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ON STARLIKENESS OF THE GENERALIZED MARCUM Q–FUNCTION

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Abstract. Our aim in this paper is to present sufficient conditions on the parameters of some classes of analytic functions related to the generalized Marcum Q-function to belong to a certain class of starlike functions. Furthermore, two classes of starlike functions related to the lower incomplete generalized hypergeometric functions are derived. Applications of these are given in the form of corollaries and examples. The key tool in the proofs of the main results are the monotonicity property for the gamma function and an inequality of the lower incomplete gamma function.

1. Preliminaries results

The celebrated and widely used the generalized Marcum Q-function is defined by

$$Q_\nu(a,b) = \frac{1}{a^{\nu-1}} \int_b^\infty t^\nu e^{-\frac{t^2+a^2}{2}} I_{\nu-1}(at)dt, \tag{1.1}$$

where $I_\nu(z)$ stands for the modified Bessel function of the first kind of the order ν, which has the power series definition [31, p. 249, Eq. 10.25.2]

$$I_\nu(x) = \sum_{k=0}^\infty \frac{(x/2)^{2k+\nu}}{k!\Gamma(k+\nu+1)}, \qquad x \in \mathbb{R},\ \nu > -1. \tag{1.2}$$

When $\nu = 1$, the function

$$Q(a,b) := Q_1(a,b) = \int_b^\infty t e^{-\frac{t^2+a^2}{2}} I_0(at)dt, \tag{1.3}$$

is known in literature as Marcum Q-function. The Marcum Q-function and the generalized Marcum Q-function, defined above are important special functions in a number of communication theory problems include the study of target detection by pulsed radars with single or multiple observations, the error probability performance of noncoherent digital communication, the outage probability of wireless communication systems, the performance analysis and capacity statistics of uncoded multiple-input multiple-output

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systems. In this regard, notable contributions can be found in [13,22,34,35] and the relevant titles therein. Also, we point out that the generalized Marcum Q-function has an important interpretation in probability theory, namely that is the complement (with respect to unity) of the cumulative distribution function of the non-central chi distribution with 2ν degrees of freedom.

The Geometric Function Theory is an important branches of complex analysis, it deals with the geometric properties of analytic functions such as starlikeness, convexity, and close-to-convexity in the open unit disk. In the last decades several researchers have studied some classes of analytic functions and have presented several interesting results with applications. The most known application is the solution of the famous Bieberbach conjecture by L. de Branges [1]. Moreover, it is worth mentioning that the researchers in the subject are interested nowadays in obtaining new theoretical methodologies and techniques with observational results together with their several applications. In recent decades, Geometric Function Theory for some special functions including have attracted the attention of many mathematicians including Fox-Wright function [25, 26], Mittag-Leffler function [8, 12, 27–30], Struve and Lommel functions [7], Modified Bessel function [24], Bessel functions and its q-analogues [2–6, 9, 36]. However, the geometric properties of a class of analytic functions related to the generalized Marcum Q-function have not been studied previously in the literature. Motivated by the above facts our main aim in this paper is to derive sufficient conditions on the parameters of the normalized form of analytic functions related to the generalized Marcum Q-function to belong to a certain class of starlike functions. In what follows, the symbol ${}_p\gamma_q[.]$ and ${}_p\Gamma_q[.]$ stands for the incomplete generalized hypergeometric functions were defined by means of the incomplete gamma functions as follows [33]:

$$\begin{aligned}{}_p\gamma_q\left[\begin{matrix}(a_1,x),a_2\cdots,a_p\\ b_1,\cdots,b_q\end{matrix}\middle|z\right] &= {}_p\gamma_q\left[\begin{matrix}(a_1,x),\mathbf{a}_{p-1}\\ \mathbf{b}_q\end{matrix}\middle|z\right]\\ &= \sum_{k=0}^{\infty}\frac{\gamma(a_1+k,x)\prod\limits_{l=2}^{p}(a_l)_k}{\Gamma(a_1)\prod\limits_{l=1}^{q}(b_l)_k}\frac{z^k}{k!},\end{aligned}\tag{1.4}$$

and

$$\begin{aligned}{}_p\Gamma_q\left[\begin{matrix}(a_1,x),a_2\cdots,a_p\\ b_1,\cdots,b_q\end{matrix}\middle|z\right] &= {}_p\Gamma_q\left[\begin{matrix}(a_1,x),\mathbf{a}_{p-1}\\ \mathbf{b}_q\end{matrix}\middle|z\right]\\ &= \sum_{k=0}^{\infty}\frac{\Gamma(a_1+k,x)\prod\limits_{l=2}^{p}(a_l)_k}{\Gamma(a_1)\prod\limits_{l=1}^{q}(b_l)_k}\frac{z^k}{k!},\end{aligned}\tag{1.5}$$

where, as usual, we make use of the following notation:

$$(a)_0=1,(a)_k=a(a+1)\cdots(a+k-1)=\frac{\Gamma(a+k)}{\Gamma(a)},$$

to denote the shifted factorial or the Pochhammer symbol and $\gamma(\nu,x)$ and $\Gamma(\nu,x)$ stands for the lower and upper incomplete gamma functions, which integral expression reads

$$\gamma(\nu,x)=\int_0^x \mathrm{e}^{-t}t^{\nu-1}\mathrm{d}t, \qquad x>0,\ \Re(\nu)>0,$$

and

$$\Gamma(\nu,x)=\int_x^\infty \mathrm{e}^{-t}t^{\nu-1}\mathrm{d}t, \qquad x>0,\ \Re(\nu)>0.$$

These two functions satisfy the following decomposition formula:

$$\gamma(\nu,x)+\Gamma(\nu,x)=\Gamma(\nu),\ \ \Re(\nu)>0. \tag{1.6}$$

We note that

$$\gamma(1,x)=1-e^{-x}, \quad x>0. \tag{1.7}$$

It is worth mentioning that the generalized Marcum Q-function can be represented in terms of the lower incomplete gamma function, reads [11, p. 39]:

$$Q_\nu(a,b)=1-e^{-\frac{a^2}{2}}\sum_{\ell=0}^\infty \frac{\gamma(\nu+\ell,b^2/2)(a^2/2)}{\ell!\Gamma(\nu+\ell)}. \tag{1.8}$$

For $b>0$ and $z\in\mathbb{C}$, we set

$$\begin{aligned}\mathbb{Q}_\nu(z,b)&=\sum_{\ell=0}^\infty \frac{\gamma(\nu+\ell,b)z^\ell}{\ell!\Gamma(\nu+\ell)}\\ &=e^z\left(1-Q_\nu(\sqrt{2z},\sqrt{2b})\right).\end{aligned} \tag{1.9}$$

According to the Cauchy-Hadamard formula, Stirling's asymptotic formula for the gamma function and the following asymptotic formula for the lower incomplete gamma function for large a [31, p. 180, Eq. (8.11.5)]:

$$\frac{\gamma(a,b)}{\Gamma(a)} \underset{a\to\infty}{\sim} (2a\pi)^{-\frac{1}{2}}(b/a)^a e^{a-b}, \tag{1.10}$$

we conclude that the function $z\mapsto \mathbb{Q}_\nu(z,b)$ defines an entire function (that is, it is absolutely convergent for all $z\in\mathbb{C}$).

Let us now recall some basic definitions and results related to the Geometric Function Theory. Let $\mathbb{H}$ denotes the class of all analytic functions in the unit disk

$$\mathbb{D}=\Big\{z\in\mathbb{C}: \quad |z|<1\Big\}.$$

Let $\mathbb{A}$ the class of analytic function $f\in\mathbb{H}$ satisfying $f(0)=f'(0)-1=0$ such that

$$f(z)=z+\sum_{\ell=2}^\infty a_\ell z^\ell, \qquad \forall z\in\mathbb{D}. \tag{1.11}$$

The function $f \in \mathbb{A}$ is called starlike function in $\mathbb{D}$, if f is univalent in $\mathbb{D}$ and $f(\mathbb{D})$ is a starlike domain with respect to the origin. The analytic characterization of the class of starlike functions is given below [14]:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \qquad \forall z \in \mathbb{D}.$$

An analytic function $f(z)$ in $\mathbb{A}$ is said to be convex in $\mathbb{D}$, if $f(z)$ is a univalent function in $\mathbb{D}$ with $f(\mathbb{D})$ as a convex domain in $\mathbb{C}$.

