)}{2} - f\left(\frac{m(2^k-1)v+u}{2^k}\right) \right] \\ & \leqslant \kappa f(u) + m(1-\kappa) f(v). \end{aligned} \tag{2.1}$$

Proof. The proof will proceed by induction on N. For $N=1$ the result is obtained directly from the first inequality in Theorem 2.1 by selecting $\eta = \frac{1}{2}$. Now assume that, for some $N \in \mathbb{N}$, (2.1) holds whenever $\kappa \in [0,\frac{1}{2^N}]$. We claim that the inequality holds for $N+1$. Note that

$$\begin{aligned} A = {} & \kappa f(u) + m(1-\kappa) f(v) \\ & - \sum_{k=1}^{N+1} 2^k \kappa \left[\frac{m f(v) + f\left(\frac{m(2^{k-1}-1)v+u}{2^{k-1}}\right)}{2} - f\left(\frac{m(2^k-1)v+u}{2^k}\right) \right] \end{aligned}$$

$$
\begin{aligned}
&= \kappa f(u)+m(1-\kappa)f(v)-2\kappa\left[\frac{f(u)+mf(v)}{2}-f\left(\frac{u+mv}{2}\right)\right]\\
&\quad -\sum_{k=2}^{N+1}2^k\kappa\left[\frac{mf(v)+f\left(\frac{m(2^{k-1}-1)v+u}{2^{k-1}}\right)}{2}-f\left(\frac{m(2^k-1)v+u}{2^k}\right)\right]\\
&= 2\kappa f\left(\frac{u+mv}{2}\right)+m(1-2\kappa)f(v)\\
&\quad -\sum_{k=1}^{N}2^{k+1}\kappa\left[\frac{mf(v)+f\left(\frac{m(2^{k}-1)v+u}{2^{k}}\right)}{2}-f\left(\frac{m(2^{k+1}-1)v+u}{2^{k+1}}\right)\right].
\end{aligned}
\tag{2.2}
$$

For simplicity, let $2\kappa = r$, $\frac{u+mv}{2}=u'$. Then (2.2) becomes

$$
\begin{aligned}
A &= rf\left(u'\right)+m(1-r)f(v)\\
&\quad -\sum_{k=1}^{N}2^k r\left[\frac{mf(v)+f\left(\frac{m(2^{k-1}-1)v+u'}{2^{k-1}}\right)}{2}-f\left(\frac{m(2^k-1)v+u'}{2^k}\right)\right]\\
&\geqslant f\left(ru'+m(1-r)v\right)\\
&= f(\kappa u+m(1-\kappa)v),
\end{aligned}
$$

here, we applied the inductive step to derive (2.2). It is important to note that when $\kappa\in[0,\frac{1}{2^{N+1}}]$ it follows that $2\kappa\in[0,\frac{1}{2^N}]$, justifying the use of the inductive step. $\square$

As a consequence of the previous theorem, we have the following corollary, which presents multiple refined terms of the equivalent form of inequality (1.2) as shown in inequality (1.3).

COROLLARY 2.7. *Let $f:[0,b]\to\mathbb{R}$ be a (A-A) m-convex function. If N is a positive integer, then for all $u,v\in[0,b]$ and $\kappa\in[\frac{2^N-1}{2^N},1]$.*

$$
\begin{aligned}
&f(m\kappa u+(1-\kappa)v)\\
&\quad +\sum_{k=1}^{N}2^k(1-\kappa)\left[\frac{mf(u)+f\left(\frac{m(2^{k-1}-1)u+v}{2^{k-1}}\right)}{2}-f\left(\frac{m(2^k-1)u+v}{2^k}\right)\right]\\
&\leqslant m\kappa f(u)+(1-\kappa)f(v).
\end{aligned}
\tag{2.3}
$$

Proof. The proof is straightforward by observing that if $\frac{2^N-1}{2^N}\leqslant\kappa\leqslant 1$ then $0\leqslant 1-\kappa\leqslant\frac{1}{2^N}$. Additionally, this can be demonstrated by exchanging κ, u, and v with $1-\kappa$, v, and u, respectively. $\square$

Using the same technique employed in the proof of Theorem 2.6 and Corollary 2.7, we obtain the following results, which offer multiple term refinements of inequality (1.6) in the context of (G-A) m-convexity.

THEOREM 2.8. *Let $f:[0,b]\to\mathbb{R}$ be a (G-A) m-convex function. If N is a positive integer. Then*

1. *If $\kappa\in[0,\frac{1}{2^N}]$, we have*

$$\begin{aligned} &f(u^{\kappa}v^{m(1-\kappa)}) \\ &+\sum_{k=1}^{N}2^k\kappa\left[\frac{mf(v)+f\left(\sqrt[2^{k-1}]{v^{m(2^{k-1}-1)}u}\right)}{2}-f\left(\sqrt[2^k]{v^{m(2^k-1)}u}\right)\right] \\ &\leqslant \kappa f(u)+m(1-\kappa)f(v). \end{aligned} \tag{2.4}$$

2. *If $\kappa\in[\frac{2^N-1}{2^N},1]$, we have*

$$\begin{aligned} &f(u^{m\kappa}v^{(1-\kappa)}) \\ &+\sum_{k=1}^{N}2^k(1-\kappa)\left[\frac{mf(u)+f\left(\sqrt[2^{k-1}]{u^{m(2^{k-1}-1)}v}\right)}{2}-f\left(\sqrt[2^k]{u^{m(2^k-1)}v}\right)\right] \\ &\leqslant m\kappa f(u)+(1-\kappa)f(v). \end{aligned} \tag{2.5}$$

2.2. The case $\kappa\notin[0,1]$

We start this section with the following reverse version of inequalities (1.2) and (1.3) applicable when $\kappa\notin[0,1]$.

