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GENERALIZED HARDY'S INEQUALITIES WITH NONLINEAR INTEGRATION LIMITS

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Abstract. In this paper, we establish two integral inequalities arising in the study of weighted norm estimates. First, we consider a sequence of measurable sets forming a partition of $\mathbb{R}^m$ and derive an upper bound for a sum involving weighted integrals of a function g, controlled by a sequence of positive numbers. Second, we prove an integral inequality involving a non-decreasing function h with $\sup(h(x)/x) < \infty$ and a power-weighted integral of f. Higher dimensional analogue of this inequality are also established.

1. Introduction

In the course of investigations in the theory of integral equations, Hilbert proved that the series

$$\sum_{m,n=1}^{\infty} \frac{a_m a_n}{m+n}$$

is convergent whenever $\sum a_m^2$ is convergent [5]. Hilbert also showed that

$$\sum_{m,n=1}^{\infty} \frac{a_m a_n}{m+n} \leqslant 2\pi \sum_{m=1}^{\infty} a_m^2.$$

This result was proved using the theory of Fourier series. In the process of giving a simpler proof of this inequality, Hardy observed that Hilbert's theorem is an easy corollary of the fact that, if $\sum_{n=1}^{\infty} a_n^2$ is convergent, then

$$\sum_{n=1}^{\infty} \left(\frac{a_1 + \cdots + a_n}{n} \right)^2$$

is also convergent [4]. Marcel Riesz generalized this theorem by proving that [4]

$$\sum_{n=1}^{\infty} \left(\frac{a_1 + a_2 + \cdots + a_n}{n} \right)^p \leqslant \left(\frac{p^2}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p, \quad \text{for } p > 1 \text{ and } a_n \geqslant 0. \tag{1}$$

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The fact that the constant $\left(\frac{p}{p-1}\right)^p$ replaces $\left(\frac{p^2}{p-1}\right)^p$ in (1) as the best possible was subsequently proved by E. Landau [8]. This inequality is now called Hardy's inequality in the literature and its integral version is given by

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p dx, \quad \text{for } p>1 \text{ and } f(x)\geqslant 0.$$

Numerous generalizations and variations of Hardy's inequality above have been developed. The modern form of the Hardy's original inequality is

$$\left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q u(x)dx\right)^{1/q} \leqslant C\left(\int_0^\infty f^p(x)v(x)dx\right)^{1/p}, \tag{2}$$

where $f(x)\geqslant 0$, u and v are weights and $1<p\leqslant q<\infty$ [10]. Then, inequality (2) holds if and only if

$$\sup_x \left(\int_x^\infty u(t)dt\right)^{1/q} \left(\int_0^x v^{1-p'}(t)dt\right)^{1/p} < \infty.$$

For power weights we have,

$$\int_0^\infty x^{-r}\left(\int_0^x f(t)dt\right)^p dx \leqslant \left(\frac{p}{r-1}\right)^p \int_0^\infty f(x)^p x^{p-r} dx, \quad r>1 \tag{3}$$

and the constant is optimal in this case. The inequality (3) is a special case of our result proved in Corollary 1. The sequence of functions used to demonstrate the sharpness of inequality (6) in Corollary 1 also applies directly to inequality (3), thereby establishing its optimality. More types of the Hardy's inequalities and its applications can be found in [1,6,7,9].

In [2], L. Bouthat *et al.* established the following discrete Hardy type inequality in which the arithmetic means of a sequence are replaced by the weighted means over nested subsets of the sequence.

THEOREM 1. [2] *Let* $\mathbb{N}$ *denote the set of positive integers, and let* $\{N_1, N_2, \ldots\}$ *be a partition of* $\mathbb{N}$. *Denote*

$$\mathbf{N}_n := N_1 \cup \cdots \cup N_n, \quad n \geqslant 1.$$

Let $(m_n)_{n\geqslant 1}$ *be a sequence of positive numbers and* $p>1$ *and* $p'>1$ *be such that* $\frac{1}{p}+\frac{1}{p'}=1$. *Define*

$$w_n := \left(\sum_{j\in N_n} m_j^{p'}\right)^{1/p'} \quad \text{and} \quad M_n := \sum_{j=1}^n w_j, \quad n\geqslant 1.$$

Suppose that

$$\rho := \sup_{n\geqslant 1}\left(w_n \sum_{j=n}^\infty \frac{1}{M_j}\right)^{1/p} < \infty.$$

Let $(a_n)_{n\geqslant 1}$ *be a sequence of complex numbers. Then,*

$$\left(\sum_{n=1}^{\infty}\left|\frac{1}{M_n}\sum_{j\in\mathbf{N}_n} m_j a_j\right|^p\right)^{1/p} \leqslant \rho\left(\sum_{n=1}^{\infty}|a_n|^p\right)^{1/p}.$$

Our first result in this note is an integral version of Theorem 1 in which the operator

$$f \mapsto \left(\frac{1}{M_n}\int_{B_n} g(x)f(x)dx\right)_{n\geqslant 1}$$

is considered, where $B_n = A_1\cup\cdots\cup A_n$ and $\{A_1,A_2,\ldots\}$ is a partition of $\mathbb{R}^m$. In Theorem 2, we show that for appropriate choice of M_n, this operator is bounded from $L^p(\mathbb{R}^m)\to \ell^p$. This generalizes Theorem 1 and, in Remark 1, we obtain Theorem 1 as a corollary of our result. Our second result is motivated by the following problem.