A function f in class $\mathbb{A}$ is called uniformly convex in $\mathbb{D}$, if, for every circular arc ς contained in $\mathbb{D}$ with center $\eta \in \mathbb{D}$, the image arc $f(\varsigma)$ is convex. This class of functions is denoted by UCV (see, for details, [32]). It was introduced by Goodman (see [16, 17]). On the other hand, Ronning [32] considered a newly-defined class of starlike functions $\mathscr{S}_p$ as follows:

$$\mathscr{S}_p = \Big\{ f : f(z) = zF'(z) \ \ (F \in UCV) \Big\}.$$

A function $f \in \mathbb{A}$ is said to be k-uniformly convex in $\mathbb{D}$, if the image of every circular arc γ, contained in $\mathbb{D}$, with center ς , where $|\varsigma| \leqslant k \ \big(k \in [0,\infty)\big)$, is convex. It is worth to mention that 1-$UCV = UCV$. This class of functions is denoted by $k-UCV$. The analytical description of $k-UCV$ can be stated as follows [18]:

$$f \in k-UCV \iff \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right|, \quad (k \geqslant 0, \ z \in \mathbb{D}).$$

The class of k-starlike functions, denoted by k-ST, were also introduced and studied by Kanas et al. [19], as follows:

$$k-\mathrm{ST} := \Big\{ f \in \mathbb{A} : f(z) = zg'(z), g \in k-UCV \Big\}.$$

The characterization for the functions from the class $k-\mathrm{ST}$ can also be described as follows [19, Theorem 2.1]:

$$f \in k-\mathrm{ST} \iff \Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|, \quad (k \geqslant 0, \ z \in \mathbb{D}).$$

For $k = 0$, we find the class of starlike functions and for $k = 1$ we obtain the class of starlike functions $\mathscr{S}_p$. Moreover, it is worth mention that if $f \in \mathbb{A}$ satisfies the following condition [19, Theorem 2.3]:

$$\sum_{\ell=2}^{\infty} [\ell + k(\ell - 1)] |a_\ell| \leqslant 1, \tag{1.12}$$

for some $k \ (0 \leqslant k < \infty)$, then f is k-starlike in $\mathbb{D}$.

The main objective of this research is to examine into a specific sufficiency criterion for the starlikeness of some analytic functions related to the function $\mathbb{Q}_\nu(z,b)$, consult (1.9), for this we consider the following normalized forms:

$$\mathfrak{Q}_\nu(z;b) = \frac{\Gamma(\nu)z}{\gamma(\nu,b)} \frac{\partial\left[z\mathbb{Q}_\nu(z,b)\right]}{\partial z} = \sum_{\ell=1}^{\infty} v_\ell(\nu,b) z^\ell, \tag{1.13}$$

where

$$v_\ell(\nu,b) = \frac{\ell\Gamma(\nu)\gamma(\nu+\ell-1,b)}{(\ell-1)!\gamma(\nu,b)\Gamma(\nu+\ell-1)}, \qquad \ell \geqslant 1. \tag{1.14}$$

In order to prove our results the following preliminary results will be helpful. The following two lemmas are due to Fejér [15].

LEMMA 1.1. *If the function* $f(z) = z + \sum_{\ell=2}^{\infty} a_\ell z^\ell$, *where* $a_\ell \geqslant 0$ *for all* $\ell \geqslant 2$, *is analytic in* $\mathbb{D}$, *and if the sequences* $(\ell a_\ell)_{\ell \geqslant 1}$ *and* $(\ell a_\ell - (\ell+1)a_{\ell+1})_{\ell \geqslant 1}$ *both are decreasing, then* f *is starlike in* $\mathbb{D}$.

LEMMA 1.2. *If the function* $f(z) = 1 + \sum_{\ell=2}^{\infty} a_\ell z^{\ell-1}$, *where* $a_\ell \geqslant 0$ *for all* $\ell \geqslant 2$, *is analytic in* $\mathbb{D}$ *and if* $(a_\ell)_{\ell \geqslant 1}$ *is a convex decreasing sequence, i.e.,* $a_\ell - 2a_{\ell+1} + a_{\ell+2} \geqslant 0$ *and* $a_\ell - a_{\ell+1} \geqslant 0$ *for all* $\ell \geqslant 1$, *then*

$$\Re(f(z)) > \frac{1}{2}, \quad \text{for all } z \in \mathbb{D}.$$

2. First set of main results

Here, we determined some sufficient conditions on the parameters of the normalized form of the generalized Marcum Q-function $\mathfrak{Q}_\nu(z;b)$ to belong to a certain class of starlike functions and k-starlike functions.

THEOREM 2.1. *Assume that the parameters* $\nu > 0$ *and* $b \in (0, b_0)$ *where* $b_0 \approx 3.47401\ldots$ *is the unique positive root of the equation* $(9x+1)e^{-x} - 1 = 0$. *Moreover, if* $7\nu\gamma(\nu,b) \leqslant 8b^\nu e^{-b}$ *then the function* $z \mapsto \mathfrak{Q}_\nu(z;b)$ *is starlike in* $\mathbb{D}$. *Furthermore, we have*

$$\Re\left(\frac{\mathfrak{Q}_\nu(z;b)}{z}\right) > \frac{1}{2} \qquad \text{for all } z \in \mathbb{D}.$$

Proof. First we prove that the sequence $\{\ell v_\ell(\nu,b)\}_{\ell \geqslant 1}$ is decreasing. Let $\ell \geqslant 2$ be fixed. From the following functional equation:

$$\gamma(\nu+1,x) = \nu\gamma(\nu,x) - x^\nu e^{-x}, \tag{2.15}$$

it follows that

$$\begin{aligned}
&\ell v_\ell(\nu,b)-(\ell+1)v_{\ell+1}(\nu,b)\\
&=\frac{\Gamma(\nu)}{(\ell-1)!\gamma(\nu,b)}\left[\frac{\ell^2\gamma(\nu+\ell-1,b)}{\Gamma(\nu+\ell-1)}-\frac{(\ell+1)^2\gamma(\nu+\ell,b)}{\ell\,\Gamma(\nu+\ell)}\right]\\
&=\frac{\Gamma(\nu)}{\ell!\gamma(\nu,b)}\left[\frac{(\ell^3-(\ell+1)^2)\gamma(\nu+\ell-1,b)}{\Gamma(\nu+\ell-1)}+\frac{(\ell+1)^2b^{\nu+\ell-1}e^{-b}}{\Gamma(\nu+\ell)}\right]\\
&=\frac{\Gamma(\nu)}{\ell!\gamma(\nu,b)}\left[\frac{(9\ell^3-8(\ell+1)^2)\gamma(\nu+\ell-1,b)}{9\Gamma(\nu+\ell-1)}+\frac{(\ell+1)^2b^{\nu+\ell-1}e^{-b}}{\Gamma(\nu+\ell)}\right.\\
&\quad\left.-\frac{(\ell+1)^2\gamma(\nu+\ell-1,b)}{9\Gamma(\nu+\ell-1)}\right].
\end{aligned} \tag{2.16}$$

Moreover, by using the following estimate [31, Eq. (8.10.2)]:

$$\gamma(\nu,x)\leqslant\frac{x^{\nu-1}(1-e^{-x})}{\nu},\qquad(\nu\geqslant1,\ x>0) \tag{2.17}$$

we obtain

$$\frac{\gamma(\nu+\ell-1,b)}{\Gamma(\nu+\ell-1)}\leqslant\frac{b^{\nu+\ell-2}(1-e^{-b})}{\Gamma(\nu+\ell)}. \tag{2.18}$$

Hence, by using the above inequality, we obtain

$$\begin{aligned}
\frac{(\ell+1)^2b^{\nu+\ell-1}e^{-b}}{\Gamma(\nu+\ell)}-\frac{(\ell+1)^2\gamma(\nu+\ell-1,b)}{9\Gamma(\nu+\ell-1)}&\geqslant\frac{(\ell+1)^2b^{\nu+\ell-2}\left[\left(9b+1\right)e^{-b}-1\right]}{9\Gamma(\nu+\ell)}\\
&\geqslant0,
\end{aligned} \tag{2.19}$$

under the hypothesis $\left(9b+1\right)e^{-b}-1\geqslant0$. Owing to the equations (2.16) and (2.19), we establish that

$$\ell v_\ell(\nu,b)\geqslant(\ell+1)v_{\ell+1}(\nu,b),\quad\text{for all }\ell\geqslant2.$$

Moreover, the hypothesis $3\nu\gamma(\nu,b)\leqslant4b^\nu e^{-b}$ implies that $2v_2(\nu,b)\leqslant v_1(\nu,b)=1$. This in turn implies that the sequences $\{\ell v_\ell(\nu,b)\}_{\ell\geqslant1}$ is decreasing. Next, we show that the sequence $\Big\{\ell v_\ell(\nu,b)-(\ell+1)v_{\ell+1}(\nu,b)\Big\}_{\ell\geqslant1}$ is decreasing. Let $\ell\geqslant2$. For convenience, we denote

$$A_\ell(\nu,b)=\ell v_\ell(\nu,b)-(\ell+1)v_{\ell+1}(\nu,b),\quad\text{for all}\quad\ell\geqslant1.$$