THEOREM 2.9. *Let $f:\mathbb{R}\to\mathbb{R}$ be a (A-A) m-convex function, and $\kappa\notin[0,1]$. Then*

1. *If $\kappa>1$, then*

$$f\left(u\nabla_{(\kappa,m)}v\right)\geqslant f(u)\nabla_{(\kappa,m)}f(v). \tag{2.6}$$

2. *If $\kappa<0$, then*

$$f\left(u\nabla_{(m,\kappa)}v\right)\geqslant f(u)\nabla_{(m,\kappa)}f(v). \tag{2.7}$$

Proof. Assume that $\kappa>1$, then we have the following

$$u=\frac{m(\kappa-1)}{\kappa}v+\frac{1}{\kappa}(\kappa u+m(1-\kappa)v),$$

then, by applying the (A-A) m-convexity, we obtain

$$f(u) \leqslant \frac{m(\kappa-1)}{\kappa} f(v) + \frac{1}{\kappa} f(\kappa u + m(1-\kappa)v),$$

which is equivalent to the following

$$f\left(u \nabla_{(\kappa,m)} v\right) \geqslant f(u) \nabla_{(\kappa,m)} f(v).$$

For the second inequality, observe that if $\kappa < 0$ then $1-\kappa > 1$. By substituting κ, u, v with $1-\kappa, v, u$ we obtain the desired inequality. $\square$

REMARK 2.10. By selecting $m = 1$ in Theorem 2.9, we recover Lemma 1 from [15].

In the following theorem, we provide a multi-term refinement of Theorem 2.9.

THEOREM 2.11. *Let $f : \mathbb{R} \to \mathbb{R}$ be a (A-A) m-convex function. If N is a positive integer and $u < v$, then for $\kappa < 0$.*

$$\begin{aligned} &\kappa f(u) + m(1-\kappa) f(v) \\ &\quad - \sum_{k=1}^{N} 2^k \kappa \left[\frac{m f(v) + f\left(\frac{m\left(2^{k-1}-1\right)v+u}{2^{k-1}}\right)}{2} - f\left(\frac{m(2^k-1)v+u}{2^k}\right) \right] \\ &\leqslant f(\kappa u + m(1-\kappa)v). \end{aligned} \tag{2.8}$$

Proof. We will prove this by induction on N. Assume that f is (A-A) $m-$convex, $u < v$ and $\kappa < 0$. Then, we have

$$\begin{aligned} &\kappa f(u) + m(1-\kappa) f(v) - 2\kappa \left[\frac{f(u) + m f(v)}{2} - f\left(\frac{u+mv}{2}\right) \right] \\ &= 2\kappa f\left(\frac{u+mv}{2}\right) + m(1-2\kappa) f(v) \\ &\leqslant f\left(m(1-2\kappa)v + 2\kappa \frac{u+mv}{2}\right) \\ &= f(\kappa u + m(1-\kappa)v), \end{aligned}$$

where Theorem 2.9 was applied, with κ and u replaced by 2κ and $\frac{u+mv}{2}$, respectively. We would like to highlight that when $u < v$ we have $\frac{u+mv}{2} < v$. Moreover, when $\kappa < 0$ we have $2\kappa < 0$, providing justification for the application of Theorem 2.9.

Now assume that, for some $N \in \mathbb{N}$, (2.8) holds whenever $u < v$ and $\kappa < 0$. We claim that the inequality holds for $N+1$. Note that

$$\begin{aligned} A = &\kappa f(u) + m(1-\kappa) f(v) \\ &+ \sum_{k=1}^{N+1} 2^k \kappa \left[\frac{m f(v) + f\left(\frac{m\left(2^{k-1}-1\right)v+u}{2^{k-1}}\right)}{2} - f\left(\frac{m(2^k-1)v+u}{2^k}\right) \right] \end{aligned}$$

$$
\begin{aligned}
&= \kappa f(u)+m(1-\kappa)f(v)-2\kappa\left[\frac{f(u)+mf(v)}{2}-f\left(\frac{u+mv}{2}\right)\right]\\
&\quad+\sum_{k=2}^{N+1}2^k\kappa\left[\frac{mf(v)+f\left(\frac{m(2^{k-1}-1)v+u}{2^{k-1}}\right)}{2}-f\left(\frac{m(2^k-1)v+u}{2^k}\right)\right]\\
&= 2\kappa f\left(\frac{u+mv}{2}\right)+m(1-2\kappa)f(v)\\
&\quad+\sum_{k=1}^{N}2^{k+1}\kappa\left[\frac{mf(v)+f\left(\frac{m(2^{k}-1)v+u}{2^{k}}\right)}{2}-f\left(\frac{m(2^{k+1}-1)v+u}{2^{k+1}}\right)\right].
\end{aligned}
\tag{2.9}
$$

For simplicity, let $2\kappa=r$, $\frac{u+mv}{2}=u'$. Then (2.9) becomes

$$
\begin{aligned}
A &= rf\left(u'\right)+m(1-r)f(v)\\
&\quad+\sum_{k=1}^{N}2^k r\left[\frac{mf(v)+f\left(\frac{m(2^{k-1}-1)v+u'}{2^{k-1}}\right)}{2}-f\left(\frac{m(2^k-1)v+u'}{2^k}\right)\right]\\
&\leqslant f\left(m(1-r)v+ru'\right)\\
&= f(\kappa u+m(1-\kappa)v),
\end{aligned}
$$

Here, we utilized the inductive step to derive (2.9). Note that when $u<v$ it follows that $u'<v$, which supports the application of the inductive step. □

COROLLARY 2.12. *Let $f:\mathbb{R}\to\mathbb{R}$ be a (A-A) m-convex function. If N is a positive integer and $v<u$, then for $\kappa>1$.*

$$
\begin{aligned}
&m\kappa f(u)+(1-\kappa)f(v)\\
&-\sum_{k=1}^{N}2^k(1-\kappa)\left[\frac{mf(u)+f\left(\frac{m(2^{k-1}-1)u+v}{2^{k-1}}\right)}{2}-f\left(\frac{m(2^k-1)u+v}{2^k}\right)\right]\\
&\leqslant f(m\kappa u+(1-\kappa)v).
\end{aligned}
\tag{2.10}
$$

REMARK 2.13. If we set $m=1$ in Theorem 2.11 and Corollary 2.12, we recover Theorem 1 from [15].

