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ABSOLUTE MONOTONICITY OF FOUR FUNCTIONS INVOLVING THE SECOND KIND OF COMPLETE ELLIPTIC INTEGRALS

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Abstract. In the study, the authors present absolute monotonicity of four functions involving the inverse hyperbolic tangent function and the second kind of complete elliptic integrals, derive four double inequalities for bounding the second kind of complete elliptic integrals, and acquire an upper bound of the Hersch–Pfluger distortion function. These inequalities improve several known ones. Moreover, the authors connect two of the four functions with normalized remainders of the Maclaurin series of two functions involving the inverse hyperbolic tangent function and the second kind of complete elliptic integrals.

1. Introduction and main results

For given complex numbers $a,b,c \in \mathbb{C}$ with $c \neq 0,-1,-2,\ldots$, the Gauss hypergeometric function ${}_2F_1$ is defined by

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}, \quad |z|<1, \tag{1} $$

where $(z)_n$ for $z \in \mathbb{C}$ is the Pochhammer symbol, also known as the rising factorial and the shifted factorial, which is defined by

$$ (z)_n = \prod_{k=0}^{n-1}(z+k) = \frac{\Gamma(n+z)}{\Gamma(z)} = \begin{cases} z(z+1)\cdots(z+n-1), & n \in \mathbb{N}; \\ 1, & n=0, \end{cases} \tag{2} $$

where

$$ \Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{\prod_{k=0}^{n}(z+k)}, \quad z \in \mathbb{C} \setminus \{0,-1,-2,\ldots\} $$

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is the classical Euler gamma function, whose reciprocal $\frac{1}{\Gamma(z)}$ is an entire function. For more information on the Gauss hypergeometric functions ${}_2F_1$, please refer to [1, Chapter 15], [6, Chapter 2], and the articles [22, 29, 40].

The complete elliptic integrals are the most important quasi-conformal mappings, they can be represented in terms of the Gauss hypergeometric functions ${}_2F_1$, and they have important applications in the theory of quasi-conformal mappings, the theory of geometric functions, and engineering. For more information, please refer to [1, 6, 51, 53] and closely related references therein.

For given $r \in (0,1)$, the complete elliptic integrals of the first and second kinds $\mathscr{K}(r)$ and $\mathscr{E}(r)$ can be represented [43, 46] respectively by

$$\begin{cases} \mathscr{K}(r) = \dfrac{\pi}{2}\,{}_2F_1\left(\dfrac{1}{2}, \dfrac{1}{2}; 1; r^2\right); \\ \mathscr{K}(0) = \dfrac{\pi}{2}; \\ \mathscr{K}(1) = \infty \end{cases}$$

and

$$\begin{cases} \mathscr{E}(r) = \dfrac{\pi}{2}\,{}_2F_1\left(-\dfrac{1}{2}, \dfrac{1}{2}; 1; r^2\right); \\ \mathscr{E}(0) = \dfrac{\pi}{2}; \\ \mathscr{E}(1) = 1. \end{cases} \tag{3}$$

For more information on some properties and applications of $\mathscr{K}(r)$ and $\mathscr{E}(r)$, please refer to [2, 3, 4, 5, 10, 27, 41, 44, 45, 52, 55, 56, 60, 61, 62, 64, 65, 70] and closely related references therein.

For $r \in (0,1)$, the conformal modular function of the Grötzsch extremum ring $\mathbb{B}^2 \backslash [0,r]$ can be represented by

$$\mu(r) = \frac{\pi}{2} \frac{\mathscr{K}\left(\sqrt{1-r^2}\right)}{\mathscr{K}(r)}. \tag{4}$$

The classical Ramanujan modular equation is defined by

$$\frac{\mathscr{K}\left(\sqrt{1-s^2}\right)}{\mathscr{K}(s)} = p\frac{\mathscr{K}\left(\sqrt{1-r^2}\right)}{\mathscr{K}(r)}, \quad p > 0. \tag{5}$$

By virtue of (4), we can rewritten (5) as

$$\mu(s) = p\mu(r), \quad p > 0. \tag{6}$$

Accordingly, the solution to the Ramanujan modular equation (6) is

$$s = \varphi_\kappa(r) = \mu^{-1}(p\mu(r)), \quad p = \frac{1}{\kappa}$$

and we call $\varphi_\kappa(r)$ the Hersch–Pfluger distortion function. It is well known that the Hersch–Pfluger distortion function $\varphi_\kappa(r)$ has important applications in quasi-conformal Schwarz lemma; see [21].

In 1970, Hübner [20] obtained the inequality

$$\varphi_\kappa(r) < r^{1/\kappa} \exp\left\{\left(1-\frac{1}{\kappa}\right)[m(r)+\ln r]\right\}$$

for $r \in (0,1)$ and $\kappa \in (1,\infty)$, where

$$m(r) = \frac{2}{\pi}(1-r^2)\mathscr{K}(r)\mathscr{K}\left(\sqrt{1-r^2}\right)$$

and we call $m(r)+\ln r$ the Hübner function. In 2008, Wang and her coauthors [58] established the inequality

$$\varphi_\kappa(r) < r^{1/\kappa} \exp\left\{\frac{4\ln 2}{\pi-2}\left(1-\frac{1}{\kappa}\right)[\mathscr{E}(r)-1]\right\}, \quad r \in (0,1). \tag{7}$$

In 2011, Guo and Qi [13] acquired the double inequality

$$\frac{\pi}{2} - \frac{1}{2}\ln\frac{(1+r)^{r-1}}{(1-r)^{r+1}} < \mathscr{E}(r) < \frac{\pi-1}{2} + \frac{1-r^2}{4r}\ln\frac{1+r}{1-r}, \quad r \in (0,1). \tag{8}$$