Owing to the functional equation of the upper incomplete gamma function (2.15), we get

$$\begin{aligned}
&\ell v_\ell(\nu,b)-2(\ell+1)v_{\ell+1}(\nu,b)+(\ell+2)v_{\ell+2}(\nu,b)\\
&\quad=\frac{\Gamma(\nu)}{(\ell-1)!\gamma(\nu,b)}\left[\frac{\ell^2\gamma(\nu+\ell-1,b)}{\Gamma(\nu+\ell-1)}+\frac{\left[(\ell+2)^2-2(\ell+1)^3\right]\gamma(\nu+\ell,b)}{\ell(\ell+1)\Gamma(\nu+\ell)}\right.\\
&\qquad\left.-\frac{(\ell+2)^2b^{\nu+\ell}e^{-b}}{\ell(\ell+1)\Gamma(\nu+\ell+1)}\right]\\
&\quad=\frac{\Gamma(\nu)\left[\ell^3(\ell+1)+(\ell+2)^2-2(\ell+1)^3\right]\gamma(\nu+\ell-1,b)}{(\ell+1)!\gamma(\nu,b)\Gamma(\nu+\ell-1)}\\
&\qquad-\frac{\Gamma(\nu)b^{\nu+\ell-1}e^{-b}}{(\ell+1)!\gamma(\nu,b)}\left(\frac{(\ell+2)^2-2(\ell+1)^3}{\Gamma(\nu+\ell)}+\frac{(\ell+2)^2be^{-b}}{\Gamma(\nu+\ell+1)}\right)\\
&\quad=\frac{\Gamma(\nu)\left[\ell^3(\ell+1)+(\ell+2)^2-2(\ell+1)^3\right]\gamma(\nu+\ell-1,b)}{(\ell+1)!\gamma(\nu,b)\Gamma(\nu+\ell-1)}\\
&\qquad+\frac{\Gamma(\nu)b^{\nu+\ell-1}e^{-b}}{(\ell+1)!\gamma(\nu,b)\Gamma(\nu+\ell)}\left(2(\ell+1)^3-(\ell+2)^2-\frac{b(\ell+2)^2}{\nu+\ell}\right)\\
&\quad=\frac{\Gamma(\nu)\left[27\ell^3(\ell+1)+27(\ell+2)^2-40(\ell+1)^3\right]\gamma(\nu+\ell-1,b)}{27(\ell+1)!\gamma(\nu,b)\Gamma(\nu+\ell-1)}\\
&\qquad+\frac{\Gamma(\nu)}{(\ell+1)!\gamma(\nu,b)}\left[\frac{b^{\nu+\ell-1}e^{-b}}{\Gamma(\nu+\ell)}\left(2(\ell+1)^3-(\ell+2)^2-\frac{b(\ell+2)^2}{\nu+\ell}\right)\right.\\
&\qquad\left.-\frac{14(\ell+1)^3\gamma(\nu+\ell-1,b)}{27\Gamma(\nu+\ell-1)}\right].
\end{aligned}\tag{2.20}$$

By using (2.18) we have

$$\frac{14(\ell+1)^3\gamma(\nu+\ell-1,b)}{27\Gamma(\nu+\ell-1)}\leqslant\frac{14(\ell+1)^3b^{\nu+\ell-2}(1-e^{-b})}{27\Gamma(\nu+\ell)}.\tag{2.21}$$

Therefore, by using the above inequality, we establish that

$$\begin{aligned}
&\frac{b^{\nu+\ell-1}e^{-b}}{\Gamma(\nu+\ell)}\left(2(\ell+1)^3-(\ell+2)^2-\frac{b(\ell+2)^2}{\nu+\ell}\right)-\frac{14(\ell+1)^3\gamma(\nu+\ell-1,b)}{27\Gamma(\nu+\ell-1)}\\
&\quad\geqslant\frac{b^{\nu+\ell-1}e^{-b}}{\Gamma(\nu+\ell)}\left(2(\ell+1)^3-(\ell+2)^2-\frac{b(\ell+2)^2}{\nu+\ell}-\frac{14(\ell+1)^3(1-e^{-b})}{27b}\right)\\
&\quad\geqslant\frac{b^{\nu+\ell-1}(\ell+1)^3e^{-b}}{\Gamma(\nu+\ell)}\left(2-\frac{(\ell+2)^2}{(\ell+1)^3}-\frac{b(\ell+2)^2}{(\nu+\ell)(\ell+1)^3}-\frac{14(1-e^{-b})}{27b}\right).
\end{aligned}\tag{2.22}$$

On the other hand, by using the fact that the function $t \mapsto \frac{(t+2)^2}{(t+1)^3}$ is decreasing on $[2,\infty)$, we conclude that

$$\frac{(\ell+2)^2}{(\ell+1)^3} + \frac{b(\ell+2)^2}{(\nu+\ell)(\ell+1)^3} \leqslant \frac{8(b+2)}{27}. \tag{2.23}$$

Hence, by combining (2.22) and (2.23), we obtain

$$2 - \frac{(\ell+2)^2}{(\ell+1)^3} - \frac{b(\ell+2)^2}{(\nu+\ell)(\ell+1)^3} - \frac{14(1-e^{-b})}{27b} \geqslant \frac{-8b^2+38b+14(e^{-b}-1)}{27b}, \tag{2.24}$$

and the last expression is non-negative by our assumption. This together with (2.20) and by using the fact that

$$\ell^3(\ell+1) + (\ell+2)^2 - 40/27(\ell+1)^3 \geqslant 0$$

for $\ell \geqslant 2$, we conclude that the sequence $\{\ell v_\ell(\nu,b)\}_{\ell\geqslant 2}$ is convex. Finally, we see that the condition $7\nu\gamma(\nu,b) \leqslant 8b^\nu e^{-b}$ implies that $4v_2(\nu,b) \leqslant v_1(\nu,b)$. Therefore, the sequence $\{\ell v_\ell(\nu,b)\}_{\ell\geqslant 1}$ is convex. Thus, it follows by Lemma 1.2 that the function $\mathfrak{Q}_\nu(z;b)$ is starlike in $\mathbb{D}$. Now, we apply Lemma 1.2 to prove that $\Re\left(\frac{\mathfrak{Q}_\nu(z;b)}{z}\right) > \frac{1}{2}$ for all $z \in \mathbb{D}$. For this, we consider the function $\tilde{\mathfrak{Q}}_\nu(z;b)$ defined by

$$\begin{aligned} \tilde{\mathfrak{Q}}_\nu(z;b) &= \frac{\mathfrak{Q}_\nu(z;b)}{z} \\ &= \sum_{\ell=1}^{\infty} v_\ell(\nu,b) z^{n-1}. \end{aligned} \tag{2.25}$$

But $\{v_\ell(\nu,b)\}_{\ell\geqslant 1}$ is a convex decreasing sequence, under the given conditions. Hence, Lemma 1.2 allows us conclude the asserted property and this completes the proof. □

Upon setting $\nu = 1$ in Theorem 2.1, in view of (1.7), we compute the following result.

COROLLARY 2.2. *If $b \in (0,b_1)$ where $b_1 \approx 0.261373\ldots$ is the unique positive root of the equation*

$$(8x+7)e^{-x} - 7 = 0,$$

then the function $z \mapsto \mathfrak{Q}_1(z;b)$ is starlike in $\mathbb{D}$.

EXAMPLE 2.3. The function $z \mapsto \mathfrak{Q}_1(z;1/4)$ is starlike in $\mathbb{D}$, see Figure 1.

THEOREM 2.4. *Let $\nu > 0$ and $b > 0$. The following assertions holds true:*
(a). If the following inequality

$$b^{\nu-1}(1-e^{-b})[(b^2+3b+1)e^b - 1] \leqslant \nu(\nu+1)\gamma(\nu,b),$$

holds true, then the function $z \mapsto \mathfrak{Q}_\nu(z;b)$ is starlike in $\mathbb{D}$.