For the rest of this section, we focus on (G-A) m-convex functions. Specifically, in the next theorem, we present a revised version of inequalities (1.5) and (1.6) for the case when $\kappa\notin[0,1]$.

THEOREM 2.14. *Let $f:\mathbb{R}\to\mathbb{R}$ be a (G-A) m-convex function, and $\kappa\notin[0,1]$. Then we have*

1. *If $\kappa>1$, then*

$$f\left(u\sharp_{(\kappa,m)}v\right)\geqslant f(u)\nabla_{(\kappa,m)}f(v). \tag{2.11}$$

2. *If $\kappa<0$, then*

$$f\left(u\sharp_{(m,\kappa)}v\right)\geqslant f(u)\nabla_{(m,\kappa)}f(v). \tag{2.12}$$

Proof. Assume that $\kappa>1$, then we have

$$u=v^{\frac{m(\kappa-1)}{\kappa}}\left(u^{\kappa}v^{m(1-\kappa)}\right)^{\frac{1}{\kappa}}$$

then, by applying the (G-A) m-convexity, we obtain

$$f(u)\leqslant\frac{m(\kappa-1)}{\kappa}f(v)+\frac{1}{\kappa}f(u^{\kappa}v^{m(1-\kappa)}),$$

which is equivalent to the following

$$f\left(u\sharp_{(\kappa,m)}v\right)\geqslant f(u)\nabla_{(\kappa,m)}f(v).$$

For the second inequality, note that if $\kappa<0$ then $1-\kappa>1$. By replacing κ,u,v with $1-\kappa,v,u$ the desired inequality follows. $\square$

By employing the same method as in the proof of Theorem 2.11, we derive the following refined version of Theorem 2.14 with multiple terms.

THEOREM 2.15. *Let $f:\mathbb{R}\to\mathbb{R}$ be a (G-A) m-convex function. If N is a positive integer and $u<v$, Then*

1. *If $\kappa<0$, we have*

$$\begin{aligned}&\kappa f(u)+m(1-\kappa)f(v)\\&-\sum_{k=1}^{N}2^{k}\kappa\left[\frac{mf(v)+f\left(\sqrt[2^{k-1}]{v^{m\left(2^{k-1}-1\right)}u}\right)}{2}-f\left(\sqrt[2^{k}]{v^{m(2^{k}-1)}u}\right)\right]\\&\leqslant f(u^{\kappa}v^{m(1-\kappa)}).\end{aligned} \tag{2.13}$$

2. *If $\kappa>1$, we have*

$$\begin{aligned}&m\kappa f(u)+(1-\kappa)f(v)\\&-\sum_{k=1}^{N}2^{k}(1-\kappa)\left[\frac{mf(u)+f\left(\sqrt[2^{k-1}]{u^{m\left(2^{k-1}-1\right)}v}\right)}{2}-f\left(\sqrt[2^{k}]{u^{m(2^{k}-1)}v}\right)\right]\\&\leqslant f(u^{m\kappa}v^{(1-\kappa)}).\end{aligned} \tag{2.14}$$

3. Further generalization using the theory of weak sub-majorization

In this section, we delve into a broader generalization of the main results established in Section 2, employing the theory of weak sub-majorization. This approach yields new results that extend several well-known inequalities from the literature regarding the concept of convexity to the realm of m-convexity. To achieve this, we first need to revisit the concept of weak sub-majorization.

We denote by $U = (U_1^*, \ldots, U_n^*)$ the vector formed from $U = (U_1, \ldots, U_n) \in \mathbb{R}^n$ by rearranging its components in non-increasing order. For two vectors $U = (U_1, \ldots, U_n)$ and $V = (V_1, \ldots, V_n)$ in $\mathbb{R}^n$, we say that V is weakly sub-majorized by U, denoted $U \succ_w V$, if

$$\sum_{i=1}^{k} U_i^* \geqslant \sum_{i=1}^{k} V_i^*$$

for all $k = 1, \ldots, n$. A key result from the theory of weak sub-majorization, which will be essential for proving our results, is presented in the following lemma.

LEMMA 3.1. [11, pp. 13] *Let $U = (U_i)_{i=1}^n$, $V = (V_i)_{i=1}^n \in \mathbb{R}^n$ and $J \subset \mathbb{R}$ be an interval containing the components of U and V. If $U \succ_w V$ and $\phi : J \to \mathbb{R}$ is a continuous increasing convex function, then*

$$\sum_{i=1}^{n} \phi(U_i) \geqslant \sum_{i=1}^{n} \phi(V_i).$$

We have the following lemma, which will allow us to derive the general form of Theorem 2.1.

LEMMA 3.2. *Let $f : [0, b] \to \mathbb{R}^+$ be a (A-A) m-convex function, $0 \leqslant \kappa \leqslant \eta \leqslant 1$ and $\eta \neq 0$. Let $U = (U_1, U_2)$ and $V = (V_1, V_2)$ be two vectors in $\mathbb{R}^2$ with components*

$$U_1 = f(u)\nabla_{(\kappa,m)} f(v), \quad U_2 = \frac{\kappa}{\eta} f\left(u\nabla_{(\eta,m)} v\right),$$
$$V_1 = f\left(u\nabla_{(\kappa,m)} v\right), \ \text{and} \ V_2 = \frac{\kappa}{\eta} f(u)\nabla_{(\eta,m)} f(v).$$

Then, we have $U \succ_w V$, namely, the vectors U^* and V^* have components satisfying that

$$V_1^* \leqslant U_1^*, \tag{3.1}$$
$$V_1^* + V_2^* \leqslant U_1^* + U_2^*. \tag{3.2}$$