In [54], Wang and his coathors obtained two double inequalities

$$\frac{\pi}{2} - \alpha_1\left[1-\left(1-r^2\right)\frac{\operatorname{arctanh} r}{r}\right] < \mathscr{E}(r) < \frac{\pi}{2} - \beta_1\left[1-\left(1-r^2\right)\frac{\operatorname{arctanh} r}{r}\right] \tag{9}$$

and

$$\frac{\pi}{2} - r\operatorname{arctanh} r - \frac{\lambda_2}{2}\ln\left(1-r^2\right) < \mathscr{E}(r) < \frac{\pi}{2} - r\operatorname{arctanh} r - \frac{\mu_2}{2}\ln\left(1-r^2\right) \tag{10}$$

for $r \in (0,1)$, where

$$\alpha_1 = \frac{3\pi}{16} = 0.589\cdots, \quad \beta_1 = \frac{\pi}{2} - 1 = 0.570\cdots,$$
$$\lambda_2 = 1, \quad \mu_2 = 2 - \frac{\pi}{4} = 1.214\cdots$$

are the best possible constants. For more information on this kind of elementary inequalities, please refer to the articles [39, 41, 63] and closely related references therein.

Let $I \subseteq \mathbb{R}$ be an infinite or finite interval. A real infinitely differentiable function $f(x)$ defined on I is said to be absolutely monotonic in $x \in I$ if and only if all of its derivatives satisfy $f^{(k)}(x) \geqslant 0$ for $k \in \mathbb{N}_0$ and $x \in I$. A real infinitely differentiable function $f(x)$ defined on I is said to be completely monotonic in $x \in I$ if and only if all of its derivatives satisfy $(-1)^k f^{(k)}(x) \geqslant 0$ for $k \in \mathbb{N}_0$ and $x \in I$. When $I = (0,\infty)$ or $I = [0,\infty)$, there have been plenty of classical investigations on absolutely (or completely, respectively) monotonic functions in [30, Chapter XIII], [59, Chapter IV], and the monograph [48]. By the way, in the papers [8, 12, 38], the authors invented the notions of logarithmically absolutely (or logarithmically completely, respectively) monotonic functions.

In the study, we will present absolute monotonicity of four functions involving the inverse hyperbolic tangent function $\operatorname{arctanh} r$ and the second kind of complete elliptic integrals $\mathscr{E}(r)$, derive four double inequalities for bounding the second kind of complete elliptic integrals $\mathscr{E}(r)$, and acquire an upper bound of the Hersch–Pfluger distortion function $\varphi_K(r)$; see (45). These inequalities improve (7), (8), (9), and (10). Moreover, we will connect two of the four functions with normalized remainders of the Maclaurin series of two functions involving the inverse hyperbolic tangent function $\operatorname{arctanh} r$ and the second kind of complete elliptic integrals $\mathscr{E}(r)$; see Remarks 2 and 4 below.

The main results of this paper can be stated in the following theorems.

THEOREM 1. *For* $r \in (0,1)$ *and*

$$a_0 = \frac{3}{8}, \quad a_1 = -\frac{3}{640}, \quad a_2 = -\frac{171}{89600}, \quad a_3 = -\frac{30161}{28672000}, \tag{11}$$

define

$$H(r) = \frac{1 - {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; r\right)}{1 - (1-r)\,{}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; r\right)} = \frac{1 - \frac{2}{\pi}\mathscr{E}(\sqrt{r})}{1 + (r-1)\frac{\operatorname{arctanh}\sqrt{r}}{\sqrt{r}}} = \sum_{n=0}^{\infty} a_n r^n.$$

Then the following conclusions are true:

1. *The limits* $H(0^+) = \frac{3}{8}$ *and* $H(1^-) = 1 - \frac{2}{\pi}$ *are valid.*

2. *For* $n \in \mathbb{N}$, *the coefficients* a_n *satisfy the recursive relation and the negativity*

$$a_n = 3\left\{\frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_{n+1}}{4[(n+1)!]^2} - \sum_{k=0}^{n-1} \frac{a_k}{[2(n-k)+1][2(n-k)+3]}\right\} < 0. \tag{12}$$

3. *The functions* $-H'(r)$ *and* $H_1(r) = \frac{1}{r}\left[\frac{3}{8} - H(r)\right]$ *are absolutely monotonic on* $(0,1)$, *with the limits* $H_1(0^+) = \frac{3}{640}$ *and* $H_1(1^-) = \frac{2}{\pi} - \frac{5}{8}$.

4. *For* $r \in (0,1)$,

$$\mathscr{E}(r) = \frac{\pi}{2} - \frac{\pi}{2}\left[1 - (1-r^2)\frac{\operatorname{arctanh} r}{r}\right] \times \left(\frac{3}{8} - \frac{3}{640}r^2 - \frac{171}{89600}r^4 - \frac{30161}{28672000}r^6 - \cdots\right). \tag{13}$$

THEOREM 2. *For $r\in(0,1)$ and*

$$b_0=1-\frac{\pi}{8},\quad b_1=\frac{15\pi-64}{384},\quad b_2=\frac{95\pi-384}{7680},\quad b_3=\frac{69195\pi-274432}{1032190},\tag{14}$$

define

$$\begin{aligned}G(r)&=\frac{\frac{\pi}{2}\left[{}_2F_1\left(-\frac{1}{2},\frac{1}{2};1;r\right)-1\right]+r\,{}_2F_1\left(\frac{1}{2},1;\frac{3}{2};r\right)}{r\,{}_2F_1(1,1;2;r)}\\&=\frac{\frac{\pi}{2}-\sqrt{r}\operatorname{arctanh}\sqrt{r}-\mathscr{E}(\sqrt{r})}{\ln(1-r)}\\&=\sum_{n=0}^{\infty}b_nr^n.\end{aligned}$$

Then the following conclusions are true:

