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NOTE ON THE NORM OF LINEAR COMBINATIONS OF A CLASS OF LINEAR OPERATORS

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Abstract. Let $H(\mathbb{D})$ be the space of analytic functions on the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$, $S(\mathbb{D}) = \{f \in H(\mathbb{D}) : f(\mathbb{D}) \subseteq \mathbb{D}\}$, $\mathscr{K}(\mathbb{D})$ the space of Cauchy transforms on $\mathbb{D}$, $\mathscr{W}_w^{(m)}(\mathbb{D})$ the mth weighted type space on $\mathbb{D}$ with the weight function w and $m \in \mathbb{N}_0$, $\mathscr{W}_{w,0}^{(m)}(\mathbb{D})$ the little mth weighted type space on $\mathbb{D}$, and

$$L_s f(z) = \sum_{k=1}^{s} \alpha_k \sum_{j=0}^{n_k} u_{j,k}(z) f^{(j)}(\varphi_{j,k}(z)), \quad z \in \mathbb{D},$$

where $s \in \mathbb{N}$, $n_k \in \mathbb{N}_0$, $k = \overline{1,s}$, $\alpha_k \in \mathbb{C}$, $k = \overline{1,s}$, $u_{j,k} \in H(\mathbb{D})$, $j = \overline{0,n_k}$, $k = \overline{1,s}$, and $\varphi_{j,k} \in S(\mathbb{D})$, $j = \overline{0,n_k}$, $k = \overline{1,s}$. We find the norm of the operator $L_s : \mathscr{K}(\mathbb{D}) \to \mathscr{W}_w^{(m)}(\mathbb{D})$ in terms of above-mentioned parameters and symbols, and present a characterization for the boundedness of the operator $L_s : \mathscr{K}(\mathbb{D}) \to \mathscr{W}_{w,0}^{(m)}(\mathbb{D})$, considerably extending several recent results in the literature.

1. Introduction

Throughout this note we denote the set of all positive natural numbers $\{1,2,3,\dots\}$ by $\mathbb{N}$, whereas the set of all nonnegative integers we denote by $\mathbb{N}_0$. By $\mathscr{D}$ we denote a domain (i.e., an open and connected set), in the complex plane $\mathbb{C}$ or in the standard Euclidean complex-vector space $\mathbb{C}^n$. The open unit disk in $\mathbb{C}$, that is, the set $\{z \in \mathbb{C} : |z| < 1\}$, we denote by $\mathbb{D}$, whereas its boundary $\{z \in \mathbb{C} : |z| = 1\}$ (i.e., the unit circle), we denote by $\partial\mathbb{D}$.

By C_j^l, where $j,l \in \mathbb{N}_0$ are such that $0 \leqslant j \leqslant l$, we denote the binomial coefficients, that is, the two-dimensional sequence of real numbers given by

$$C_j^l = \frac{l!}{j!(l-j)!},$$

where $j,l \in \mathbb{N}_0$, $0 \leqslant j \leqslant l$, and where, as usual, we regard that the following relation holds $0! = 1$ (we prefer the notation instead of the frequently used ones C_l^j and $\binom{l}{j}$).

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If $p,q\in\mathbb{N}_0$ are such that $p>q$ and $(a_j)_{j\in I}$, $I\subseteq\mathbb{N}_0$, is a finite or infinite sequence of real or complex numbers, then we also use the following convention

$$\sum_{j=p}^{q} a_j = 0.$$

If $s,t\in\mathbb{N}_0$ satisfy the condition $s\leqslant t$, then if we write $k=\overline{s,t}$ it means that $s\leqslant k\leqslant t$ and $k\in\mathbb{N}_0$.

By $H(\mathscr{D})$ we denote the linear space of analytic (i.e., holomorphic) functions on a domain $\mathscr{D}$, whereas by $C(\mathscr{D})$ we denote the linear space of continuous functions on the domain. For some basic information on analytic functions in one or several complex variables consult, for instance, the following books: [1, 19, 20]. For some information on various spaces of analytic functions on domains in $\mathbb{C}$, as well as on domains in $\mathbb{C}^n$ and linear operators acting on them, see, for instance [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 38, 39, 40, 41, 42, 43, 44, 45] and the related references therein.