Proof. First, the inequality (3.2) follows directly from Theorem 2.1. To establish inequality (3.1), it is necessary to prove that $U_1 \geqslant V_i$ for all $i = 1, 2$. Indeed, we have

$U_1 \geqslant V_1$ due to the fact that f is an $m-$convex function. For $U_1 \geqslant V_2$, we have

$$\begin{aligned} U_1 - V_2 &= f(u)\nabla_{(\kappa,m)} f(v) - \frac{\kappa}{\eta} f(u)\nabla_{(\eta,m)} f(v) \\ &= \kappa f(u) + m(1-\kappa) f(v) - \frac{\kappa}{\eta}(\eta f(u) + m(1-\eta) f(v)) \\ &= m\left(1 - \frac{\kappa}{\eta}\right) f(v) \geqslant 0. \end{aligned}$$

Therefore, we can affirm that inequality (3.1) holds, completing the proof. $\square$

In a similar manner, we can demonstrate the following lemma.

LEMMA 3.3. *Let $f : [0,b] \to \mathbb{R}^+$ be a (A-A) m-convex function and $0 \leqslant \kappa \leqslant \eta < 1$. Let $U = (U_1, U_2)$ and $V = (V_1, V_2)$ be two vectors in $\mathbb{R}^2$ with components*

$$\begin{aligned} U_1 &= f(u)\nabla_{(m,\kappa)} f(v), \quad U_2 = \frac{1-\kappa}{1-\eta} f\left(u\nabla_{(m,\eta)} v\right), \\ V_1 &= f\left(u\nabla_{(m,\kappa)} v\right), \ \textit{and} \ V_2 = \frac{1-\kappa}{1-\eta} f(u)\nabla_{(m,\eta)} f(v). \end{aligned}$$

Then, we have $V \succ_w U$, namely, the vectors U^* and V^* have components satisfying that

$$V_1^* \geqslant U_1^*, \tag{3.3}$$
$$V_1^* + V_2^* \geqslant U_1^* + U_2^*. \tag{3.4}$$

As a result of the previous two lemmas, in conjunction with Lemma 3.1, we obtain the following generalization of Theorem 2.1.

THEOREM 3.4. *Let $f : [0,b] \to \mathbb{R}^+$ be a (A-A) m-convex function, $0 \leqslant \kappa \leqslant \eta < 1$, $\eta \neq 0$ and $\phi : J \to \mathbb{R}$ is a continuous increasing convex function. Then for any $u, v \in [0,b]$ we have*

$$\phi \circ f\left(u\nabla_{(\kappa,m)} v\right) + \phi\left(\frac{\kappa}{\eta} f(u)\nabla_{(\eta,m)} f(v)\right) - \phi\left(\frac{\kappa}{\eta} f\left(u\nabla_{(\eta,m)} v\right)\right) \leqslant \phi\left(f(u)\nabla_{(\kappa,m)} f(v)\right),$$

and

$$\begin{aligned} \phi\left(f(u)\nabla_{(m,\kappa)} f(v)\right) \leqslant \phi \circ f\left(u\nabla_{(m,\kappa)} v\right) &+ \phi\left(\frac{1-\kappa}{1-\eta} f(u)\nabla_{(m,\eta)} f(v)\right) \\ &- \phi\left(\frac{1-\kappa}{1-\eta} f\left(u\nabla_{(m,\eta)} v\right)\right). \end{aligned}$$

Proof. Let us examine two vectors defined as follows

$$U = (U_1, U_2) \quad \text{and} \quad V = (V_1, V_2)$$

defined as in Lemma 3.2. According to Lemma 3.2, we have $V \prec_w U$. By applying Lemma 3.1 to the function ϕ, we obtain the inequality:

$$\phi(U_1)+\phi(U_2) \geqslant \phi(V_1)+\phi(V_2),$$

which can be rewritten as,

$$\phi(U_1)-\phi(V_1) \geqslant \phi(V_2)-\phi(U_2).$$

This concludes the proof. Using the same approach, we can establish the other inequality. $\square$

In particular, if we choose $\phi(x)=x^\lambda$ for $\lambda \geqslant 1$, we obtain the following corollary.

COROLLARY 3.5. *Let $f:[0,b] \to \mathbb{R}^+$ be a (A-A) m-convex function, $0 \leqslant \kappa \leqslant \eta < 1$, $\eta \neq 0$, and $\lambda \geqslant 1$. Then for any $u,v \in [0,b]$, we have*

$$f^\lambda\left(u\nabla_{(\kappa,m)}v\right)+\left(\frac{\kappa}{\eta}\right)^\lambda\left(\left(f(u)\nabla_{(\eta,m)}f(v)\right)^\lambda - f^\lambda\left(u\nabla_{(\eta,m)}v\right)\right) \leqslant \left(f(u)\nabla_{(\kappa,m)}f(v)\right)^\lambda,$$

and

$$\left(f(u)\nabla_{(m,\kappa)}f(v)\right)^\lambda \leqslant f^\lambda\left(u\nabla_{(m,\kappa)}v\right)+\left(\frac{1-\kappa}{1-\eta}\right)^\lambda\left(\left(f(u)\nabla_{(m,\eta)}f(v)\right)^\lambda - f^\lambda\left(u\nabla_{(m,\eta)}v\right)\right).$$

REMARK 3.6. By setting $m=1$ in the previous theorem, we can derive the principal result from the paper [14]. This demonstrates the significance of our result.

By selecting $\phi(x)=\exp(x)$, we derive the following corollary, which provides a refined additive term for the notion of (G-A) m-convex functions.