1. *The limits $G(0^+)=1-\frac{\pi}{8}$ and $G(1^-)=\frac{1}{2}$ are valid.*
2. *For $n\in\mathbb{N}$, the coefficients b_n satisfy the recursive relation and the negativity*

$$b_n=\frac{\pi}{2}\frac{\left(-\frac{1}{2}\right)_{n+1}\left(\frac{1}{2}\right)_{n+1}}{[(n+1)!]^2}+\frac{1}{2n+1}-\sum_{k=0}^{n-1}\frac{b_k}{n-k+1}<0.\tag{15}$$

3. *The functions $-G'(r)$ and $G_1(r)=\frac{1}{r}\left[1-\frac{\pi}{8}-G(r)\right]$ are absolutely monotonic on $(0,1)$, with the limits $G_1(0^+)=\frac{64-15\pi}{384}$ and $G_1(1^-)=\frac{4-\pi}{8}$.*
4. *For $r\in(0,1)$,*

$$\begin{aligned}\mathscr{E}(r)=\frac{\pi}{2}-r\operatorname{arctanh}r-\Bigg(1-\frac{\pi}{8}+\frac{15\pi-64}{384}r^2\\+\frac{95\pi-384}{7680}r^4+\frac{69195\pi-274432}{1032190}r^6+\cdots\Bigg)\ln\left(1-r^2\right).\end{aligned}\tag{16}$$

We will prove these main conclusions in Section 3 below.

2. A lemma

In the proofs of our main results, we will need the formulas

$${}_2F_1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\quad c\ne0,-1,-2,\ldots,\quad \Re(c-a-b)>0\tag{17}$$

and

$$z^{b-a}\frac{\Gamma(z+a)}{\Gamma(z+b)}\sim1+\frac{(a-b)(a+b-1)}{2z}+\cdots,\quad z\to\infty.\tag{18}$$

These two formulas can be found in [1, pp. 257 and 556, Entries 6.1.47 and 15.1.20] and [49, pp. 66–68, Section 3.6.2], respectively.

LEMMA 1. *For $n \in \mathbb{N}$, the sequence*

$$C_n = \frac{(2n+1)(2n+3)(4n+5)}{n+2} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_{n+1}}{[(n+1)!]^2}$$

is increasing in $n \in \mathbb{N}$, with the limit $\lim_{n\to\infty} C_n = \frac{16}{\pi}$. Consequently, for $n \in \mathbb{N}$,

$$\frac{135}{32} \frac{n+2}{(2n+1)(2n+3)(4n+5)} \leqslant \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_{n+1}}{[(n+1)!]^2} < \frac{16}{\pi} \frac{n+2}{(2n+1)(2n+3)(4n+5)}.$$

Proof. Due to $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and the asymptotic formula (18), we obtain the limit

$$\begin{aligned}
\lim_{n\to\infty} C_n &= \lim_{n\to\infty} \frac{(2n+1)(2n+3)(4n+5)\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_{n+1}}{(n+2)[(n+1)!]^2} \\
&= \lim_{n\to\infty} \frac{(2n+1)(2n+3)(4n+5)\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{3}{2}\right)}{(n+2)\left[\Gamma\left(\frac{1}{2}\right)\right]^2[\Gamma(n+2)]^2} \\
&= \lim_{n\to\infty} \frac{(2n+1)(2n+3)(4n+5)}{(n+2)\left[\Gamma\left(\frac{1}{2}\right)\right]^2 n^2} \\
&= \frac{16}{\pi}.
\end{aligned}$$

By virtue of the definition in (2), we arrive at

$$\begin{aligned}
\frac{C_{n+1}}{C_n} &= \frac{(2n+3)(2n+5)(4n+9)\left(\frac{1}{2}\right)_{n+1} \left(\frac{1}{2}\right)_{n+2}}{(2n+1)(2n+3)(4n+5)\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_{n+1}} \frac{(n+2)[(n+1)!]^2}{(n+3)[(n+2)!]^2} \\
&= \frac{(2n+5)(2n+3)(4n+9)}{4(4n+5)(n+3)(n+2)} \\
&= \frac{16n^3+100n^2+204n+135}{16n^3+100n^2+196n+120} \\
&= 1 + \frac{8n+15}{16n^3+100n^2+196n+120} \\
&> 1.
\end{aligned}$$

Accordingly, the sequence C_n is increasing in $n \in \mathbb{N}$.

In light of the increasing property of the sequence C_n in $n \in \mathbb{N}$ and in view of the limit $\lim_{n\to\infty} C_n = \frac{16}{\pi}$, we acquire $\frac{135}{32} = C_1 \leqslant C_n < \frac{16}{\pi}$. The proof of Lemma 1 is thus complete. □

3. Proofs of main results

We are now in a position to prove Theorems 1 and 2.

3.1. Proof of Theorem 1

Let

$$h_1(r) = 1 - {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; r\right)$$

and

$$h_2(r) = 1 - (1-r)\,{}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; r\right).$$

Then $H(r) = \frac{h_1(r)}{h_2(r)} = \sum_{n=0}^{\infty} a_n r^n$. Making use of the definition in (1), we derive

$$h_1(r) = 1 - \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{[(n)!]^2} r^n = \frac{1}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_{n+1}}{[(n+1)!]^2} r^{n+1}$$

and

$$h_2(r) = 1 - (1-r) \sum_{n=0}^{\infty} \frac{1}{2n+1} r^n = \sum_{n=0}^{\infty} \frac{2}{(2n+1)(2n+3)} r^{n+1}.$$

Accordingly, we deduce

$$H(r) = \frac{1}{4} \frac{\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{1}{2})_{n+1}}{[(n+1)!]^2} r^n}{\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+3)} r^n} = \sum_{n=0}^{\infty} a_n r^n,$$

which can be reformulated as

$$\frac{1}{4} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_{n+1}}{[(n+1)!]^2} r^n = \sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+3)} r^n \sum_{n=0}^{\infty} a_n r^n.$$

As a result, we arrive at

$$\begin{aligned} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_{n+1}}{4[(n+1)!]^2} &= \sum_{k=0}^{n} \frac{a_k}{[2(n-k)+1][2(n-k)+3]} \\ &= \frac{a_n}{3} + \sum_{k=0}^{n-1} \frac{a_k}{[2(n-k)+1][2(n-k)+3]}, \end{aligned} \tag{19}$$

which is equivalent to the recursive relation in (12).