Given a non-decreasing function h on $(0,\infty)$, with $h(0)=0$, let g and u be positive functions on $(0,\infty)$, and consider the operator T given by

$$Tf(x) := \frac{1}{u(x)}\int_0^{h(x)} g(t)f(t)dt, \quad x>0.$$

We are interested in finding the conditions on u,h and g so that T is bounded on $L^p(0,\infty)$. When $u(x)=x^{-r/p}$ and $g(x)=x^{\frac{r-p}{p}}$, $r>1$, we show in Theorem 3 that T is bounded on $L^p(0,\infty)$ whenever $\sup_x \frac{h(x)}{x}$ is bounded.

Let $B(0,r)$ denote the ball centered at 0 with radius r and $|B(0,r)|$ be its measure. In [3], Grafakos *et al.* proved the following higher dimensional version of the Hardy's inequality.

$$\left(\int_{\mathbb{R}^n}\left(\frac{1}{|B(0,|x|)|}\int_{B(0,|x|)}|f(y)|dy\right)^p dx\right)^{1/p} \leqslant \frac{p}{p-1}\left(\int_{\mathbb{R}^n}|f(y)|^p dy\right)^{1/p},$$

and the constant $p/(p-1)$ is the best possible. Motivated by this, in Theorem 4 we discuss the higher dimensional analogue of Theorem 3, where we consider the operator

$$f \mapsto \frac{1}{|B(0,|\cdot|)|^{r/p}}\int_{B(0,h(|\cdot|))} f(t)dt.$$

2. Main results

In this section, we present our main results along with their proofs.

THEOREM 2. *Let* $A_1,A_2,\ldots$ *be a partition of* $\mathbb{R}^m$ *such that each* A_i *is measurable. Let* $B_n=\cup_{i=1}^n A_i$ *and* $p,p'\in(1,\infty)$ *be such that* $\frac{1}{p}+\frac{1}{p'}=1$. *For* $g>0$ *defined on* $\mathbb{R}^m$, *define*

$$w_i = \left(\int_{A_i} g(x)^{p'}dx\right)^{1/p'}.$$

If M_n is a sequence of positive numbers such that

$$\rho^p := \sup_i \sum_{n=i}^{\infty} \frac{w_i(\sum_{j=1}^n w_j)^{p/p'}}{M_n^p} < \infty.$$

Then

$$\left(\sum_{n=1}^{\infty}\left|\frac{1}{M_n}\int_{B_n} g(x)f(x)dx\right|^p\right)^{1/p} \leqslant \rho\left(\int_{\mathbb{R}^m}|f(x)|^P dx\right)^{1/p}. \tag{4}$$

Proof. Write

$$\int_{B_n}|g(x)f(x)|dx = \sum_{i=1}^{n}\int_{A_i}|g(x)f(x)|dx.$$

By using the definition of w_i and Hölder's inequality, we find

$$\int_{B_n}|g(x)f(x)|dx \leqslant \sum_{i=1}^{n} w_i\left(\int_{A_i}|f(x)|^p dx\right)^{1/p}.$$

Write $w_i = w_i^{1/p'} w_i^{1/p}$ and use once again Hölder's inequality to obtain

$$\int_{B_n}|g(x)f(x)|dx \leqslant \left(\sum_{i=1}^{n} w_i\right)^{1/p'}\left(\sum_{i=1}^{n} w_i\int_{A_i}|f(x)|^p\,dx\right)^{1/p}.$$

Therefore,

$$\begin{aligned}\left|\frac{1}{M_n}\int_{B_n} g(x)f(x)dx\right|^p &\leqslant \left(\frac{1}{M_n}\right)^p\left(\sum_{j=1}^{n} w_j\right)^{p/p'}\sum_{i=1}^{n} w_i\int_{A_i}|f(x)|^p\,dx\\ &= \sum_{i=1}^{n} W_{i,n}\int_{A_i}|f(x)|^p dx,\end{aligned}$$

where,

$$W_{i,n} := \frac{w_i\left(\sum_{j=1}^n w_j\right)^{p/p'}}{M_n^p}.$$

Hence,

$$\begin{aligned}\sum_{n=1}^{\infty}\left|\frac{1}{M_n}\int_{B_n} g(x)f(x)dx\right|^p &\leqslant \sum_{n=1}^{\infty}\sum_{i=1}^{n} W_{i,n}\int_{A_i}|f(x)|^p dx\\ &= \sum_{i=1}^{\infty}\sum_{n=i}^{\infty} W_{i,n}\int_{A_i}|f(x)|^p dx\\ &\leqslant \sum_{i=1}^{\infty}\rho^p\int_{A_i}|f(x)|^p dx\\ &= \rho^p\int_{\mathbb{R}^m}|f(x)|^p dx.\end{aligned}$$