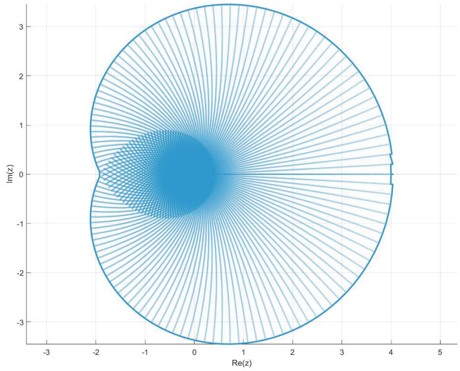

Figure 1: *Image of the open unit disk under the function* $\mathfrak{Q}_1(z;1/4)$.

(b). If the following inequality

$$b^{\nu-1}(1-e^{-b})\left[(b+1)e^{b}-1\right]\leqslant \nu(\nu+1)\gamma(\nu,b),$$

holds true, then the function $z\mapsto \mathfrak{Q}_\nu(z;b)$ *is starlike in*

$$\mathbb{D}_{\frac{1}{2}}=\left\{z: z\in\mathbb{C} \text{ and } |z|<\frac{1}{2}\right\}.$$

Proof. (a). To show that the function $z\mapsto \mathfrak{Q}_\nu(z;b)$ is starlike in $\mathbb{D}$, it suffices to establish that

$$\Re\left(\frac{z[\mathfrak{Q}_\nu(z;b)]'}{\mathfrak{Q}_\nu(z;b)}\right)>0, \qquad \text{for all } z\in\mathbb{D}.$$

For this objective in view, it suffices to establish that

$$\left|\frac{z[\mathfrak{Q}_\nu(z;b)]'-\mathfrak{Q}_\nu(z;b)}{\mathfrak{Q}_\nu(z;b)}\right|<1, \qquad \text{for all } z\in\mathbb{D}.$$

Let $z\in\mathbb{D}$. Bearing in mind the fact that the function $t\mapsto\frac{1}{\Gamma(t)}$ is decreasing on $[2,\infty)$, according to (1.13) and (2.17), we establish that

$$\begin{aligned}\left|\frac{z[\mathfrak{Q}_\nu(z;b)]'-\mathfrak{Q}_\nu(z;b)}{z}\right| &< \frac{\Gamma(\nu)}{\gamma(\nu,b)}\sum_{\ell=1}^{\infty}\frac{(\ell+1)\gamma(\nu+\ell,b)}{(\ell-1)!\Gamma(\nu+\ell)}\\ &\leqslant \frac{\Gamma(\nu)b^{\nu-1}(1-e^{-b})}{\gamma(\nu,b)}\sum_{\ell=1}^{\infty}\frac{(\ell+1)b^{\ell}}{(\ell-1)!\Gamma(\nu+\ell+1)}\\ &\leqslant \frac{b^{\nu-1}(1-e^{-b})}{\nu(\nu+1)\gamma(\nu,b)}\sum_{\ell=1}^{\infty}\frac{(\ell+1)b^{\ell}}{(\ell-1)!}\\ &= \frac{b^{\nu}(b+2)(e^{b}-1)}{\nu(\nu+1)\gamma(\nu,b)}.\end{aligned} \tag{2.26}$$

Moreover, from (1.13) and (2.17), it follows that

$$\begin{aligned}\left|\frac{\mathfrak{Q}_\nu(z;b)}{z}\right| &> 1-\frac{\Gamma(\nu)}{\gamma(\nu,b)}\sum_{\ell=2}^{\infty}\frac{\ell\gamma(\nu+\ell-1,b)}{(\ell-1)!\Gamma(\nu+\ell-1)}\\ &\geqslant 1-\frac{\Gamma(\nu)b^{\nu-1}(1-e^{-b})}{\gamma(\nu,b)}\sum_{\ell=1}^{\infty}\frac{(\ell+1)b^\ell}{\ell!\Gamma(\nu+\ell+1)}\\ &\geqslant 1-\frac{b^{\nu-1}(1-e^{-b})}{\nu(\nu+1)\gamma(\nu,b)}\sum_{\ell=1}^{\infty}\frac{(\ell+1)b^\ell}{\ell!}\\ &=\frac{\nu(\nu+1)\gamma(\nu,b)-b^{\nu-1}(1-e^{-b})[(b+1)e^b-1]}{\nu(\nu+1)\gamma(\nu,b)}.\end{aligned} \tag{2.27}$$

By virtue of relations (2.26) and (2.27), for any $z\in\mathbb{D}$, we infer

$$\left|\frac{z[\mathfrak{Q}_\nu(z;b)]'-\mathfrak{Q}_\nu(z;b)}{\mathfrak{Q}_\nu(z;b)}\right|<1, \qquad \text{for all } z\in\mathbb{D},$$

where we have made use of the given hypothesis.

(b). Let $z\in\mathbb{D}$. Again, from (1.13) and (2.17) and using the fact that the function $t\mapsto\frac{1}{\Gamma(t)}$ is decreasing on $[2,\infty)$, we infer

$$\begin{aligned}\left|\frac{\mathfrak{Q}_\nu(z;b)-z}{z}\right| &< \frac{b^\nu(1-e^{-b})\Gamma(\nu)}{\gamma(\nu,b)}\sum_{\ell=2}^{\infty}\frac{\ell b^{\ell-1}}{(\ell-1)!\Gamma(\nu+\ell)}\\ &\leqslant\frac{b^\nu(1-e^{-b})}{\nu(\nu+1)\gamma(\nu,b)}\sum_{\ell=2}^{\infty}\frac{\ell b^{\ell-1}}{(\ell-1)!}\\ &=\frac{b^\nu(1-e^{-b})\left[(b+1)e^b-1\right]}{\nu(\nu+1)\gamma(\nu,b)}\\ &\leqslant 1,\end{aligned} \tag{2.28}$$

under the given hypothesis. On the other hand, according to MacGregor [20], if a function $f\in\mathbb{A}$ such that

$$\left|\frac{f(z)-z}{z}\right|<1, \qquad \forall z\in\mathbb{D},$$

then f is starlike in $\mathbb{D}_{\frac{1}{2}}$. Therefore we conclude the asserted result. □

Choosing $\nu=1$ in part (a) of Theorem 2.4, in view of (1.7), we compute the following result:

COROLLARY 2.5. *If $b\in(0,b_2)$ where $b_2\approx 0.339950\ldots$ is the unique positive root of the equation*

$$3-(x^2+3x+1)e^x=0,$$

then the function $z\mapsto\mathfrak{Q}_1(z;b)$ is starlike in $\mathbb{D}$.

EXAMPLE 2.6. The function $z \mapsto \mathfrak{Q}_1(z;1/3)$ is starlike in $\mathbb{D}$, see Figure 2.

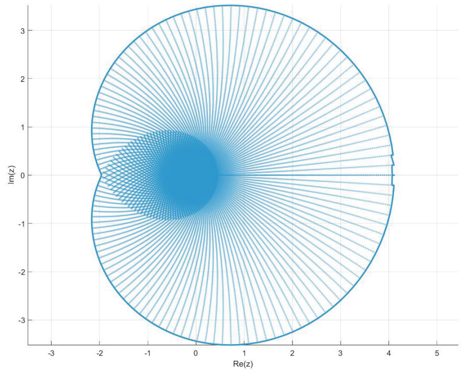

Figure 2: *Image of the open unit disk under the function* $\mathfrak{Q}_1(z;1/3)$.

Taking $\nu = 1$ in part (b) of Theorem 2.4, in view of (1.7), we obtain the following result:

COROLLARY 2.7. *If* $b \in (0, b_3)$ *where* $b_3 \approx 0.617642\ldots$ *is the unique positive root of the equation*

$$3-(x+1)e^x = 0,$$

then the function $z \mapsto \mathfrak{Q}_1(z;b)$ *is starlike in* $\mathbb{D}_{\frac{1}{2}}$.

EXAMPLE 2.8. The function $z \mapsto \mathfrak{Q}_1(z;61/100)$ is starlike in $\mathbb{D}_{\frac{1}{2}}$, see Figure 3.

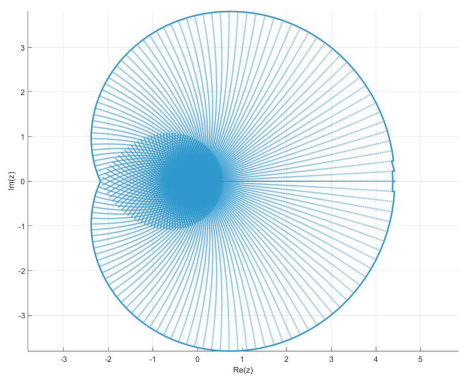

Figure 3: *Image of the open unit disk under the function* $\mathfrak{Q}_1(z;61/100)$.