Let

$$A_n^{(1)} = \frac{(2n+1)(2n+3)}{4} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_{n+1}}{[(n+1)!]^2}, \quad n \in \mathbb{N}.$$

Then, by virtue of the definition in (2), we find

$$\frac{A_{n+1}^{(1)}}{A_n^{(1)}} = \frac{(2n+3)(2n+5)}{4(n+2)^2} = \frac{4n^2+16n+15}{4n^2+16n+16} < 1, \quad n \in \mathbb{N}.$$

Hence, the sequence $A_n^{(1)}$ decreases in $n \in \mathbb{N}$.

On the other hand, let

$$A_n^{(2)} = (2n+1)(2n+3)\sum_{k=0}^{n} \frac{a_k}{[2(n-k)+1][2(n-k)+3]}, \quad n \in \mathbb{N}.$$

Due to the last equation in (19), we see that $A_n^{(1)} = A_n^{(2)}$ for $n \in \mathbb{N}$. As a result, the sequence $A_n^{(2)}$ decreases in $n \in \mathbb{N}$. Furthermore, we have

$$\begin{aligned}
\frac{A_{n+1}^{(2)} - A_n^{(2)}}{2n+3} &= (2n+5)\sum_{k=1}^{n+1} \frac{a_k}{[2(n-k)+3][2(n-k)+5]} \\
&\quad - (2n+1)\sum_{k=1}^{n} \frac{a_k}{[2(n-k)+1][2(n-k)+3]} \\
&= \frac{(2n+5)a_{n+1}}{3} + \sum_{k=1}^{n}\left(\frac{(2n+5)a_k}{[2(n-k)+3][2(n-k)+5]}\right. \\
&\quad \left. - \frac{(2n+1)a_k}{[2(n-k)+1][2(n-k)+3]}\right) \\
&= \frac{(2n+5)a_{n+1}}{3} - 8\sum_{k=1}^{n} \frac{ka_k}{[2(n-k)+1][2(n-k)+3][2(n-k)+5]} \\
&< 0.
\end{aligned}$$

Consequently, we conclude

$$a_{n+1} < \frac{24}{2n+5}\sum_{k=1}^{n} \frac{ka_k}{[2(n-k)+1][2(n-k)+3][2(n-k)+5]}, \quad n \in \mathbb{N}. \tag{20}$$

In view of the recursive relation in (12), it is easy to deduce the four values in (11). By induction and with the aid of the inequality (20), it is not difficult to prove that $a_n < 0$ for $n \in \mathbb{N}$.

It is clear that $H(0^+) = a_0 = \frac{3}{8}$. In light of the formula (17), we acquire

$$\begin{aligned}
\lim_{r\to 1^-} H(r) = \lim_{r\to 1^-} \frac{1 - {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; r\right)}{1-(1-r)\,{}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; r\right)} &= 1 - {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; 1\right) \\
&= 1 - \frac{[\Gamma(1)]^2}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)} \\
&= 1 - \frac{2}{\pi}.
\end{aligned}$$

Since $a_n < 0$ for $n \in \mathbb{N}$, considering the series expansion $H(r) = \sum_{n=0}^{\infty} a_n r^n$, we obtain

$$-H'(r) = -\sum_{n=0}^{\infty}(n+1)a_{n+1}r^n = \sum_{n=0}^{\infty}|(n+1)a_{n+1}|r^n$$

and

$$H_1(r) = \frac{1}{r}\left[\frac{3}{8} - H(r)\right] = -\sum_{n=0}^{\infty} a_{n+1}r^n = \sum_{n=0}^{\infty}|a_{n+1}|r^n.$$

Consequently, the functions $-H'(r)$ and $H_1(r)$ are absolutely monotonic on $(0,1)$.

It is ready that $H_1(0^+) = |a_1| = \frac{3}{640}$ and $H_1(1^-) = \frac{3}{8} - H(1^-) = \frac{2}{\pi} - \frac{5}{8}$.

In [11, p. 61], [34, p. 473, Eq. 83], and [49, p. 109], we find

$$\frac{\operatorname{arctanh} r}{r} = {}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; r^2\right).$$

Hence, we have

$$1 - \left(1 - r^2\right)\frac{\operatorname{arctanh} r}{r} = 1 - (1 - r^2)\,{}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; r^2\right). \tag{21}$$

Accordingly, we derive

$$H\left(r^2\right) = \frac{1 - \frac{2}{\pi}\mathscr{E}(r)}{1 - \left(1 - r^2\right)\frac{\operatorname{arctanh} r}{r}} = \sum_{n=0}^{\infty} a_n r^{2n}.$$