Therefore,

$$\left(\sum_{n=1}^{\infty}\left|\frac{1}{M_n}\int_{B_n} g(x)f(x)dx\right|^p\right)^{1/p} \leqslant \rho\left(\int_{\mathbb{R}^m}|f(x)|^p dx\right)^{1/p}. \quad \square$$

REMARK 1. Theorem 1 can be obtained from Theorem 2. To see this, let $\mathbb{N} = N_1 \cup N_2 \cup \cdots$ be a partition of $\mathbb{N}$ and define $A_n := \cup_{i\in N_n}[i-1,i)$. Corresponding to the sequences $(a_n)_{n\geqslant 1}$ and $(m_n)_{n\geqslant 1}$, define

$$f(x) = \begin{cases} 0, & \text{if } x\in(-\infty,0), \\ a_n, & \text{if } x\in[n-1,n). \end{cases}$$

and

$$g(x) = \begin{cases} 0, & \text{if } x\in(-\infty,0), \\ m_n, & \text{if } x\in[n-1,n). \end{cases}$$

Therefore, from (4) we get

$$\left(\sum_{n=1}^{\infty}\left|\frac{1}{M_n}\sum_{j\in N_1\cup\cdots\cup N_n} m_j a_j\right|^p\right)^{1/p} \leqslant \rho\left(\sum_{n=1}^{\infty}|a_n|^p\right)^{1/p}.$$

THEOREM 3. *Let h be a non-decreasing function on $(0,\infty)$ such that $h(0)=0$. Fix $p>1$ and $r>1$. Then*

$$\int_0^{\infty} x^{-r}\left(\int_0^{h(x)} t^{\frac{r-p}{p}} f(t)dt\right)^p dx \leqslant c^{r-1}\left(\frac{p}{r-1}\right)^p \int_0^{\infty} f(x)^p dx, \tag{5}$$

for all $f:\mathbb{R}\to[0,\infty)$, where $c=\sup_x \frac{h(x)}{x}$.

Proof. Note that, setting $f(x)x^{1-r/p}$ instead of $f(x)$, inequality (5) is equivalent to

$$\int_0^{\infty} x^{-r}\left(\int_0^{h(x)} f(t)dt\right)^p dx \leqslant c^{r-1}\left(\frac{p}{r-1}\right)^p \int_0^{\infty} f(x)^p x^{p-r}dx.$$

Observe that

$$\begin{aligned}\left(\int_0^{\infty} x^{-r}\left(\int_0^{h(x)} f(t)dt\right)^p dx\right)^{1/p} &= \left(\int_0^{\infty} x^{p-r}\left(\int_0^{h(x)} \frac{f(t)}{x}dt\right)^p dx\right)^{1/p} \\ &\leqslant \left(\int_0^{\infty}\left(\int_0^{c} x^{1-r/p} f(sx)ds\right)^p dx\right)^{1/p}.\end{aligned}$$

Using Minkowski's integral inequality and the change of variable $sx\mapsto y$, we get

$$\begin{aligned}&\left(\int_0^{\infty} x^{-r}\left(\int_0^{h(x)} f(t)dt\right)^p dx\right)^{1/p} \leqslant \int_0^{c}\left(\int_0^{\infty} x^{p-r} f(sx)^p dx\right)^{1/p} ds \\ &= \int_0^{c}\left(\int_0^{\infty}\left(\frac{y}{s}\right)^{p-r} f(y)^p \frac{dy}{s}\right)^{1/p} ds = c^{(r-1)/p}\frac{p}{r-1}\left(\int_0^{\infty} y^{p-r} f(y)^p dy\right)^{1/p},\end{aligned}$$

which proves our assertion. □

Note that, when $h(x) = x$ this result gives the weighted Hardy's inequality (3) with the sharp constant $p/(r-1)$. In the following corollary, we show that for $h(x) = kx$, $k \geqslant 0$, the above inequality is sharp.

COROLLARY 1. *For $k > 0$,*

$$\int_0^\infty x^{-r}\left(\int_0^{kx} f(t)dt\right)^p dx \leqslant k^{r-1}\left(\frac{p}{r-1}\right)^p \int_0^\infty f(x)^p x^{p-r} dx, \tag{6}$$

in which the constant is optimal.