The set

$$\{f\in H(\mathscr{D}): f(\mathscr{D})\subseteq\mathscr{D}\}$$

(analytic self-maps of a domain $\mathscr{D}$), we denote by $S(\mathscr{D})$, whereas by $W(\mathscr{D})$ we denote the set

$$\{w\in C(\mathscr{D}): w(z)>0, z\in\mathscr{D}\},$$

that is, the set of the, so called, weight functions on $\mathscr{D}$.

By $\mathfrak{M}$ we denote the set of all complex Borel measures. If $\mu\in\mathfrak{M}$, then by $\|\mu\|$ we denote the total variation of the complex Borel measure μ. For some basic information and classical results on the Borel measures and related topics consult, for instance, [19].

If for an $f\in H(\mathbb{D})$ we have

$$f(z)=\int_{\partial\mathbb{D}}\frac{d\mu(\zeta)}{1-\overline{\zeta}z}, \tag{1}$$

for $z\in\mathbb{D}$, where μ is a complex Borel measure on $\partial\mathbb{D}$, we say that it belongs to the class of Cauchy transforms. The class of all the Cauchy transforms on $\mathbb{D}$ we denote by $\mathscr{K}(\mathbb{D})=\mathscr{K}$.

For an $f\in H(\mathbb{D})$ the representation given in (1) need not be unique, because of which a norm on the space is defined in a specific way. Namely, it is shown that the following quantity

$$\|f\|_{\mathscr{K}}=\inf_{\mu\in\mathfrak{M}}\left\{\|\mu\| \;:\; f(z)=\int_{\partial\mathbb{D}}\frac{d\mu(\zeta)}{1-\overline{\zeta}z}\right\},$$

is a norm on the class of Cauchy transforms, and with the norm the class of transforms becomes a Banach space. For more information on this and related topics see, for instance, [5].

In order to generalize several spaces of analytic functions on the domains in $\mathbb{C}$, such as the weighted type space, the Bloch type space, and the Zygmund type space, in [26] we defined the space $\mathscr{W}_w^{(m)}(\mathbb{D}) = \mathscr{W}_w^{(m)}$ as follows

$$\mathscr{W}_w^{(m)} = \Big\{ f \in H(\mathbb{D}) : \|f\|_{\mathscr{W}_w^{(m)}} := \sum_{k=0}^{m-1} |f^{(k)}(0)| + \sup_{z\in\mathbb{D}} w(z)|f^{(m)}(z)| < \infty \Big\},$$

where $m \in \mathbb{N}_0$ and $w \in W(\mathbb{D})$. We called the space the mth weighted type space of holomorphic function. The quantity $\|\cdot\|_{\mathscr{W}_w^{(m)}}$ is a norm on the space $\mathscr{W}_w^{(m)}(\mathbb{D})$, and with this norm it becomes a Banach space.

Its subspace denoted by $\mathscr{W}_{w,0}^{(m)}(\mathbb{D}) = \mathscr{W}_{w,0}^{(m)}$, which consists of all $f \in H(\mathbb{D})$ such that

$$\lim_{|z|\to 1} w(z)|f^{(m)}(z)| = 0,$$

was also introduced in [26]. We called the space the little mth weighted type space of holomorphic function.

There have been many investigations devoted to some concrete linear operators from or to them as it was the case, for example, in the following papers: [8, 26, 28, 44, 45].

Beside the integral type operators (see, for instance, [8, 9, 10, 11, 18, 24], as well as the related references therein), the composition operator

$$C_\varphi(f) = f \circ \varphi,$$

where $\varphi \in S(\mathscr{D})$, the multiplication operator

$$M_u(f) = uf,$$

where $u \in H(\mathscr{D})$, and the differentiation operator

$$Df = f',$$

are some of the most investigated concrete linear operators on many subspaces of $H(\mathscr{D})$ (the first two operators on a domain $\mathscr{D} \subseteq \mathbb{C}^n$, whereas the third one on a domain $\mathscr{D} \subseteq \mathbb{C}$).