Combining (12) and (21) yields (13). The proof of Theorem 1 is thus complete. □

3.2. Proof of Theorem 2

Let

$$g_1(r) = \frac{\pi}{2}\left[{}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; r\right) - 1\right] + r\,{}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; r\right)$$

and

$$g_2(r) = r\,{}_2F_1(1, 1; 2; r).$$

Then $G(r) = \frac{g_1(r)}{g_2(r)} = \sum_{n=0}^{\infty} b_n r^n$. By the definition in (1), we have

$$g_1(r) = \sum_{n=0}^{\infty}\left[\frac{\pi}{2}\frac{\left(-\frac{1}{2}\right)_{n+1}\left(\frac{1}{2}\right)_{n+1}}{[(n+1)!]^2} + \frac{1}{2n+1}\right]r^{n+1}$$

and

$$g_2(r) = r\,{}_2F_1(1, 1; 2; r) = \sum_{n=0}^{\infty}\frac{1}{n+1}r^{n+1}.$$

Accordingly, we obtain

$$G(r) = \frac{\sum_{n=0}^{\infty}\left[\frac{\pi}{2}\frac{(-\frac{1}{2})_{n+1}(\frac{1}{2})_{n+1}}{[(n+1)!]^2} + \frac{1}{2n+1}\right]r^n}{\sum_{n=0}^{\infty}\frac{1}{n+1}r^n} = \sum_{n=0}^{\infty} b_n r^n,$$

that is,

$$\sum_{n=0}^{\infty}\left[\frac{\pi}{2}\frac{\left(-\frac{1}{2}\right)_{n+1}\left(\frac{1}{2}\right)_{n+1}}{[(n+1)!]^2}+\frac{1}{2n+1}\right]r^n=\sum_{n=0}^{\infty}\frac{1}{n+1}r^n\sum_{n=0}^{\infty}b_nr^n. \tag{22}$$

In other words,

$$\frac{\pi}{2}\frac{\left(-\frac{1}{2}\right)_{n+1}\left(\frac{1}{2}\right)_{n+1}}{[(n+1)!]^2}+\frac{1}{2n+1}=\sum_{k=0}^{n}\frac{b_k}{n-k+1}=b_n+\sum_{k=0}^{n-1}\frac{b_k}{n-k+1}. \tag{23}$$

As a result, we derive the recursive relation in (15).

Set

$$B_n^{(1)}=(n+1)\left[\frac{\pi}{2}\frac{\left(-\frac{1}{2}\right)_{n+1}\left(\frac{1}{2}\right)_{n+1}}{[(n+1)!]^2}+\frac{1}{2n+1}\right].$$

Making use of Lemma 1, we reveal

$$\begin{aligned}B_{n+1}^{(1)}-B_n^{(1)}&=\frac{\pi}{2}\left[\frac{(n+2)(-1/2,n+2)\left(\frac{1}{2}\right)_{n+2}}{[(n+2)!]^2}-\frac{(n+1)\left(-\frac{1}{2}\right)_{n+1}\left(\frac{1}{2}\right)_{n+1}}{[(n+1)!]^2}\right]\\&\quad-\frac{1}{(2n+1)(2n+3)}\\&=\frac{\pi}{2}\frac{\left(-\frac{1}{2}\right)_{n+1}\left(\frac{1}{2}\right)_{n+1}}{[(n+1)!]^2}\left[\frac{(2n+1)(2n+3)}{4(n+2)}-(n+1)\right]\\&\quad-\frac{1}{(2n+1)(2n+3)}\\&=\frac{\pi}{16}\frac{(4n+5)\left(\frac{1}{2}\right)_{n}\left(\frac{1}{2}\right)_{n+1}}{(n+2)[(n+1)!]^2}-\frac{1}{(2n+1)(2n+3)}\\&<0.\end{aligned}$$

This means that $B_n^{(1)}$ is decreasing in $n\in\mathbb{N}$.

On the other hand, let

$$B_n^{(2)}=(n+1)\sum_{k=0}^{n}\frac{b_k}{n-k+1}.$$

By the equations in (23), we see easily that $B_n^{(1)}=B_n^{(2)}$. Hence, the sequence $B_n^{(2)}$ is decreasing in $n\in\mathbb{N}$. As a result, we obtain

$$\begin{aligned}B_{n+1}^{(2)}-B_n^{(2)}&=(n+2)\sum_{k=0}^{n+1}\frac{b_k}{n-k+2}-(n+1)\sum_{k=0}^{n}\frac{b_k}{n-k+1}\\&=(n+2)b_{n+1}+\sum_{k=1}^{n}\left[\frac{(n+2)b_k}{n-k+2}-\frac{(n+1)b_k}{n-k+1}\right]\\&=(n+2)b_{n+1}-\sum_{k=1}^{n}\frac{kb_k}{(n-k+1)(n-k+2)}\\&<0.\end{aligned}$$

Consequently, we have

$$b_{n+1} < \frac{1}{n+2} \sum_{k=1}^{n} \frac{k b_k}{(n-k+1)(n-k+2)}. \tag{24}$$

By the equation (22), we have $b_0 = 1 - \frac{\pi}{8}$. With the help of the recursive relation in (15), we derive the rest values in (14). By induction and (24), it is not difficult to verify that $b_n < 0$ for $n \in \mathbb{N}$.

By the L'Hôpital rule and $\frac{\mathrm{d}\mathscr{E}(r)}{\mathrm{d}r} = \frac{\mathscr{E}(r)-\mathscr{K}(r)}{r}$, we acquire the limit

$$\begin{aligned}
\lim_{r\to 1^-} G(r) &= \lim_{r\to 1^-} \frac{\frac{\pi}{2}\left[{}_2F_1\left(-\frac{1}{2},\frac{1}{2};1;r\right)-1\right] + r\,{}_2F_1\left(\frac{1}{2},1;\frac{3}{2};r\right)}{r\,{}_2F_1(1,1;2;r)} \\
&= \lim_{r\to 1^-} \frac{(r-1)[\mathscr{K}(\sqrt{r})-\mathscr{E}(\sqrt{r})] + (1-r)\sqrt{r}\operatorname{arctanh}\sqrt{r}}{2r} + \frac{1}{2} \\
&= \frac{1}{2}.
\end{aligned}$$

Since $b_n < 0$ for $n \in \mathbb{N}$, by the expression $G(r) = \sum_{n=0}^{\infty} b_n r^n$, we arrive at

$$-G'(r) = -\sum_{n=0}^{\infty} (n+1) b_{n+1} r^n = \sum_{n=0}^{\infty} |(n+1) b_{n+1}| r^n$$

and

$$G_1(r) = \frac{1}{r}\left[1 - \frac{\pi}{8} - G(r)\right] = -\sum_{n=0}^{\infty} b_{n+1} r^n = \sum_{n=0}^{\infty} |b_{n+1}| r^n.$$

Accordingly, the funcitons $-G'(r)$ and $G_1(r)$ are absolutely monotonic on $(0,1)$. Moreover, it is obvious that the limits

$$G_1(0^+) = |b_1| = \frac{64 - 15\pi}{384}$$

and

$$G_1(1^-) = 1 - \frac{\pi}{8} - G(1^-) = \frac{4-\pi}{8}$$

are valid.