COROLLARY 3.7. *Let $f:[0,b] \to \mathbb{R}^+$ be a (A-G) m-convex function, $0 \leqslant \kappa \leqslant \eta < 1$, $\eta \neq 0$. Then for any $u,v \in [0,b]$, we have*

$$f\left(u\nabla_{(\kappa,m)}v\right)+\left(f(u)\sharp_{(\eta,m)}f(v)\right)^{\frac{\kappa}{\eta}} - f^{\frac{\kappa}{\eta}}\left(u\nabla_{(\eta,m)}v\right) \leqslant f(u)\sharp_{(\kappa,m)}f(v),$$

and

$$f(u)\sharp_{(m,\kappa)}f(v) \leqslant f\left(u\nabla_{(m,\kappa)}v\right)+\left(f(u)\nabla_{(m,\eta)}f(v)\right)^{\frac{1-\kappa}{1-\eta}} - f^{\frac{1-\kappa}{1-\eta}}\left(u\nabla_{(m,\eta)}v\right).$$

REMARK 3.8. By setting $m=1$ in the previous corollary, we recover a version of the main result from [14] concerning log-convex functions. This result leads to refined versions of certain classical inequalities for unitarily invariant norms, including Hölder's inequality.

Using the same approach as in Lemma 3.2, we have the following lemma, which can be utilized to prove similar results for the concept of (G-A) m-positive convex functions.

LEMMA 3.9. *Let $f:[0,b]\to\mathbb{R}^+$ be (G-A) m-convex function, $0\leqslant\kappa\leqslant\eta\leqslant 1$, $\eta\neq 0$. Let $U=(U_1,U_2)$ and $V=(V_1,V_2)$ be two vectors in $\mathbb{R}^2$ with components*

$$U_1=f(u)\nabla_{(\kappa,m)}f(v),\quad U_2=\frac{\kappa}{\eta}f\left(u\sharp_{(\eta,m)}v\right),$$
$$V_1=f\left(u\sharp_{(\kappa,m)}v\right),\ \text{and}\ V_2=\frac{\kappa}{\eta}f(u)\nabla_{(\eta,m)}f(v).$$

Then, we have $U\succ_w V$, namely, the vectors U^* and V^* have components satisfying that

$$V_1^*\leqslant U_1^*, \tag{3.5}$$
$$V_1^*+V_2^*\leqslant U_1^*+U_2^*. \tag{3.6}$$

In a similar manner, we have the following lemma.

LEMMA 3.10. *Let $f:[0,b]\to\mathbb{R}^+$ be (G-A) m-convex function and $0\leqslant\kappa\leqslant\eta<1$. Let $U=(U_1,U_2)$ and $V=(V_1,V_2)$ be two vectors in $\mathbb{R}^2$ with components*

$$U_1=f(u)\nabla_{(m,\kappa)}f(v),\quad U_2=\frac{1-\kappa}{1-\eta}f\left(u\sharp_{(m,\eta)}v\right),$$
$$V_1=f\left(u\sharp_{(m,\kappa)}v\right),\ \text{and}\ V_2=\frac{1-\kappa}{1-\eta}f(u)\nabla_{(m,\eta)}f(v).$$

Then, we have $V\succ_w U$, namely, the vectors V^* and U^* have components satisfying that

$$V_1^*\geqslant U_1^*, \tag{3.7}$$
$$V_1^*+V_2^*\geqslant U_1^*+U_2^*. \tag{3.8}$$

As a consequence of the preceding two lemmas, combined with Lemma 3.1, we obtain the following generalization of Theorem 2.3.

THEOREM 3.11. *Let $f:[0,b]\to\mathbb{R}^+$ be a (G-A) m-convex function, $0\leqslant\kappa\leqslant\eta<1$, $\eta\neq 0$ and ϕ be a convex function. Then for any $u,v\in[0,b]$ we have*

$$\phi\circ f\left(u\sharp_{(\kappa,m)}v\right)+\phi\left(\frac{\kappa}{\eta}f(u)\nabla_{(\eta,m)}f(v)\right)-\phi\left(\frac{\kappa}{\eta}f\left(u\sharp_{(\eta,m)}v\right)\right)\leqslant\phi\left(f(u)\nabla_{(\kappa,m)}f(v)\right),$$

and

$$\phi\left(f(u)\nabla_{(m,\kappa)}f(v)\right)\leqslant\phi\circ f\left(u\sharp_{(m,\kappa)}v\right)+\phi\left(\frac{1-\kappa}{1-\eta}f(u)\nabla_{(m,\eta)}f(v)\right)$$
$$-\phi\left(\frac{1-\kappa}{1-\eta}f\left(u\sharp_{(m,\eta)}v\right)\right).$$

Specifically, by selecting $\phi(x)=x^\lambda$ for $\lambda\geqslant 1$, we derive the following corollary.

COROLLARY 3.12. *Let $f:[0,b]\to\mathbb{R}^+$ be a (A-A) m-convex function, $0\leqslant\kappa\leqslant\eta<1,\ \eta\neq 0$ and $\lambda\geqslant 1$. Then for any $u,v\in[0,b]$, we have*

$$f^{\lambda}\left(u\sharp_{(\kappa,m)}v\right)+\left(\frac{\kappa}{\eta}\right)^{\lambda}\left(\left(f(u)\nabla_{(\eta,m)}f(v)\right)^{\lambda}-f^{\lambda}\left(u\sharp_{(\eta,m)}v\right)\right)\leqslant\left(f(u)\nabla_{(\kappa,m)}f(v)\right)^{\lambda},$$

and

$$\left(f(u)\nabla_{(m,\kappa)}f(v)\right)^{\lambda}\leqslant f^{\lambda}\left(u\sharp_{(m,\kappa)}v\right)+\left(\frac{1-\kappa}{1-\eta}\right)^{\lambda}\left(\left(f(u)\nabla_{(m,\eta)}f(v)\right)^{\lambda}-f^{\lambda}\left(u\sharp_{(m,\eta)}v\right)\right).$$

By choosing $\phi(x)=\exp(x)$, we obtain the following corollary, which offers a refined additive term for the concept of (G-G) m-convex functions.