There are some other operators such as the operator of radial differentiation or of the partial differentiation, which are investigated instead of the differentiation operator D, on domains in $\mathbb{C}^n$. See also the recently introduced linear operators in [15], [30] and [32].

There have been some considerable recent interest in investigating products of above-mentioned linear operators as it was the case, for example, in [3, 4, 8, 21, 22, 23, 35, 36, 38, 39]. The product $M_u \circ C_\varphi$, that is, the weighted composition operator has been studied a lot (see, e.g., [2, 12, 13, 23, 27] and the related references therein).

Some investigations of the products of differentiation and composition operators DC_φ and $C_\varphi D$ were conducted somewhat later (see, for instance, [16, 28, 42] and the references therein).

Their generalization defined as follows

$$D^n_{\varphi,u} f = u(f^{(n)} \circ \varphi), \tag{2}$$

where $n \in \mathbb{N}_0$, $\varphi \in S(\mathbb{D})$ and $u \in H(\mathbb{D})$, was studied, for instance, in [3, 13, 29, 38, 39, 41, 43].

The following generalization of the products of differentiation and composition operators DC_φ and $C_\varphi D$

$$T_{\varphi,u_1,u_2} f(z) = u_1(z) f(\varphi(z)) + u_2(z) f'(\varphi(z)), \quad z \in \mathbb{D}, \tag{3}$$

where $u_1, u_2 \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$, was studied first in [35]. The operator (3) was later investigated also between some spaces of analytic functions on the upper half-plane in [34]. The operator was a good prototype of a concrete linear operator containing the composition, multiplication and differentiation operator for some investigations on spaces of holomorphic functions.

In [36] we generalized the operator (3) by including the derivatives of any order as follows

$$T^n_{\varphi,u_1,u_2} f(z) = u_1(z) f^{(n)}(\varphi(z)) + u_2(z) f^{(n+1)}(\varphi(z)), \quad z \in \mathbb{D} \tag{4}$$

where $n \in \mathbb{N}_0$, $u_1, u_2 \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$.

For some other investigations of the operators given in (3) and (4), and their relatives see, for instance, [4, 6, 7, 14, 30, 32, 40] and the related references quoted therein).

Having published [36] the present author have noticed that the results therein can be easily generalized for the case of the following more general operator

$$T^n_{\varphi,\vec{u}} f(z) = \sum_{j=0}^{n} u_j(z) f^{(j)}(\varphi(z)), \quad z \in \mathbb{D}, \tag{5}$$

where $n \in \mathbb{N}_0$, $u_j \in H(\mathbb{D})$, $j = \overline{0,n}$, and $\varphi \in S(\mathbb{D})$, and informed some of his collaborators about it.

In our recent papers [30] and [32] we have introduced some multi-dimensional generalizations and close relatives of the above-mentioned linear operators, including the operator in (5).

This operator can be extended further as follows

$$T^n_{\vec{\varphi},\vec{u}} f(z) = \sum_{j=0}^{n} u_j(z) f^{(j)}(\varphi_j(z)), \quad z \in \mathbb{D}, \tag{6}$$

where $n \in \mathbb{N}_0$, $u_j \in H(\mathbb{D})$, $j = \overline{0,n}$, and $\varphi_j \in S(\mathbb{D})$, $j = \overline{0,n}$.

Finding the norm of a linear operator is one of the basic problems in operator theory (see, for instance, [17, 19, 20, 37], where some formulas for the norms of many linear operators can be found).

For some recent results on calculating the norm of some concrete linear operators between spaces of analytic functions on various domains see, for instance, [9, 12, 23, 25, 27, 31, 33].

Motivated by above-mentioned line of numerous investigations of products of concrete linear operators acting between spaces of analytic functions on domains in $\mathbb{C}$ and $\mathbb{C}^n$, in this short note we generalize several results in the literature, including some in [31] and [33], by finding the norm of the linear operator $L_s : \mathscr{K}(\mathbb{D}) \to \mathscr{W}_w^{(m)}(\mathbb{D})$, which is defined as follows:

$$L_s f(z) = \sum_{k=1}^{s} \alpha_k \sum_{j=0}^{n_k} u_{j,k}(z) f^{(j)}(\varphi_{j,k}(z)), \quad z \in \mathbb{D}, \tag{7}$$

where $s \in \mathbb{N}$, $n_k \in \mathbb{N}_0$, $k = \overline{1,s}$, $\alpha_k \in \mathbb{C}$, $k = \overline{1,s}$, $u_{j,k} \in H(\mathbb{D})$, $j = \overline{0,n_k}$, $k = \overline{1,s}$, and $\varphi_{j,k} \in S(\mathbb{D})$, $j = \overline{0,n_k}$, $k = \overline{1,s}$.