THEOREM 2.9. *Assume that* $\nu, b > 0$ *and* $k \geqslant 0$. *If the following inequality*

$$b^{\nu-1}(1-e^{-b})\left[(k+1)(b^2+2b)e^b+(b+1)e^b-1\right] \leqslant \nu(\nu+1)\gamma(\nu,b),$$

is valid, then the function $z \mapsto \mathfrak{Q}_\nu(z;b)$ *is* k*-starlike in* $\mathbb{D}$.

Proof. In order to establish the required result, it suffices to show that

$$\sum_{\ell=2}^{\infty}\big(\ell+k(\ell-1)\big)\nu_\ell(\nu,b) < 1, \quad \text{for all} \quad k \geqslant 0.$$

Moreover, by virtue of (1.13) and (2.17), we establish that

$$\begin{aligned}
\sum_{\ell=2}^{\infty}\big(\ell+k(\ell-1)\big)\nu_\ell(\nu,b) &= \frac{\Gamma(\nu)}{\gamma(\nu,b)}\sum_{\ell=2}^{\infty}\frac{\ell\big(\ell+k(\ell-1)\big)\gamma(\nu+\ell-1,b)}{(\ell-1)!\Gamma(\nu+\ell-1)} \\
&\leqslant \frac{b^{\nu-1}(1-e^{-b})\Gamma(\nu)}{\gamma(\nu,b)}\sum_{\ell=2}^{\infty}\frac{\ell\big(\ell+k(\ell-1)\big)b^{\ell-1}}{(\ell-1)!\Gamma(\nu+\ell)} \\
&\leqslant \frac{b^{\nu-1}(1-e^{-b})}{\nu(\nu+1)\gamma(\nu,b)}\sum_{\ell=2}^{\infty}\frac{\ell\big(\ell+k(\ell-1)\big)b^{\ell-1}}{(\ell-1)!} \\
&= \frac{b^{\nu-1}(1-e^{-b})}{\nu(\nu+1)\gamma(\nu,b)}\left(\sum_{\ell=2}^{\infty}\frac{\ell^2 b^{\ell-1}}{(\ell-1)!}+k\sum_{\ell=2}^{\infty}\frac{\ell b^{\ell-1}}{(\ell-2)!}\right).
\end{aligned} \tag{2.29}$$

Moreover, straightforward calculation would yield

$$\sum_{\ell=2}^{\infty}\frac{\ell b^{\ell-1}}{(\ell-2)!} = (b^2+2b)e^b. \tag{2.30}$$

Further, we have

$$\begin{aligned}
\sum_{\ell=2}^{\infty}\frac{\ell^2 b^{\ell-1}}{(\ell-1)!} &= \sum_{\ell=2}^{\infty}\frac{\ell(\ell-1)b^{\ell-1}}{(\ell-1)!}+\sum_{\ell=2}^{\infty}\frac{\ell b^{\ell-1}}{(\ell-1)!} \\
&= \sum_{\ell=2}^{\infty}\frac{\ell b^{\ell-1}}{(\ell-2)!}+\sum_{\ell=2}^{\infty}\frac{\ell b^{\ell-1}}{(\ell-1)!} \\
&= (b^2+3b+1)e^b-1,
\end{aligned} \tag{2.31}$$

Inserting (2.30) and (2.31) into (2.29), we establish that

$$\begin{aligned}
\sum_{\ell=2}^{\infty}\big(\ell+k(\ell-1)\big)\nu_\ell(\nu,b) &\leqslant \frac{b^{\nu-1}(1-e^{-b})\left[(k+1)(b^2+2b)e^b+(b+1)e^b-1\right]}{\nu(\nu+1)\gamma(\nu,b)} \\
&\leqslant 1,
\end{aligned}$$

where we have made use of the given hypothesis. □

By taking $k=1$ in Theorem 2.9, we compute the following result.

COROLLARY 2.10. *Let* $\nu > 0$ *and* $b > 0$. *If the following inequality holds true:*

$$b^{\nu-1}(1-e^{-b})\left[(2b^2+5b+1)e^b-1\right] \leqslant \nu(\nu+1)\gamma(\nu,b),$$

then $\mathfrak{Q}_\nu(z;b) \in \mathscr{S}_p$.

By setting $\nu = 1$ in the above Corollary, in view of (1.7), we obtain the following results.

COROLLARY 2.11. *Let* $b \in (0,b_4)$ *where* $b_4 \approx 0.245367\ldots$ *is the unique positive root of the equation*

$$3-(2x^2+5x+1)e^x = 0,$$

then $\mathfrak{Q}_1(z;b) \in \mathscr{S}_p$.

EXAMPLE 2.12. The function $\mathfrak{Q}_1(z;24/100) \in \mathscr{S}_p$.

REMARK 2.13. If we set $k = 0$ in Theorem 2.9, we re-obtain the result asserted by part (a) of Theorem 2.4.

3. Second set of main results

In this section, we present one of our main results which gives a sufficient condition for the Alexander transform Λ_f where $f(z) = \mathfrak{Q}_\nu(\nu,b)(z)$ to be in the family of starlike functions. The Alexander transform of f is defined on $\mathbb{D}$ by

$$\begin{aligned}\Lambda_f(z) &= \int_0^z \frac{(f*l)(t)}{t}dt = \int_0^z \frac{f(t)}{t}dt \\ &= z+\sum_{\ell=2}^{\infty}\frac{a_\ell}{\ell}z^\ell = (f*h)(z),\end{aligned} \tag{3.32}$$

where the functions f,l and h are defined by

$$f(z) = z+\sum_{\ell=2}^{\infty}a_\ell z^\ell, \quad l(z) = \frac{z}{1-z} = \sum_{\ell=1}^{\infty}z^\ell,$$

and

$$h(z) = -\log(1-z) = \sum_{\ell=1}^{\infty}\frac{z^\ell}{\ell}.$$

We note that the convolution $f*g$, or Hadamard product (see, for instance, the book of P.L. Duren [14]), of two power series

$$f(z) = z+\sum_{\ell=2}^{\infty}a_\ell z^\ell,$$

$$g(z) = z+\sum_{\ell=2}^{\infty}b_\ell z^\ell,$$

is defined as the power series

$$(f*g)(z)=z+\sum_{\ell=2}^{\infty}a_\ell b_\ell z^\ell.$$

Our first main results of this section is the following theorem.

THEOREM 3.1. *Assume that the parameters* $\nu>0$ *and* $b\in(0,b_5)$ *where* $b_5\approx 0.338462\ldots$ *is the unique positive root of the equation*

$$-12x^2+10x+7e^{-x}-7=0.$$

If the following inequality

$$\nu\gamma(\nu,b)\leqslant 2b^\nu e^{-b},$$

holds true, then the function $z\mapsto\Lambda_{\mathfrak{Q}_\nu(.,b)}(z)$ *is starlike in* $\mathbb{D}$. *Moreover, we have*

$$\Re\left(\frac{\Lambda_{\mathfrak{Q}_\nu(.,b)}(z)}{z}\right)>\frac{1}{2},\quad \text{for all } z\in\mathbb{D}.$$

Proof. From (1.13), the Alexander transform of the function $\mathfrak{Q}_\nu(z;b)$ takes the following form:

$$\Lambda_{\mathfrak{Q}_\nu(.,b)}(z)=\sum_{\ell=1}^{\infty}w_\ell(\nu,b)z^\ell, \tag{3.33}$$

where the sequence $\{w_\ell(\nu,b)\}_{\ell\geqslant1}$ is defined by

$$w_\ell(\nu,b)=\frac{v_\ell(\nu,b)}{\ell},\qquad \ell\geqslant1. \tag{3.34}$$

The inequality $w_1(\nu,b)-2w_2(\nu,b)\geqslant0$ holds because the parameters ν and b satisfy $\nu\gamma(\nu,b)\leqslant 2b^\nu e^{-b}$. Next, we show that the sequence $\{\ell w_\ell(\nu,b)\}_{\ell\geqslant2}$ is decreasing. Fix any $\ell\geqslant2$. From (2.15) it follows that

$$\ell w_\ell(\nu,b)-(\ell+1)w_\ell(\nu,b)=\frac{\Gamma(\ell)}{\ell!\gamma(\nu,b)}\left(\frac{(\ell^2-\ell-1)\gamma(\nu+\ell-1,b)}{\Gamma(\nu+\ell-1)}+\frac{(\ell+1)b^{\nu+\ell-1}e^{-b}}{\Gamma(\nu+\ell)}\right),$$

and the last expression is non-negative for all $\ell\geqslant2$.