COROLLARY 3.13. *Let $f:[0,b]\to\mathbb{R}^+$ be a (G-G) m-convex function, $0\leqslant\kappa\leqslant\eta<1,\ \eta\neq 0$. Then for any $u,v\in[0,b]$, we have*

$$f\left(u\sharp_{(\kappa,m)}v\right)+\left(f(u)\sharp_{(\eta,m)}f(v)\right)^{\frac{\kappa}{\eta}}-f^{\frac{\kappa}{\eta}}\left(u\sharp_{(\eta,m)}v\right)\leqslant f(u)\sharp_{(\kappa,m)}f(v),$$

and

$$f(u)\sharp_{(m,\kappa)}f(v)\leqslant f\left(u\sharp_{(m,\kappa)}v\right)+\left(f(u)\sharp_{(m,\eta)}f(v)\right)^{\frac{1-\kappa}{1-\eta}}-f^{\frac{1-\kappa}{1-\eta}}\left(u\sharp_{(m,\eta)}v\right).$$

REMARK 3.14. The previous corollary proved an additive refinement term version of Theorem 6 from [16].

4. On Jensen's type inequality for (A-A) m-convex functions

This section aims to improve the well known Jensen's type inequality of the class of (A-A) m-convex functions.

The well-known Jensen-type inequality for (A-A) m-convex functions [13], is given as follows

$$f\left(\sum_{k=1}^{n}m\lambda_k u_k\right)\leqslant\sum_{k=1}^{n}m\lambda_k f(u_k),\quad \sum_{k=1}^{n}\lambda_k=1. \tag{4.1}$$

As a consequence of the previous inequality, the following inequality extends (1.2) to n parameters, as demonstrated by the subsequent inequality in [13],

$$f\left(\lambda_1u_1+\sum_{k=2}^{n}m\lambda_k u_k\right)\leqslant\lambda_1 f(u_1)+\sum_{k=2}^{n}m\lambda_k f(u_k),\quad \sum_{k=1}^{n}\lambda_k=1. \tag{4.2}$$

We explore the following functional associated with Jensen's type inequality for (A-A) m-convex functions can be expressed as follows

$$\mathscr{J}_m(f,\mathbf{u},\lambda)=\sum_{k=1}^{n}m\lambda_k f(u_k)-f\left(\sum_{k=1}^{n}m\lambda_k u_k\right),\quad \sum_{k=1}^{n}\lambda_k=1. \tag{4.3}$$

where $\mathbf{u} = (u_1, u_2, \dots, u_n) \in [0,b]^n$ and $f : [0,b] \to \mathbb{R}$ is a (A-A) m-convex function. The corresponding non-weighted functional, i.e. when $\lambda = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)$, is defined by

$$\mathscr{J}_m(f,\mathbf{u}) = \sum_{k=1}^{n} \frac{m}{n} f(u_k) - f\left(\sum_{k=1}^{n} \frac{m}{n} u_k\right). \tag{4.4}$$

Under the assumption that f is (A-A) m-convex, it is evident that the functionals $\mathscr{J}_m(f,\mathbf{u},\lambda)$ and $\mathscr{J}_m(f,\mathbf{u})$ are non-negative. For the rest of this section, we examine the following:

$$\{u_1, \dots, u_n\} \subset [0,b], \quad \text{and} \quad \{\lambda_1, \dots, \lambda_n, \eta_1, \dots, \eta_n\} \subset (0,1) \text{ such that}$$

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \eta_i = 1.$$

In the following theorem, we provide refined and reversed versions of Jensen's type inequality for (A-A) m-convex functions, as mentioned in inequality (4.1).

THEOREM 4.1. *Let $f : [0,b] \to \mathbb{R}$ be a (A-A) m-convex function, the following inequalities*

$$r\mathscr{J}_m(f,\mathbf{u},\eta) \leqslant \mathscr{J}_m(f,\mathbf{u},\lambda) \leqslant R\mathscr{J}_m(f,\mathbf{u},\eta) \tag{4.5}$$

hold for all $\mathbf{u} \in [0,b]^n$, where $r = \min_{1\leqslant i\leqslant n}\left\{\frac{\lambda_i}{\eta_i}\right\}, R = \max_{1\leqslant i\leqslant n}\left\{\frac{\lambda_i}{\eta_i}\right\}$.

Proof. We begin by proving the first inequality. Consider the following

$$\begin{aligned}
I &= \sum_{k=1}^{n} m\lambda_k f(u_k) - r\left(\sum_{k=1}^{n} m\eta_k f(u_k) - f\left(\sum_{k=1}^{n} m\eta_k u_k\right)\right) \\
&= \sum_{k=1}^{n} m(\lambda_k - r\eta_k) f(u_k) + rf\left(\sum_{k=1}^{n} m\eta_k u_k\right) \\
&\geqslant f\left(\sum_{k=1}^{n} m(\lambda_k - r\eta_k) u_k + r\sum_{k=1}^{n} m\eta_k u_k\right) \\
&= f\left(\sum_{k=1}^{n} m\lambda_k u_k\right),
\end{aligned}$$

where the first inequality is obtained from inequality (4.2), which is equivalent to the first inequality of our theorem. For the second inequality, we have

$$\begin{aligned}
I &= \sum_{k=1}^{n} m\eta_k f(u_k) - \frac{1}{R}\left(\sum_{k=1}^{n} m\lambda_k f(u_k) - f\left(\sum_{k=1}^{n} m\lambda_k u_k\right)\right) \\
&= \sum_{k=1}^{n} m\frac{(R\eta_k - \lambda_k)}{R} f(u_k) + \frac{1}{R} f\left(\sum_{k=1}^{n} m\lambda_k u_k\right)
\end{aligned}$$

$$\geqslant f\left(\sum_{k=1}^{n} m\frac{(R\eta_k-\lambda_k)}{R}u_k+\frac{1}{R}\sum_{k=1}^{n} m\lambda_k u_k\right)$$
$$=f\left(\sum_{k=1}^{n} m\eta_k u_k\right),$$

where the first inequality is derived again from inequality (4.2), which is equivalent to the second inequality of our theorem. □

REMARK 4.2. By choosing $m=1$ in Theorem 4.1, we recover the Dragomir Jensen-type inequalities oubtained in the main result of [2].