Beside presenting a formula for the norm of the operator $L_s : \mathscr{K}(\mathbb{D}) \to \mathscr{W}_w^{(m)}(\mathbb{D})$, which immediately gives a characterization for the boundedness of the operator, here we also present a characterization for the boundedness of the operator $L_s : \mathscr{K}(\mathbb{D}) \to \mathscr{W}_{w,0}^{(m)}(\mathbb{D})$, that is, in the case where the space $\mathscr{W}_w^{(m)}(\mathbb{D})$ is replaced by its little type counterpart $\mathscr{W}_{w,0}^{(m)}(\mathbb{D})$.

2. Main results

First we quote the following auxiliary result (see, e.g., [28, 29]).

LEMMA 1. *Let $\mathscr{D}$ be a domain in $\mathbb{C}$, $n \in \mathbb{N}_0$, $u, \varphi, g \in H(\mathscr{D})$. Then*

$$(u(g \circ \varphi))^{(n)}(z) = \sum_{k=0}^{n} g^{(k)}(\varphi(z)) \sum_{l=k}^{n} C_l^n u^{(n-l)}(z) P_{l,k}(\varphi(z)),$$

for $z \in \mathscr{D}$, where

$$P_{l,k}(\varphi(z)) = \sum_{k_1,k_2,\dots,k_l} \frac{l!}{k_1!k_2!\cdots k_l!} \prod_{j=1}^{l} \left(\frac{\varphi^{(j)}(z)}{j!} \right)^{k_j},$$

and the sum is taken over all non negative integers $k_1, k_2, \dots, k_l$ such that $k = k_1 + k_2 + \cdots + k_l$ and $k_1 + 2k_2 + \cdots + lk_l = l$.

The following theorem is our main result in this note.

THEOREM 1. *Let $s, m \in \mathbb{N}$, $n_k \in \mathbb{N}_0$, $k = \overline{1,s}$, $\alpha_k \in \mathbb{C}$, $k = \overline{1,s}$, $u_{j,k} \in H(\mathbb{D})$, $j = \overline{0,n_k}$, $k = \overline{1,s}$, and $\varphi_{j,k} \in S(\mathbb{D})$, $j = \overline{0,n_k}$, $k = \overline{1,s}$, $w \in W(\mathbb{D})$,*

$$P_1(z,\zeta) := \left| \sum_{k=1}^{s} \alpha_k \sum_{j=0}^{n_k} \sum_{i=0}^{m} \frac{w(z)(j+i)!(\overline{\zeta})^{j+i}}{(1-\overline{\zeta}\varphi_{j,k}(z))^{j+i+1}} \sum_{l=i}^{m} C_l^m u_{j,k}^{(m-l)}(z) P_{l,i}(\varphi_{j,k}(z)) \right| \tag{8}$$

and

$$P_2(\zeta) := \sum_{t=0}^{m-1} \left| \sum_{k=1}^{s} \alpha_k \sum_{j=0}^{n_k} \sum_{i=0}^{t} \frac{(i+j)!(\overline{\zeta})^{i+j}}{(1-\overline{\zeta}\varphi_{j,k}(0))^{i+j+1}} \sum_{l=i}^{t} C_l^t u_{j,k}^{(t-l)}(0) P_{l,i}(\varphi_{j,k}(0)) \right|.$$