Consequently, the sequence $\{\ell w_\ell(\nu,b)\}_{\ell\geqslant1}$ is decreasing. Next, we establish that the sequence $\{\ell w_\ell(\nu,b)-(\ell+1)w_{\ell+1}(\nu,b)\}_{\ell\geqslant1}$ is decreasing. For convenience we denote

$$B_\ell(\nu,b)=\ell w_\ell(\nu,b)-(\ell+1)w_{\ell+1}(\nu,b)\quad \text{for each } \ell\geqslant1.$$

It follows from (2.15) and (2.17) that

$$\begin{aligned}B_\ell(\nu,b)-B_{\ell+1}(\nu,b) &= \frac{\Gamma(\nu)\big(\ell^3-\ell^2-3\ell\big)\gamma(\nu+\ell-1,b)}{(\ell+1)!\gamma(\nu,b)\Gamma(\nu+\ell-1)}+\frac{2(\ell+1)\Gamma(\nu)b^{\nu+\ell-1}e^{-b}}{\ell!\gamma(\nu,b)\Gamma(\nu+\ell)}\\ &\quad -\frac{(\ell+2)\Gamma(\nu)b^{\nu+\ell-1}e^{-b}}{(\ell+1)!\gamma(\nu,b)\Gamma(\nu+\ell)}-\frac{(\ell+2)\Gamma(\nu)b^{\nu+\ell}e^{-b}}{(\ell+1)!\gamma(\nu,b)\Gamma(\nu+\ell+1)}\\ &= \frac{\Gamma(\nu)\big(\ell^3-\ell^2-3\ell+7/2\big)\gamma(\nu+\ell-1)}{(\ell+1)!\gamma(\nu,b)\Gamma(\nu+\ell-1)}-\frac{7\Gamma(\nu)\gamma(\nu+\ell-1)}{2(\ell+1)!\gamma(\nu,b)\Gamma(\nu+\ell-1)}\\ &\quad +\frac{\Gamma(\nu)b^{\nu+\ell-1}e^{-b}}{\ell!\gamma(\nu,b)\Gamma(\nu+\ell)}\left(2(\ell+1)-\frac{\ell+2}{\ell+1}-\frac{b(\ell+2)}{\nu+\ell}\right)\\ &\geqslant \frac{\Gamma(\nu)\big(\ell^3-\ell^2-3\ell+7/2\big)\gamma(\nu+\ell-1)}{(\ell+1)!\gamma(\nu,b)\Gamma(\nu+\ell-1)}\\ &\quad +\frac{\Gamma(\nu)b^{\nu+\ell-1}e^{-b}}{\ell!\gamma(\nu,b)\Gamma(\nu+\ell)}\left(2(\ell+1)-\frac{\ell+2}{\ell+1}-\frac{b(\ell+2)}{\nu+\ell}-\frac{7(1-e^{-b})}{2b(\ell+1)}\right).\end{aligned} \tag{3.35}$$

On the other hand, by using the fact that the function $t\mapsto\frac{t+2}{t+1}+\frac{\alpha(t+2)}{t}+\frac{\beta}{t+1}$ is decreasing on $[1,\infty)$ for each $\alpha,\beta>0$ we conclude that

$$\frac{\ell+2}{\ell+1}+\frac{b(\ell+2)}{\ell}+\frac{7(1-e^{-b})}{2(\ell+1)}\leqslant\frac{12b^2+6b-7e^{-b}+7}{4b},\qquad \ell\geqslant 1. \tag{3.36}$$

Hence, by (3.35) and (3.36) we infer

$$\begin{aligned}B_\ell(\nu,b)-B_{\ell+1}(\nu,b)\geqslant&\frac{\Gamma(\nu)\big(\ell^3-\ell^2-3\ell+7/2\big)\gamma(\nu+\ell-1)}{(\ell+1)!\gamma(\nu,b)\Gamma(\nu+\ell-1)}\\ &+\frac{\Gamma(\nu)b^{\nu+\ell-1}(-12b^2+10b+7e^{-b}-7)e^{-b}}{4b\ell!\gamma(\nu,b)\Gamma(\nu+\ell)},\end{aligned} \tag{3.37}$$

where the last inequality is non-negative for all $\ell\geqslant 1$. Therefore, the sequence

$$\{\ell v_\ell(\nu,b)-(\ell+1)v_{\ell+1}(\nu,b)\}_{\ell\geqslant 1},$$

is decreasing. Thanks to Lemma 1.1 and Lemma 1.2, we deduce that $\Lambda_{\mathfrak{Q}_\nu}$ is starlike in $\mathbb{D}$ and $\Re\left(\frac{\Lambda_{\mathfrak{Q}_\nu(.,b)}}{z}\right)>\frac{1}{2}$ for all $z\in\mathbb{D}$. $\square$

Taking $\nu=1$ in the above Theorem we compute the following result.

COROLLARY 3.2. *Under the assumptions of Theorem 3.1, the function* $z\mapsto\Lambda_{\mathfrak{Q}_1(.,b)}(z)$ *is starlike in* $\mathbb{D}$*. Moreover, we have*

$$\Re\left(\frac{\Lambda_{\mathfrak{Q}_1(.,b)}(z)}{z}\right)>\frac{1}{2},\quad \text{for all } z\in\mathbb{D}.$$

EXAMPLE 3.3. The function $z \mapsto \Lambda_{\mathfrak{Q}_1(.,1/3)}(z)$ is starlike in $\mathbb{D}$, see Figure 4.

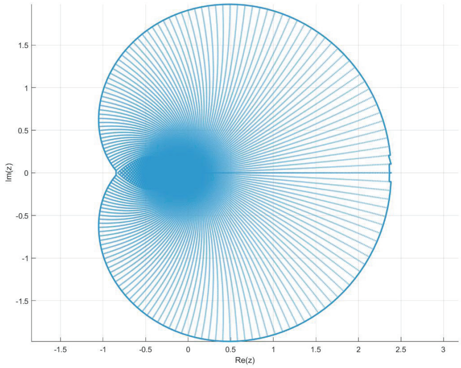

Figure 4: *Image of the open unit disk under the function* $\Lambda_{\mathfrak{Q}_1(.,1/3)}(z)$.

THEOREM 3.4. *Let $\nu > 0$ and $b > 0$. The following assertions are true:*
(a). If the following inequality

$$b^{\nu-1}(1-e^{-b})[(b+1)e^b - 1] \leqslant \nu(\nu+1)\gamma(\nu,b),$$

is valid, then the function $z \mapsto \Lambda_{\mathfrak{Q}_\nu(.,b)}(z)$ is starlike in $\mathbb{D}$.
(b). If the following inequality

$$b^{\nu-1}(1-e^{-b})(e^b - 1) \leqslant \nu(\nu+1)\gamma(\nu,b),$$

is valid, then the function $z \mapsto \Lambda_{\mathfrak{Q}_\nu(.,b)}(z)$ is starlike in $\mathbb{D}_{\frac{1}{2}}$.

Proof. (a). Let $z \in \mathbb{D}$. From (3.33) and (2.17), we obtain

$$\begin{aligned}
\left| \frac{z\left[\Lambda_{\mathfrak{Q}_\nu(.,b)}\right]'(z) - \Lambda_{\mathfrak{Q}_\nu(.,b)}(z)}{\Lambda_{\mathfrak{Q}_\nu(.,b)}(z)} \right| &< \frac{\Gamma(\nu)}{\gamma(\nu,b)} \sum_{\ell=1}^{\infty} \frac{\gamma(\nu+\ell,b)}{(\ell-1)!\Gamma(\nu+\ell)} \\
&\leqslant \frac{b^{\nu-1}\Gamma(\nu)(1-e^{-b})}{\gamma(\nu,b)} \sum_{\ell=1}^{\infty} \frac{b^\ell}{(\ell-1)!\Gamma(\nu+\ell+1)} \\
&\leqslant \frac{b^{\nu-1}(1-e^{-b})}{\nu(\nu+1)\gamma(\nu,b)} \sum_{\ell=1}^{\infty} \frac{b^\ell}{(\ell-1)!} \\
&= \frac{b^\nu(e^b-1)}{\nu(\nu+1)\gamma(\nu,b)}.
\end{aligned} \tag{3.38}$$

Furthermore, by applying the inequality (2.17), we establish that

$$\begin{aligned}\left|\frac{\Lambda_{\mathfrak{Q}_{\nu}(.,b)}(z)}{z}\right| &> 1-\frac{\Gamma(\nu)}{\gamma(\nu,b)}\sum_{\ell=1}^{\infty}\frac{\gamma(\nu+\ell,b)}{\ell!\Gamma(\nu+\ell)}\\ &\geqslant 1-\frac{b^{\nu-1}\Gamma(\nu)(1-e^{-b})}{\gamma(\nu,b)}\sum_{\ell=1}^{\infty}\frac{b^{\ell}}{\ell!\Gamma(\nu+\ell+1)}\\ &\geqslant 1-\frac{b^{\nu-1}(1-e^{-b})}{\nu(\nu+1)\gamma(\nu,b)}\sum_{\ell=1}^{\infty}\frac{b^{\ell}}{\ell!}\\ &=\frac{\nu(\nu+1)\gamma(\nu,b)-b^{\nu-1}(1-e^{-b})(e^{b}-1)}{\nu(\nu+1)\gamma(\nu,b)}.\end{aligned} \tag{3.39}$$

Therefore, by combining (3.38) and (3.39), we conclude

$$\left|\frac{z\left[\Lambda_{\mathfrak{Q}_{\nu}(.,b)}\right]'(z)-\Lambda_{\mathfrak{Q}_{\nu}(.,b)}(z)}{\Lambda_{\mathfrak{Q}_{\nu}(.,b)}(z)}\right|<1, \quad \text{for all } z\in\mathbb{D},$$

and the last expression is valid by our assumption.