By the definition in (1), we acquire

$$-\ln(1-r) = \sum_{n=0}^{\infty} \frac{1}{n+1} r^{n+1} = r\,{}_2F_1(1,1;2;r). \tag{25}$$

Combining this with (3) gives

$$G(r^2) = \frac{\mathscr{E}(r) - \frac{\pi}{2} + r \operatorname{arctanh} r}{-\ln(1-r^2)} = \sum_{n=0}^{\infty} b_n r^{2n}.$$

Consequently, we deduce the expansion (16). The proof of Theorem 2 is thus complete. □

4. Corolaries and remarks

In this section, we deduce several corollaries from Theorems 1 and 2, while we list several remarks on our main results.

COROLLARY 1. *Let the sequence a_n for $n \in \mathbb{N}_0$ be defined as in Theorem 1 and let*

$$\alpha_n = |a_{n+1}|, \quad \beta_n = \frac{2}{\pi} - 1 + \sum_{k=0}^{n} a_k, \quad P_n(r) = \sum_{k=0}^{n+1} a_k r^{2k}.$$

Then the function

$$H_{2,n}(r) = \frac{1}{r^{n+1}} \left[\sum_{k=0}^{n} a_k r^k - H(r) \right] \tag{26}$$

is absolutely monotonic from $(0,1)$ onto $(\alpha_n, \alpha_n + \beta_n)$. Consequently, the double inequalities

$$\frac{\pi}{2} \left\{ 1 - \left[1 - (1-r^2) \frac{\operatorname{arctanh} r}{r} \right] \left[P_n(r) - \alpha_{n+1} r^{2n+4} \right] \right\} < \mathscr{E}(r) < \frac{\pi}{2} \left\{ 1 - \left[1 - (1-r^2) \frac{\operatorname{arctanh} r}{r} \right] \left[P_n(r) - \beta_{n+1} r^{2n+4} \right] \right\} \tag{27}$$

and

$$\frac{\pi}{2} \left\{ 1 - \left[1 - (1-r^2) \frac{\operatorname{arctanh} r}{r} \right] \left(\frac{3}{8} - \frac{3}{640} r^2 - \frac{171}{89600} r^4 - \frac{179200 - 56591\pi}{89600\pi} r^6 \right) \right\} > \mathscr{E}(r) > \frac{\pi}{2} \left\{ 1 - \left[1 - (1-r^2) \frac{\operatorname{arctanh} r}{r} \right] \left(\frac{3}{8} - \frac{3}{640} r^2 - \frac{171}{89600} r^4 - \frac{30161}{28672000} r^6 \right) \right\} \tag{28}$$

are sound for $r \in (0,1)$.

Proof. Since $H(r) = \sum_{n=0}^{\infty} a_n r^n$, by Theorem 1, we arrive at

$$H_{2,n}(r) = \frac{1}{r^{n+1}} \left(\sum_{k=0}^{n} a_k r^k - \sum_{k=0}^{\infty} a_k r^k \right) = -\sum_{k=n+1}^{\infty} a_k r^{k-n-1} = \sum_{k=0}^{\infty} |a_{k+n+1}| r^k.$$

Accordingly, the function $H_{2,n}(r)$ is absolutely monotonic on $(0,1)$.

By Theorem 1, we acquire

$$H_{2,n}(0^+) = |a_{n+1}|, \quad H_{2,n}(1^-) = \sum_{k=0}^{n} a_k - H(1^-) = \frac{2}{\pi} - 1 + \sum_{k=0}^{n} a_k.$$

Since $H_{2,n}(r)$ is increasing and convex on $(0,1)$, we discover

$$|a_{n+1}| + |a_{n+2}| r < H_{2,n}(r) < |a_{n+1}| + \left[\sum_{k=0}^{n} a_k - H(1^-) - |a_{n+1}| \right] r = |a_{n+1}| + \beta_{n+1} r. \tag{29}$$

In view of (4), (21), and (29), we derive the double inequality (27).

Taking $n=1$ in (27) leads to the double inequality (28). The proof of Corollary 1 is complete. □

REMARK 1. The double inequality (27) is better than (8) and (9).

REMARK 2. Suppose that a real infinitely differentiable function $f(x)$ has a formal Maclaurin power series expansion

$$\sum_{j=0}^{\infty} f^{(j)}(0)\frac{x^j}{j!}. \tag{30}$$

If $f^{(k+1)}(0)\ne 0$ for some $k\in\mathbb{N}_0$, then we call the function

$$T_k[f(x)]=\begin{cases}\dfrac{1}{f^{(k+1)}(0)}\dfrac{(k+1)!}{x^{k+1}}\left[f(x)-\displaystyle\sum_{j=0}^{k} f^{(j)}(0)\frac{x^j}{j!}\right], & x\ne 0\\ 1, & x=0\end{cases} \tag{31}$$

the normalized remainder, or normalized tail, of the Maclaurin series expansion (30). This concept was invented by the second author and his coworkers starting from April 2023. The first several articles implicitly discussed about the normalized remainders are [23, 24, 25, 37]. The latest articles explicitly considered about the normalized remainders are [7, 22, 26, 31, 32, 35, 50, 57, 66, 67, 68, 69], especially the review and research article [36].