Then, the operator $L_s : \mathscr{K} \to \mathscr{W}_w^{(m)}$ *is bounded if and only if*

$$P := \sup_{\zeta\in\partial\mathbb{D}} \sup_{z\in\mathbb{D}} \big(P_1(z,\zeta)+P_2(\zeta)\big) < \infty, \tag{9}$$

and if the operator is bounded, then

$$\|L_s\|_{\mathscr{K}\to\mathscr{W}_w^{(m)}} = P. \tag{10}$$

Proof. Assume $L_s : \mathscr{K} \to \mathscr{W}_w^{(m)}$ is bounded. Let $\zeta \in \partial\mathbb{D}$ and

$$f_\zeta(z) = \frac{1}{1-\overline{\zeta}z}, \quad z \in \mathbb{D}. \tag{11}$$

Since

$$f_\zeta(z) = \int_{\partial\mathbb{D}} \frac{d\delta_\zeta(\eta)}{1-\overline{\eta}z}, \quad z \in \mathbb{D},$$

where δ_ζ is the measure concentrated at ζ such that $\|\delta_\zeta\| = 1$, and the measure is positive, it is well known that

$$\|f_\zeta\|_{\mathscr{K}} = \|\delta_\zeta\|, \quad \zeta \in \partial\mathbb{D},$$

from which together with the boundedness of the operator, it follows that

$$\sum_{t=0}^{m-1} \left|(L_s f_\zeta)^{(t)}(0)\right| + \sup_{z\in\mathbb{D}} w(z)|(L_s f_\zeta)^{(m)}(z)| = \|L_s f_\zeta\|_{\mathscr{W}_w^{(m)}} \leqslant \|L_s\|_{\mathscr{K}\to\mathscr{W}_w^{(m)}},$$

for any $\zeta \in \partial\mathbb{D}$.

From this fact and the relations

$$\sum_{t=0}^{m-1} \left|(L_s f_\zeta)^{(t)}(0)\right| = \sum_{t=0}^{m-1} \left| \sum_{k=1}^{s} \alpha_k \sum_{j=0}^{n_k} \sum_{i=0}^{t} \frac{(i+j)!(\overline{\zeta})^{i+j}}{(1-\overline{\zeta}\varphi_{j,k}(0))^{i+j+1}} \sum_{l=i}^{t} C_l^t u_{j,k}^{(t-l)}(0) P_{l,i}(\varphi_{j,k}(0)) \right|$$

and

$$w(z)|(L_s f_\zeta)^{(m)}(z)| = \left| \sum_{k=1}^{s} \alpha_k \sum_{j=0}^{n_k} \sum_{i=0}^{m} \frac{w(z)(j+i)!(\overline{\zeta})^{j+i}}{(1-\overline{\zeta}\varphi_{j,k}(z))^{j+i+1}} \sum_{l=i}^{m} C_l^m u_{j,k}^{(m-l)}(z) P_{l,i}(\varphi_{j,k}(z)) \right|,$$

for $\zeta \in \partial\mathbb{D}$ and $z \in \mathbb{D}$, we get

$$P \leqslant \|L_s\|_{\mathscr{K}\to\mathscr{W}_w^{(m)}}. \tag{12}$$

Now, assume that (9) holds. For any $f \in \mathscr{K}$, there is a unique $\mu \in \mathfrak{M}$ such that

$$\|f\|_{\mathscr{K}} = \|\mu\| \tag{13}$$

and (1) holds, and consequently

$$f^{(j)}(z)=j!\int_{\partial\mathbb{D}}\frac{(\overline{\zeta})^{j}d\mu(\zeta)}{(1-\overline{\zeta}z)^{j+1}},\quad z\in\mathbb{D}, \tag{14}$$

for $j\in\mathbb{N}_0$.