(b). Let $z\in\mathbb{D},$ with the help of (3.33) and (2.17), yields

$$\begin{aligned}\left|\frac{\Lambda_{\mathfrak{Q}_{\nu}(.,b)}(z)-z}{z}\right| &< \frac{\Gamma(\nu)}{\gamma(\nu,b)}\sum_{\ell=2}^{\infty}\frac{\gamma(\nu+\ell-1,b)}{(\ell-1)!\Gamma(\nu+\ell-1)}\\ &\leqslant \frac{b^{\nu-1}(1-e^{-b})\Gamma(\nu)}{\gamma(\nu,b)}\sum_{\ell=2}^{\infty}\frac{b^{\ell-1}}{(\ell-1)!\Gamma(\nu+\ell)}\\ &\leqslant \frac{b^{\nu-1}(1-e^{-b})}{\nu(\nu+1)\gamma(\nu,b)}\sum_{\ell=2}^{\infty}\frac{b^{\ell-1}}{(\ell-1)!}\\ &=\frac{b^{\nu-1}(1-e^{-b})(e^{b}-1)}{\nu(\nu+1)\gamma(\nu,b)}\leqslant 1,\end{aligned} \tag{3.40}$$

under the given condition (b). This ends the proof of Theorem 3.4. □

Specifying $\nu=1$ in Theorem 3.4, thanks to (1.7), we compute the following corollaries.

COROLLARY 3.5. *Let $b\in(0,b_6)$ where $b_6\approx 0.617642\ldots$ is the unique positive root of the equation*

$$3-(x+1)e^{x}=0,$$

then the function $z\mapsto\Lambda_{\mathfrak{Q}_1(.,b)}(z)$ is starlike in $\mathbb{D}$.

COROLLARY 3.6. *If $b\in(0,\log(3))$ then the function $z\mapsto\Lambda_{\mathfrak{Q}_1(.,b)}(z)$ is starlike in $\mathbb{D}_{\frac{1}{2}}$.*

EXAMPLE 3.7. The function $z\mapsto\Lambda_{\mathfrak{Q}_1(.,3/5)}(z)$ is starlike in $\mathbb{D}$ and the function $z\mapsto\Lambda_{\mathfrak{Q}_1(.,\log(2))}(z)$ is starlike in $\mathbb{D}_{\frac{1}{2}}$, see Figure 5 and Figure 6.

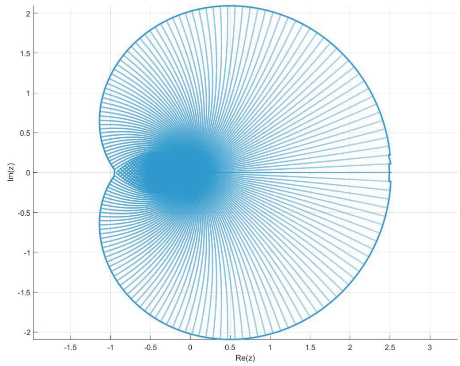

Figure 5: *Image of the open unit disk under the function* $\Lambda_{\mathfrak{Q}_1(.,3/5)}(z)$.

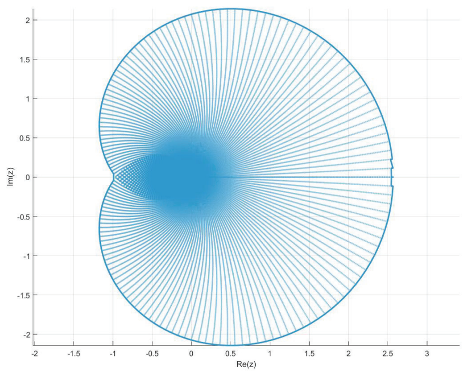

Figure 6: *Image of the open unit disk under the function* $\Lambda_{\mathfrak{Q}_1(.,\log(2))}(z)$.

THEOREM 3.8. *Assume that* $\nu, b > 0$ *and* $k \geqslant 0$. *If the following inequality*

$$b^{\nu-1}(1-e^{-b})\left[(b+1+kb)e^b-1\right] \leqslant \nu(\nu+1)\gamma(\nu,b),$$

is valid, then the function $z \mapsto \Lambda_{\mathfrak{Q}_\nu(.,b)}(z)$ *is* k*-starlike in* $\mathbb{D}$.

Proof. According to the analytic characterizations of k-starlike functions, to prove that the function $\Lambda_{\mathfrak{Q}_\nu(.,b)}(z)$ is k-starlike in $\mathbb{D}$ it is enough to prove that the following inequality

$$\sum_{\ell=2}^{\infty}\big(\ell+k(\ell-1)\big)w_\ell(\nu,b) < 1,$$

holds true for each $k \geqslant 0$. According to (3.33) and (2.17), we obtain

$$\begin{aligned}
\sum_{\ell=2}^{\infty}\big(\ell+k(\ell-1)\big)w_{\ell}(\nu,b) &= \frac{\Gamma(\nu)}{\gamma(\nu,b)}\sum_{\ell=2}^{\infty}\frac{\big(\ell+k(\ell-1)\big)\gamma(\nu+\ell-1,b)}{(\ell-1)!\Gamma(\nu+\ell-1)} \\
&\leqslant \frac{b^{\nu-1}(1-e^{-b})\Gamma(\nu)}{\gamma(\nu,b)}\sum_{\ell=2}^{\infty}\frac{\big(\ell+k(\ell-1)\big)b^{\ell-1}}{(\ell-1)!\Gamma(\nu+\ell)} \\
&\leqslant \frac{b^{\nu-1}(1-e^{-b})}{\nu(\nu+1)\gamma(\nu,b)}\sum_{\ell=2}^{\infty}\frac{\big(\ell+k(\ell-1)\big)b^{\ell-1}}{(\ell-1)!} \\
&= \frac{b^{\nu}(1-e^{-b})}{\nu(\nu+1)\gamma(\nu,b)}\left(\sum_{\ell=2}^{\infty}\frac{\ell b^{\ell-1}}{(\ell-1)!}+k\sum_{\ell=2}^{\infty}\frac{b^{\ell-1}}{(\ell-2)!}\right) \\
&= \frac{b^{\nu-1}(1-e^{-b})\left[(b+1+kb)e^{b}-1\right]}{\nu(\nu+1)\gamma(\nu,b)} \\
&\leqslant 1,
\end{aligned} \tag{3.41}$$

where we have made use of the given hypothesis. □

By taking $k=1$ in Theorem 3.8, we compute the following result.

COROLLARY 3.9. *Let* $\nu>0$ *and* $b>0$.
If the following inequality holds true:

$$b^{\nu-1}(1-e^{-b})\left[(2b+1)e^{b}-1\right] \leqslant \nu(\nu+1)\gamma(\nu,b),$$

then $\Lambda_{\mathfrak{Q}_{\nu}(.,b)}(z)\in\mathscr{S}_p$.

By setting $\nu=1$ in the above Corollary, in view of (1.7), we obtain the following results.

COROLLARY 3.10. *Let* $b\in(0,b_7)$ $b_7\approx 0.453295\ldots$ *is the unique positive root of the equation*

$$3-(2x+1)e^{x}=0,$$

then $\Lambda_{\mathfrak{Q}_{1}(.,b)}(z)\in\mathscr{S}_p$.

EXAMPLE 3.11. The function $\Lambda_{\mathfrak{Q}_{1}(.,45/10)}(z)\in\mathscr{S}_p$.

REMARK 3.12. If we set $k=0$ in Theorem 3.8, we easily get the result asserted by part (a) of Theorem 3.4.