COROLLARY 4.3. *Let $f:[0,b]\to\mathbb{R}$ be a (A-A) m-convex function, the following inequalities*

$$nr_0\mathscr{J}_m(f,\mathbf{u})\leqslant\mathscr{J}_m(f,\mathbf{u},\lambda)\leqslant nR_0\mathscr{J}_m(f,\mathbf{u}) \tag{4.6}$$

hold for all $\mathbf{u}\in[0,b]^n$, where $r_0=\min_{1\leqslant i\leqslant n}\{\lambda_i\}, R_0=\max_{1\leqslant i\leqslant n}\{\lambda_i\}$.

The following lemma will enable us to derive the general form of Theorem 4.1.

LEMMA 4.4. *Let $f:[0,b]\to\mathbb{R}^+$ be (A-A) m-convex function. Let $U=(U_1,U_2)$ and $V=(V_1,V_2)$ be two vectors in $\mathbb{R}^2$ with components*

$$U_1=\sum_{k=1}^{n} m\lambda_k f(u_k),\quad U_2=rf\left(\sum_{k=1}^{n} m\lambda_k u_k\right),$$
$$V_1=f\left(\sum_{k=1}^{n} m\lambda_k u_k\right),\ \text{ and }\ V_2=r\sum_{k=1}^{n} m\eta_k f(u_k).$$

Then, we have $U\succ_w V$, namely, the vectors U^* and V^* have components satisfying that

$$V_1^*\leqslant U_1^*, \tag{4.7}$$
$$V_1^*+V_2^*\leqslant U_1^*+U_2^*. \tag{4.8}$$

Proof. First, the inequality (4.8) follows directly from Theorem 4.1. In order to establish inequality (4.7), we need to prove that $U_1\geqslant V_i$ for all $i=1,2$. Indeed, we have $U_1\geqslant V_1$ due to the Jensen's type inequality for m-convex functions. For $U_1\geqslant V_2$, we have

$$U_1-V_2=\sum_{k=1}^{n} m\lambda_k f(u_k)-r\sum_{k=1}^{n} m\eta_k f(u_k)$$
$$=m\sum_{k=1}^{n}(\lambda_k-r\eta_k)f(u_k)\geqslant 0.$$

Hence, we affirm that inequality (4.7) is valid, which completes the proof. □

We also require the following lemma, which can be proven in a similar manner.

LEMMA 4.5. *Let $f:[0,b]\to\mathbb{R}^+$ be (A-A) m-convex function. Let $U=(U_1,U_2)$ and $V=(V_1,V_2)$ be two vectors in $\mathbb{R}^2$ with components*

$$U_1=\sum_{k=1}^{n} m\lambda_k f(u_k),\quad U_2=Rf\left(\sum_{k=1}^{n} m\eta_k u_k\right),$$
$$V_1=R\sum_{k=1}^{n} m\eta_k f(u_k),\ \text{and}\ V_2=f\left(\sum_{k=1}^{n} m\lambda_k u_k\right),$$

Then, we have $V\succ_w U$, namely, the vectors U^* and V^* have components satisfying that

$$V_1^*\geqslant U_1^*, \tag{4.9}$$
$$V_1^*+V_2^*\geqslant U_1^*+U_2^*. \tag{4.10}$$

In the next theorem, we introduce a generalization of Theorem 4.1 through the concept of weak sub-majorization in Section 3. The proof is similar to that used in Theorem 3.4.

THEOREM 4.6. *Let $f:[0,b]\to\mathbb{R}^+$ be a (A-A) m-convex function, and $\phi:J\to\mathbb{R}$ is a continuous increasing convex function. Then we have*

$$\mathscr{J}_{m,\phi}(f,\mathbf{u},\eta,r)\leqslant\phi\left(\sum_{k=1}^{n} m\lambda_k f(u_k)\right)-\phi\circ f\left(\sum_{k=1}^{n} m\lambda_k u_k\right)\leqslant\mathscr{J}_{m,\phi}(f,\mathbf{u},\eta,R), \tag{4.11}$$

where,

$$\mathscr{J}_{m,\phi}(f,\mathbf{u},\eta,t)=\phi\left(t\sum_{k=1}^{n} m\eta_k f(u_k)\right)-\phi\circ f\left(t\sum_{k=1}^{n} m\eta_k u_k\right). \tag{4.12}$$

REMARK 4.7. By choosing $m=1$ in Theorem 4.6, we derive the main result from [9].

5. Application to refined Hermite-Hadamard inequality

In this section, we concentrate on the Hermite-Hadamard type inequality for (A-A) and (G-A) m-convex functions.

The well-known Hermite-Hadamard inequality [6, 7] establishes that for a convex function $f:[u,v]\to\mathbb{R}$

$$f\left(\frac{u+v}{2}\right)\leqslant\frac{1}{v-u}\int_u^v f(\xi)d\xi\leqslant\frac{f(u)+f(v)}{2}, \tag{5.1}$$

This inequality, even in its simplest form, has attracted the attention of many researchers due to its applications across various fields.

It is well-established that the right-hand side of the Hermite-Hadamard inequality can be further refined as follows (see, for example, [8, 18]).

$$\frac{1}{v-u}\int_u^v f(\xi)d\xi \leqslant \frac{1}{2}\left[f\left(\frac{u+v}{2}\right)+\frac{f(u)+f(v)}{2}\right], \tag{5.2}$$

it directly follows that the residual (the distance between the two sides) in the right-hand inequality of (5.1) is greater than that in the left-hand inequality.

The right-hand side of inequality (5.1), has bent extended to the notion of (A-A) m-convex functions as follows.