Lemma 1 and (14) imply

$$\begin{aligned}
w(z)|(L_sf)^{(m)}(z)| &= w(z)\Big|\sum_{k=1}^{s}\alpha_k\sum_{j=0}^{n_k}(u_{j,k}(z)f^{(j)}(\varphi_{j,k}(z)))^{(m)}\Big|\\
=&w(z)\Big|\sum_{k=1}^{s}\alpha_k\sum_{j=0}^{n_k}\sum_{i=0}^{m}f^{(j+i)}(\varphi_{j,k}(z))\sum_{l=i}^{m}C_l^m u_{j,k}^{(m-l)}(z)P_{l,i}(\varphi_{j,k}(z))\Big|\\
=&\Big|\sum_{k=1}^{s}\alpha_k\sum_{j=0}^{n_k}\sum_{i=0}^{m}\int_{\partial\mathbb{D}}\frac{w(z)(j+i)!(\overline{\zeta})^{j+i}d\mu(\zeta)}{(1-\overline{\zeta}\varphi_{j,k}(z))^{j+i+1}}\sum_{l=i}^{m}C_l^m u_{j,k}^{(m-l)}(z)P_{l,i}(\varphi_{j,k}(z))\Big|\\
\leqslant&\int_{\partial\mathbb{D}}\Big|\sum_{k=1}^{s}\alpha_k\sum_{j=0}^{n_k}\sum_{i=0}^{m}\frac{w(z)(j+i)!(\overline{\zeta})^{j+i}}{(1-\overline{\zeta}\varphi_{j,k}(z))^{j+i+1}}\sum_{l=i}^{m}C_l^m u_{j,k}^{(m-l)}(z)P_{l,i}(\varphi_{j,k}(z))\Big|d|\mu|(\zeta)\\
=&\int_{\partial\mathbb{D}}P_1(z,\zeta)d|\mu|(\zeta).
\end{aligned} \tag{15}$$

Employing some standard calculations, known inequalities, as well as Lemma 1 it follows that

$$\begin{aligned}
\sum_{t=0}^{m-1}\Big|(L_sf)^{(t)}(0)\Big| &= \sum_{t=0}^{m-1}\Big|\sum_{k=1}^{s}\alpha_k\sum_{j=0}^{n_k}(u_{j,k}(z)f^{(j)}(\varphi_{j,k}(z)))^{(t)}\Big|_{z=0}\\
=&\sum_{t=0}^{m-1}\Big|\sum_{k=1}^{s}\alpha_k\sum_{j=0}^{n_k}\sum_{i=0}^{t}f^{(i+j)}(\varphi_{j,k}(0))\sum_{l=i}^{t}C_l^t u_{j,k}^{(t-l)}(0)P_{l,i}(\varphi_{j,k}(0))\Big|\\
=&\sum_{t=0}^{m-1}\Big|\sum_{k=1}^{s}\alpha_k\sum_{j=0}^{n_k}\sum_{i=0}^{t}\int_{\partial\mathbb{D}}\frac{(i+j)!(\overline{\zeta})^{i+j}d\mu(\zeta)}{(1-\overline{\zeta}\varphi_{j,k}(0))^{i+j+1}}\sum_{l=i}^{t}C_l^t u_{j,k}^{(t-l)}(0)P_{l,i}(\varphi_{j.k}(0))\Big|\\
\leqslant&\int_{\partial\mathbb{D}}\sum_{t=0}^{m-1}\Big|\sum_{k=1}^{s}\alpha_k\sum_{j=0}^{n_k}\sum_{i=0}^{t}\frac{(i+j)!(\overline{\zeta})^{i+j}}{(1-\overline{\zeta}\varphi_{j,k}(0))^{i+j+1}}\sum_{l=i}^{t}C_l^t u_{j,k}^{(t-l)}(0)P_{l,i}(\varphi_{j,k}(0))\Big|d|\mu|(\zeta)\\
=&\int_{\partial\mathbb{D}}P_2(\zeta)d|\mu|(\zeta).
\end{aligned} \tag{16}$$

Relations (13), (15) and (16) imply

$$\begin{aligned}
\sum_{k=0}^{m-1}\Big|(L_sf)^{(k)}(0)\Big|+w(z)|(L_sf)^{(m)}(z)| &\leqslant \int_{\partial\mathbb{D}}(P_2(\zeta)+P_1(z,\zeta))d|\mu|(\zeta)\\
&\leqslant \int_{\partial\mathbb{D}}\sup_{\zeta\in\partial\mathbb{D}}\sup_{z\in\mathbb{D}}\big(P_2(\zeta)+P_1(z,\zeta)\big)d|\mu|(\zeta)\\
&= \sup_{\zeta\in\partial\mathbb{D}}\sup_{z\in\mathbb{D}}\big(P_2(\zeta)+P_1(z,\zeta)\big)\|\mu\|_{\mathscr{K}}\\
&=P\|f\|_{\mathscr{K}},
\end{aligned} \tag{17}$$