4. The third set of main results

Our aim in this section is to find sufficient conditions for the function

$$z \mapsto ze^{z}(1-Q_{\nu}(\sqrt{2z},\sqrt{2b})),$$

to be starlike in the open unit disk. Furthermore, two classes of starlike functions related to the lower incomplete hypergeometric function ${}_p\gamma_p[z]$ are derived.

THEOREM 4.1. *Under the hypotheses of part (a) of Theorem 3.4 or if* $\mathfrak{Q}_{\nu}(1;b) \leqslant 2$, *then the function* $z \mapsto ze^{z}(1-Q_{\nu}(\sqrt{2}z^{\frac{1}{2}},\sqrt{2b}))$ *is starlike in* $\mathbb{D}$.

Proof. For convenience, let us write

$$\tilde{\mathfrak{Q}}_{\nu}(t;b) = \frac{\mathfrak{Q}_{\nu}(t;b)}{t} = \sum_{\ell=0}^{\infty} \rho_{\ell}(\nu,b)t^{\ell} \quad \text{where} \quad \rho_{\ell}(\nu,b) = \nu_{\ell+1}(\nu,b).$$

A simple computation leads us to

$$\begin{aligned} \int_0^z \tilde{\mathfrak{Q}}_{\nu}(t;b)dt &= \sum_{\ell=0}^{\infty} \frac{\rho_{\ell}(\nu,b)z^{\ell+1}}{\ell+1} \\ &= z + \sum_{\ell=2}^{\infty} \frac{\nu_{\ell}(\nu,b)z^{\ell}}{\ell} \\ &= z + \sum_{\ell=2}^{\infty} w_{\ell}(\nu,b)z^{\ell}, \end{aligned} \tag{4.42}$$

where the sequence $\{w_{\ell}(\nu,b)\}_{\ell\geqslant 1}$ is given by (3.34). In view of the inequality (1.12) (when $k=0$) we need only show that

$$\sum_{\ell=2}^{\infty} \ell w_{\ell}(\nu,b) < 1.$$

We note that

$$\sum_{\ell=2}^{\infty} \ell w_{\ell}(\nu,b) = \sum_{\ell=2}^{\infty} \nu_{\ell}(\nu,b) = \mathfrak{Q}_{\nu}(1;b) - 1 \leqslant 1$$

if $\mathfrak{Q}_{\nu}(1;b) \leqslant 2$. Moreover, taking $k=0$ in (3.41), we establish that

$$\sum_{\ell=2}^{\infty} \ell w_{\ell}(\nu,b) \leqslant 1,$$

if the following inequality

$$b^{\nu-1}(1-e^{-b})[(b+1)e^{b}-1] \leqslant \nu(\nu+1)\gamma(\nu,b),$$

is valid. Then, we conclude that the function $z \mapsto \int_0^z \frac{\mathfrak{Q}_\nu(t;b)}{t}dt$ is starlike. Moreover, it follows from (1.13) and (1.9) that

$$\begin{aligned}\int_0^z \frac{\mathfrak{Q}_\nu(t;b)}{t}dt &= \frac{\Gamma(\nu)}{\gamma(\nu,b)} \int_0^z [t\mathbb{Q}_\nu(t,b))]'\,dt \\ &= \frac{\Gamma(\nu)}{\gamma(\nu,b)} z\mathbb{Q}_\nu(z,b) \\ &= \frac{\Gamma(\nu)}{\gamma(\nu,b)} \left[ze^z(1-Q_\nu(\sqrt{2}z^{\frac{1}{2}},\sqrt{2b}))\right].\end{aligned}$$

This completes the proof of Theorem 4.1. □

If we take $\nu = 1,$ then the Theorem 4.1 yield.

COROLLARY 4.2. *Under the assumptions of Corollary 3.5, the function* $z \mapsto ze^z(1-Q(\sqrt{2z},\sqrt{2b}))$ *is starlike in* $\mathbb{D}$.

THEOREM 4.3. *Consider that one of the following assertions is valid:*
(a). The parameters $\nu, b > 0$ *satisfy the conditions given by*

$$2b^{\nu-1}\big(\cosh(b)-1\big) \leqslant \nu(\nu+1)\gamma(\nu,b).$$

(b). The parameters $\nu, b > 0$ *satisfy* $\Lambda_{\mathfrak{Q}_\nu(.,b)}(1) \leqslant 2$.
Then the function

$$z \mapsto z \cdot {}_2\gamma_2\left[\begin{matrix}(\nu,b),1\\ \nu,2\end{matrix}\middle| z\right],$$

is starlike in $\mathbb{D}$.

Proof. For convenience, let us write

$$\tilde{\Lambda}_{\mathfrak{Q}_\nu(.,b)}(t) = \frac{\Lambda_{\mathfrak{Q}_\nu(.,b)}(t)}{t} = \sum_{\ell=0}^{\infty} \tilde{\rho}_\ell(\nu,b)t^\ell \quad \text{where} \quad \tilde{\rho}_\ell(\nu,b) = w_{\ell+1}(\nu,b).$$

A direct computation gives

$$\int_0^z \tilde{\Lambda}_{\mathfrak{Q}_\nu(.,b)}(t)dt = z + \sum_{\ell=2}^{\infty} \chi_\ell(\nu,b)z^\ell, \tag{4.43}$$

where $\{\chi_\ell(\nu,b)\}_{\ell\geqslant 1}$ is defined by

$$\chi_\ell(\nu,b) = \frac{w_\ell(\nu,b)}{\ell}, \quad \text{for all } \ell \geqslant 1.$$

Owing to the inequality (2.17), we establish that

$$\begin{aligned}\sum_{\ell=2}^{\infty}\ell\chi_\ell(\nu,b)&=\sum_{\ell=2}^{\infty}w_\ell(\nu,b)\\&\leqslant\frac{b^{\nu-1}(1-e^{-b})\Gamma(\nu)}{\gamma(\nu,b)}\sum_{\ell=2}^{\infty}\frac{b^{\ell-1}}{(\ell-1)!\Gamma(\nu+\ell)}\\&\leqslant\frac{2b^{\nu-1}(\cosh(b)-1)}{\nu(\nu+1)\gamma(\nu,b)}\\&\leqslant 1,\end{aligned}\tag{4.44}$$

under the given hypothesis. Therefore, the function $\int_0^z\frac{\Lambda_{\mathfrak{Q}_\nu(.,b)}(t)}{t}dt$ is starlike in $\mathbb{D}$. However, we have

$$\begin{aligned}\int_0^z\frac{\Lambda_{\mathfrak{Q}_\nu(.,b)}(t)}{t}dt&=\frac{\Gamma(\nu)z}{\gamma(\nu,b)}\sum_{\ell=0}^{\infty}\frac{\gamma(\nu+\ell,b)z^\ell}{(\ell+1)!\Gamma(\nu+\ell)}\\&=\frac{\Gamma(\nu)z}{\gamma(\nu,b)}\sum_{\ell=0}^{\infty}\frac{\gamma(\nu+\ell,b)\Gamma(\ell+1)}{\Gamma(\ell+2)\Gamma(\nu+\ell)}\frac{z^\ell}{\ell!}\\&=\frac{\Gamma(\nu)z}{\gamma(\nu,b)}{}_2\gamma_2\left[\begin{matrix}(\nu,b),1\\ \nu,2\end{matrix}\middle|z\right].\end{aligned}\tag{4.45}$$

(b). In view of the first equation (4.44), we find that

$$\sum_{\ell=2}^{\infty}\ell\chi_\ell(\nu,b)=\sum_{\ell=2}^{\infty}w_\ell(\nu,b)=\Lambda_{\mathfrak{Q}_\nu(.,b)}(1)-1\leqslant 1,$$

where we have made use of the given hypothesis. □

Taking in the part (a) of the above Theorem the values $\nu=1$, we obtain the following corollary.

COROLLARY 4.4. *If* $0<b<\log(3)$, *then the function* $z\mapsto z\cdot{}_1\gamma_1\left[\begin{matrix}(1,b)\\ 2\end{matrix}\middle|z\right]$ *is starlike in* $\mathbb{D}$.

5. Conclusion

In our present paper, we have established some sufficient conditions so that some classes of functions related to the generalized Marcum Q-function belong to the class of starlike functions. Furthermore, two classes of starlike functions related to the upper incomplete generalized hypergeometric function ${}_p\gamma_p$ are derived. The various results, which we have obtained in our present investigation, are believed to be new, and their importance is illustrated by several interesting consequences and examples.

New research directions can be formulated for other special functions, which has a power series expressed in terms of the lower or upper incomplete gamma functions. However, these goals will be addressed and presented in future work.

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