Proof. For $h(x) = kx$, we trivially have $\sup_x \frac{h(x)}{x} = k$. Using Theorem 3, we get

$$\int_0^\infty x^{-r}\left(\int_0^{kx} f(t)dt\right)^p dx \leqslant k^{r-1}\left(\frac{p}{r-1}\right)^p \int_0^\infty f(x)^p x^{p-r} dx.$$

To show that the constant is optimal, it is enough to show that there exist a sequence of functions (f_n) such that the quotient

$$\frac{\int_0^\infty x^{-r}\left(\int_0^{kx} f_n(t)dt\right)^p dx}{\int_0^\infty f_n(x)^p x^{p-r} dx} \to k^{r-1}\left(\frac{p}{r-1}\right)^p \quad \text{as } n \to \infty.$$

Let χ_A denote the characteristic function of the set A. Consider

$$f_n(x) = x^{\frac{-1}{p}+\frac{1}{n}} x^{\frac{r-p}{p}} \chi_{(0,a)}(x), \quad \text{for some } a > 0.$$

Then

$$\int_0^\infty f_n(x)^p x^{p-r} dx = \frac{n a^{\frac{p}{n}}}{p}$$

and

$$\int_0^\infty x^{-r}\left(\int_0^{kx} f(t)dt\right)^p dx = \frac{n k^{r-1+\frac{p}{n}} p^p}{(r-1+\frac{p}{n})^p p}\left(\frac{a}{k}\right)^{\frac{p}{n}}.$$

Therefore,

$$\frac{\int_0^\infty x^{-r}\left(\int_0^{kx} f(t)dt\right)^p dx}{\int_0^\infty y^{p-r} f(y)^p dy} = \frac{k^{r-1+\frac{1}{n}} p^p}{(r-1+\frac{1}{n}p)^p} \to k^{r-1}\left(\frac{p}{r-1}\right)^p \quad \text{as } n \to \infty. \qquad \square$$

We now give the higher dimensional analogue of Theorem 3.

THEOREM 4. *Let h be a non-decreasing function on $(0,\infty)$ satisfying $h(0) = 0$ and $\sup_x \frac{h(x)}{x} = c < \infty$. Fix $n > 1$ and let $f : \mathbb{R}^n \to [0,\infty)$. Then*

$$\int_{\mathbb{R}^n} \frac{1}{|B(0,|x|)|^r}\left(\int_{B(0,h(|x|))} f(t)dt\right)^p dx \leqslant k \int_{\mathbb{R}^n} f(x)^p |x|^{n(p-r)} dx,$$

where

$$k = \left(\frac{2\pi^{n/2}}{\Gamma(n/2)}\right)^{p-r} n^{r-2p} c^{r-1} \left(\frac{p}{r-1}\right)^{p}.$$

Proof. Using polar coordinates (see [11]), write

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{1}{|B(0,|x|)|^r} \left(\int_{B(0,h(|x|))} f(t)dt\right)^p dx \\
&= m(S^{n-1}) \int_0^\infty \frac{1}{|B(0,s)|^r} \left(\int_0^{h(s)} \int_{S^{n-1}} f(l\theta) d\theta l^{n-1} dl\right)^p s^{n-1} ds.
\end{aligned}$$

Now, apply Hölder's inequality in the variable θ to get

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{1}{|B(0,|x|)|^r} \left(\int_{B(0,h(|x|))} f(t)dt\right)^p dx \\
&\leqslant m(S^{n-1})^p \int_0^\infty \frac{1}{|B(0,s)|^r} \left(\int_0^{h(s)} \left(\int_{S^{n-1}} f(l\theta)^p d\theta\right)^{1/p} l^{n-1} dl\right)^p s^{n-1} ds.
\end{aligned}$$

Applying Theorem 3 to the function

$$F(l) := \left(\int_{S^{n-1}} f(l\theta)^p d\theta\right)^{1/p} l^{n-1},$$

we finally obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{1}{|B(0,|x|)|^r} \left(\int_{B(0,h(|x|))} f(t)dt\right)^p dx \\
&\leqslant \left(\frac{2\pi^{n/2}}{\Gamma(n/2)}\right)^{p-r} n^{r-2p} c^{r-1} \left(\frac{p}{r-1}\right)^p \int_0^\infty \int_{S^{n-1}} f(l\theta)^p d\theta l^{n-1} l^{n(p-r)} dl \\
&= \left(\frac{2\pi^{n/2}}{\Gamma(n/2)}\right)^{p-r} n^{r-2p} c^{r-1} \left(\frac{p}{r-1}\right)^p \int_{\mathbb{R}^n} f(x)^p |x|^{n(p-r)} dx. \quad \square
\end{aligned}$$

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