THEOREM 5.1. ([3]) *Let $f:[0,\infty)\to\mathbb{R}$ be a (A-A) m-convex function. If $0\leqslant u<v<\infty$ and $f\in L_1[u,v]$. This leads to the following inequality.*

$$\frac{1}{v-u}\int_u^v f(\xi)d\xi \leqslant \min\left\{\frac{f(u)+mf\left(\frac{v}{m}\right)}{2},\frac{f(v)+mf\left(\frac{u}{m}\right)}{2}\right\}. \tag{5.3}$$

As a consequence of Theorem 2.6 and Corollary 2.7, we obtain the following refinement of inequality (5.3).

THEOREM 5.2. *Let $f:[0,\infty)\to\mathbb{R}$ be a (A-A) m-convex function. If $0\leqslant u<v<\infty$ and $f\in L_1([u,v])$. This leads to the following inequality.*

$$\begin{aligned}&\frac{1}{v-u}\int_u^v f(\xi)d\xi\\&\quad+\frac{1}{2^{2N-1}}\sum_{k=1}^{N}2^k\left[\frac{mf\left(\frac{v}{m}\right)+f\left(\frac{(2^{k-1}-1)v+u}{2^{k-1}}\right)}{2}-f\left(\frac{(2^k-1)v+u}{2^k}\right)\right]\\&\leqslant\frac{f(u)+mf\left(\frac{v}{m}\right)}{2}.\end{aligned} \tag{5.4}$$

Proof. Based on Theorem 2.6, by substituting v with $\frac{v}{m}$, we arrive at the following result:

$$\begin{aligned}&f(\kappa u+(1-\kappa)v)\\&\quad+\sum_{k=1}^{N}2^k\kappa\left[\frac{mf\left(\frac{v}{m}\right)+f\left(\frac{(2^{k-1}-1)v+u}{2^{k-1}}\right)}{2}-f\left(\frac{(2^k-1)v+u}{2^k}\right)\right]\times\chi_{[0,\frac{1}{2^N}]}\\&\leqslant\kappa f(u)+m(1-\kappa)f(v).\end{aligned} \tag{5.5}$$

Thus, by integrating both sides of the inequality from 0 to 1, we obtain the desired result. □

In a similar manner, we can derive the following inequality.

THEOREM 5.3. *Let* $f:[0,\infty)\to\mathbb{R}$ *be a (A-A) m-convex function. If* $0\leqslant u<v<\infty$ *and* $f\in L_1([u,v])$. *This leads to the following inequality.*

$$\begin{aligned}&\frac{1}{v-u}\int_u^v f(\xi)d\xi\\&+\frac{1}{2^{2N-1}}\sum_{k=1}^{N}2^k\left[\frac{mf\left(\frac{u}{m}\right)+f\left(\frac{(2^{k-1}-1)u+v}{2^{k-1}}\right)}{2}-f\left(\frac{(2^k-1)u+v}{2^k}\right)\right]\\ \leqslant\ &\frac{f(v)+mf\left(\frac{u}{m}\right)}{2}. \end{aligned}\tag{5.6}$$

REMARK 5.4. By choosing $m=1$ and $N=1$ in the two theorems above, we recover inequality (5.2).

In the following theorem, we establish the Hermite-Hadamard inequality for the class of (G-A) m-convex functions.

THEOREM 5.5. *Let* $f:[0,\infty)\to\mathbb{R}$ *be a (G-A) m-convex function. If* $0\leqslant u<v<\infty$ *and* $f\in L_1[u,v]$. *This leads to the following inequality.*

$$\frac{1}{\ln(v)-\ln(u)}\int_u^v\frac{f(\xi)}{\xi}d\xi\leqslant\min\left\{\frac{f(u)+mf(\sqrt[m]{v})}{2},\frac{f(v)+mf(\sqrt[m]{u})}{2}\right\}.\tag{5.7}$$

Proof. By leveraging the property that f is an (G-A) m-convex function and substituting v with $\sqrt[m]{v}$ in inequality (1.5) and u with $\sqrt[m]{u}$ n inequality (1.6), we obtain

$$f\left(u^{\kappa}v^{(1-\kappa)}\right)\leqslant\kappa f(u)+m(1-\kappa)f(\sqrt[m]{v}),\tag{5.8}$$

and

$$f\left(u^{\kappa}v^{(1-\kappa)}\right)\leqslant m\kappa f(\sqrt[m]{u})+(1-\kappa)f(v).\tag{5.9}$$

Therefore, by integrating both sides of the two inequalities from 0 to 1, we achieve the desired result. □

Using the same method employed in Theorem 5.2, we obtain the following refinement of the Hermite-Hadamard inequality for (G-A) m-convex functions.

THEOREM 5.6. *Let* $f:[0,b]\to\mathbb{R}$ *be a (G-A) m-convex function. If N is a posi-*

tive integer. Then

$$\begin{aligned}
&\frac{1}{\ln(v)-\ln(u)}\int_u^v \frac{f(\xi)}{\xi}d\xi\\
&+\frac{1}{2^{2N-1}}\sum_{k=1}^{N}2^k\left[\frac{mf(v)+f\left(\sqrt[2^{k-1}]{v^{(2^{k-1}-1)}u}\right)}{2}-f\left(\sqrt[2^k]{v^{(2^k-1)}u}\right)\right]\\
\leqslant\ &\frac{f(u)+mf\left(\sqrt[m]{v}\right)}{2}, \qquad (5.10)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\ln(v)-\ln(u)}\int_u^v \frac{f(\xi)}{\xi}d\xi\\
&+\frac{1}{2^{2N-1}}\sum_{k=1}^{N}2^k\left[\frac{mf(u)+f\left(\sqrt[2^{k-1}]{u^{(2^{k-1}-1)}v}\right)}{2}-f\left(\sqrt[2^k]{u^{(2^k-1)}v}\right)\right]\\
\leqslant\ &\frac{f(v)+mf\left(\sqrt[m]{u}\right)}{2}. \qquad (5.11)
\end{aligned}$$

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