The Stirling numbers of the second kind $S(j,\ell)$ can be analytically generated by

$$\left(\frac{e^z-1}{z}\right)^{\ell}=\sum_{j=0}^{\infty}\frac{S(j+\ell,\ell)}{\binom{j+\ell}{\ell}}\frac{z^j}{j!},\quad \ell\geqslant 0; \tag{32}$$

see [28, Example 2.76] and [47, Eq. (9.59)]. The r-associate Stirling numbers of the second kind $S_r(j,\ell)$ for $j\geqslant\ell\geqslant 0$ and $r\geqslant 0$ are defined [17, p. 303, Eq. (1.2)] by

$$\left(e^z-\sum_{j=0}^{r}\frac{z^j}{j!}\right)^{\ell}=\left(\sum_{j=r+1}^{\infty}\frac{z^j}{j!}\right)^{\ell}=\ell!\sum_{j=(r+1)\ell}^{\infty}S_r(j,\ell)\frac{z^j}{j!}; \tag{33}$$

see also [42, Section 1.12]. The equation (33) can be reformulated as

$$\begin{aligned}(T_r[e^z])^{\ell}&=\left[\frac{(r+1)!}{z^{r+1}}\left(e^z-\sum_{j=0}^{r}\frac{z^j}{j!}\right)\right]^{\ell}\\&=\frac{\ell![(r+1)!]^{\ell}}{[(r+1)\ell]!}\sum_{j=0}^{\infty}\frac{S_r(j+(r+1)\ell,\ell)}{\binom{j+(r+1)\ell}{j}}\frac{z^j}{j!}\end{aligned} \tag{34}$$

for $\ell,r\in\mathbb{N}_0$. The equation (32) is a special case $r=0$ of (34). Consequently, the integer powers of the normalized remainder $T_r[e^z]$ for $r\in\mathbb{N}_0$, which was investigated in [7, 35] and [36, Section 1.7], can be regarded as the generating functions of the Stirling numbers of the second kind $S(j,\ell)$ and the r-associate Stirling numbers of the second kind $S_r(j,\ell)$ for $j \geqslant \ell \geqslant 0$ and $r \geqslant 0$.

The Bernoulli numbers and polynomials B_j and $B_j(t)$ for $j \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$ are generated [49, p. 3] by

$$\frac{z}{\mathrm{e}^z - 1} = \sum_{j=0}^{\infty} B_j \frac{z^j}{j!} = 1 - \frac{z}{2} + \sum_{j=1}^{\infty} B_{2j} \frac{z^{2j}}{(2j)!}, \quad |z| < 2\pi \tag{35}$$

and

$$\frac{z\,\mathrm{e}^{zt}}{\mathrm{e}^z - 1} = \sum_{j=0}^{\infty} B_j(t) \frac{z^j}{j!}, \quad |z| < 2\pi. \tag{36}$$

In the thesis [16] and the papers [9, 14, 15, 18, 19], among other things, Carlitz and Howard defined the sequences and polynomials A_j, $A_j(t)$, $A_{k,j}$, and $A_{k,j}(t)$ by

$$\frac{z^2}{2} \frac{1}{\mathrm{e}^z - z - 1} = \sum_{j=0}^{\infty} A_j \frac{z^j}{j!}, \tag{37}$$

$$\frac{z^2}{2} \frac{\mathrm{e}^{zt}}{\mathrm{e}^z - z - 1} = \sum_{j=0}^{\infty} A_j(t) \frac{z^j}{j!}, \tag{38}$$

$$\frac{z^k}{k!} \frac{1}{\mathrm{e}^z - \sum_{j=0}^{k-1} \frac{z^j}{j!}} = \sum_{j=0}^{\infty} A_{k,j} \frac{z^j}{j!}, \tag{39}$$

and

$$\frac{z^k}{k!} \frac{\mathrm{e}^{zt}}{\mathrm{e}^z - \sum_{j=0}^{k-1} \frac{z^j}{j!}} = \sum_{j=0}^{\infty} A_{k,j}(t) \frac{z^j}{j!} \tag{40}$$

for $k \in \mathbb{N}$, and they examined a lot of algebraic properties of these sequences and polynomials. It is easy to see that $A_j(0) = A_j$, $A_{1,j} = B_j$, $A_{2,j} = A_j$, $A_{k,j}(0) = A_{k,j}$, $A_{1,j}(0) = B_j$, $A_{2,j}(0) = A_j$, and $A_{1,j}(t) = B_j(t)$ for $j \in \mathbb{N}_0$. The sequence A_j for $j \in \mathbb{N}_0$ was recently investigated in [7, Remark 2] and [42, Section 2]. In terms of the notation $T_{k-1}[\mathrm{e}^z]$, the equation (40) can be reformulated as

$$\frac{\mathrm{e}^{zt}}{T_{k-1}[\mathrm{e}^z]} = \sum_{j=0}^{\infty} A_{k,j}(t) \frac{z^j}{j!}.$$

Meanwhile, the generating functions of the sequences and polynomials B_j, $B_j(t)$, A_j, $A_j(t)$, and $A_{k,j}$ in (35), (36), (37), (38), and (39) can be respectively rewritten as $\frac{1}{T_0[\mathrm{e}^z]}$, $\frac{\mathrm{e}^{zt}}{T_0[\mathrm{e}^z]}$, $\frac{1}{T_1[\mathrm{e}^z]}$, $\frac{\mathrm{e}^{zt}}{T_1[\mathrm{e}^z]}$, and $\frac{1}{T_{k-1}[\mathrm{e}^z]}$ for $k \in \mathbb{N}$.

We say that the series $\sum_{j=0}^{\infty} a_j$ envelops the number A if the relations

$$\left| A - \sum_{j=0}^{n} a_j \right| < |a_{n+1}|, \quad n \in \mathbb{N}_0$$

are satisfied. For more detailed information, please refer to [33, Chapter 4]. The normalized remainder defined in (31) has close connection to the well-known "enveloping series", a topic covered in [33, Chapter 4].