for $z \in \mathbb{D}$, from which the boundedness of $L_s : \mathcal{K} \to \mathcal{W}_w^{(m)}$ easily follows, as well as

$$\|L_s\|_{\mathcal{K} \to \mathcal{W}_w^{(m)}} \leqslant P. \tag{18}$$

Inequalities (12) and (18) yield (10). $\square$

The following theorem presents a characterization for the boundedness of $L_s : \mathcal{K} \to \mathcal{W}_{w,0}^{(m)}$.

THEOREM 2. *Let $s, m \in \mathbb{N}$, $n_k \in \mathbb{N}_0$, $k = \overline{1,s}$, $\alpha_k \in \mathbb{C}$, $k = \overline{1,s}$, $u_{j,k} \in H(\mathbb{D})$, $j = \overline{0,n_k}$, $k = \overline{1,s}$, and $\varphi_{j,k} \in S(\mathbb{D})$, $j = \overline{0,n_k}$, $k = \overline{1,s}$, and $w \in W(\mathbb{D})$. Then, the operator $L_s : \mathcal{K} \to \mathcal{W}_{w,0}^{(m)}$ is bounded if and only if (9) holds and*

$$\lim_{|z|\to 1} P_1(z,\zeta) = 0 \tag{19}$$

for every $\zeta \in \partial\mathbb{D}$.

Proof. If $L_s : \mathcal{K} \to \mathcal{W}_{w,0}^{(m)}$ is bounded, then so is $L_s : \mathcal{K} \to \mathcal{W}_w^{(m)}$. Employing Theorem 1 we obtain (9).

Further, we also have $L_s f_\zeta \in \mathcal{W}_{w,0}^{(m)}$, for every $f_\zeta \in \mathcal{K}$, $\zeta \in \partial\mathbb{D}$, defined in (11), which means that

$$\lim_{|z|\to 1} w(z)|(L_s f_\zeta)^{(m)}(z)| = \lim_{|z|\to 1} P_1(z,\zeta) = 0,$$

for every $\zeta \in \partial\mathbb{D}$.

Let (9) and (19) hold. Then, Theorem 1 implies the boundedness of $L_s : \mathcal{K} \to \mathcal{W}_w^{(m)}$.

Now note that for any $f \in \mathcal{K}$ we have that (1) holds for some Borel measure μ, so (15) implies

$$w(z)|(L_s f)^{(m)}(z)| \leqslant \int_{\partial\mathbb{D}} P_1(z,\zeta)\,d|\mu|(\zeta). \tag{20}$$

Condition (19) says that the integrand in (20) tends to zero for each $\zeta \in \partial\mathbb{D}$, as $|z| \to 1$.

Beside this, we also have

$$P_1(z,\zeta) \leqslant P < \infty,$$

for $z \in \mathbb{D}$ and $\zeta \in \partial\mathbb{D}$, where P is defined in (9).

An application of the dominated convergence theorem implies the following relation

$$\lim_{|z|\to 1} \int_{\partial\mathbb{D}} P_1(z,\zeta)\,d|\mu|(\zeta) = 0.$$

This fact together with (20) implies

$$\lim_{|z|\to 1} w(z)|(L_s f)^{(m)}(z)| = 0.$$

Thus, for every $f \in \mathcal{K}$ we have $L_s f \in \mathcal{W}_{w,0}^{(m)}$, implying the boundedness of $L_s : \mathcal{K} \to \mathcal{W}_{w,0}^{(m)}$. $\square$

Conclusion

We show that it is possible to calculate norm of any linear combination of concrete linear operators, which many authors call Stević-Sharma type operators, from the space of Cauchy transforms to a general weighted-type space, continuing a long line of previous investigations of product type operators on spaces of analytic functions. The study in this article suggests further investigations of linear combinations of various concrete linear operators.

Authors contributions. This work is based on some ideas by the author of this paper, who has done all the research and writing by himself.

Use of AI tools declaration. The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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