In terms of the notion of normalized remainders mentioned above, the function $H_{2,n}(r)$ defined in (26) can be reformulated as

$$T_n[H(r)] = -\frac{1}{a_{n+1}} H_{2,n}(r), \quad r \in (0,1)$$

for $n \in \mathbb{N}_0$. Considering the negativity in (12) and employing Corollary 1, we conclude that the normalized remainder $T_n[H(r)]$ for $n \in \mathbb{N}_0$ is absolutely monotonic from $(0,1)$ onto $\left(1, 1 + \frac{\beta_n}{\alpha_n}\right)$.

COROLLARY 2. *Let the sequence b_n for $n \in \mathbb{N}_0$ be defined as in Theorem 2. Set*

$$\delta_n = |b_{n+1}|, \quad \lambda_n = -\frac{1}{2} + \sum_{k=0}^{n} b_k, \quad Q_n(r) = \sum_{k=0}^{n+1} b_k r^{2k}$$

for $r \in (0,1)$. Then the function

$$G_{2,n}(r) = \frac{1}{r^{n+1}} \left[\sum_{k=0}^{n} b_k r^k - G(r) \right] \tag{41}$$

is absolutely monotonic from $(0,1)$ onto $(\delta_n, \delta_n + \lambda_n)$. Consequently, for $r \in (0,1)$, the double inequalities

$$\begin{aligned} \frac{\pi}{2} - r \operatorname{arctanh} r - \left[Q_n(r) - \lambda_{n+1} r^{2n+4}\right] \ln\left(1 - r^2\right) < \mathscr{E}(r) \\ < \frac{\pi}{2} - r \operatorname{arctanh} r - \left[Q_n(r) - \delta_{n+1} r^{2n+4}\right] \ln\left(1 - r^2\right) \end{aligned} \tag{42}$$

and

$$\begin{aligned} \left(\frac{8-\pi}{4} - \frac{64-15\pi}{192} r^2 - \frac{384-95\pi}{3840} r^4 - \frac{2716-85\pi}{3840} r^6 \right) \frac{\ln\left(1-r^2\right)}{2} \\ > \frac{\pi}{2} - r \operatorname{arctanh} r - \mathscr{E}(r) > \\ \left(\frac{8-\pi}{4} - \frac{64-15\pi}{192} r^2 - \frac{384-95\pi}{3840} r^4 - \frac{274432-69195\pi}{516095} r^6 \right) \frac{\ln\left(1-r^2\right)}{2} \end{aligned} \tag{43}$$

are valid.

Proof. From Theorem 2, it follows that

$$G_{2,n}(r) = \frac{1}{r^{n+1}} \left(\sum_{k=0}^{n} b_k r^k - \sum_{k=0}^{\infty} b_k r^k \right) = -\sum_{k=n+1}^{\infty} b_k r^{k-n-1} = \sum_{k=0}^{\infty} |b_{k+n+1}| r^k.$$

Consequently, the function $G_{2,n}(r)$ is absolutely monotonic on $(0,1)$ and it is easy to see that the limits $G_{2,n}(0^+) = |b_{n+1}|$ and

$$G_{2,n}(1^-) = \sum_{k=0}^{n} b_k - G(1^-) = \sum_{k=0}^{n} b_k - \frac{1}{2}$$

are valid.

Since the function $G_{2,n}(r)$ is increasing and convex on $(0,1)$, we deduce the double inequality

$$\begin{aligned} |b_{n+1}| + |b_{n+2}|r < G_{2,n}(r) &< |b_{n+1}| + \left[\sum_{k=0}^{n} b_k - G(1^-) - |b_{n+1}|\right] r \\ &= |b_{n+1}| + \lambda_{n+1} r. \end{aligned} \tag{44}$$

Combining (4), (25), and (44) leads to (42).

Taking $n = 1$ in (42) results in (43). The proof of Coroillary 2 is complete. □

REMARK 3. The double inequality (42) is better than (10).

REMARK 4. As discussed in Remark 2, in terms of the notion of normalized remainders defined by (31), the function $G_{2,n}(r)$ defined in (41) can be reformulated as

$$T_n[G(r)] = -\frac{1}{b_{n+1}} G_{2,n}(r), \quad r \in (0,1)$$

for $n \in \mathbb{N}_0$. Making use of the negativity in (15) and utilizing Corollary 2, we reveal that the normalized remainder $T_n[G(r)]$ for $n \in \mathbb{N}_0$ is absolutely monotonic from $(0,1)$ onto $\left(1, 1 + \frac{\lambda_n}{\delta_n}\right)$.

Making use of (7) and Corollaries 1 and 2, we derive the following corollary.

COROLLARY 3. *For $r \in (0,1)$ and $\kappa \in (1,\infty)$,*

$$\varphi_\kappa(r) < r^{1/\kappa} \exp\left\{ \frac{4\ln 2}{\pi - 2} \left(1 - \frac{1}{\kappa}\right) \min\{S(r), T(r)\} \right\}, \tag{45}$$

where a_n and b_n for $n \in \mathbb{N}_0$ are defined as in Theorems 1 and 2,

$$S(r) = \frac{\pi}{2} - 1 - \frac{\pi}{2}\left[1 - \left(1 - r^2\right)\frac{\operatorname{arctanh} r}{r}\right]\left[\sum_{k=0}^{n+1} a_k r^{2k} - \left(\sum_{k=0}^{n+1} a_k - 1 + \frac{2}{\pi}\right) r^{2n+4}\right],$$

and

$$T(r) = \frac{\pi}{2} - 1 - r \operatorname{arctanh} r - \left(\sum_{k=0}^{n+1} b_k r^{2k} + b_{n+2} r^{2n+4}\right) \ln\left(1 - r^2\right).$$

REMARK 5. The upper bound in (45) is better than the corresponding one in (7).

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