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THE UPPER BOUNDS OF NON–REAL EIGENVALUES FOR INDEFINITE p–LAPLACIAN WITH GENERAL SEPARATED BOUNDARY CONDITIONS

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Abstract. In this paper, the upper bounds of non-real eigenvalues of indefinite p-Laplacian problems with general Sturm-Liouville (S-L) type separated boundary conditions are studied. The upper bounds of imaginary parts and absolute values of non-real eigenvalues are given by using the method of bounded variation.

1. Introduction

We study the following one-dimensional p-Laplacian eigenvalue problem

$$-\Delta_p y+q[y]^{p-1}=\lambda w[y]^{p-1},\ \ x\in[0,1], \tag{1.1}$$

with the Sturm-Liouville type boundary conditions

$$\begin{aligned} B_1y&:=\cos\alpha y(0)-\sin\alpha y'(0)=0,\\ B_2y&:=\cos\beta y(1)-\sin\beta y'(1)=0, \end{aligned} \tag{1.2}$$

where $\alpha,\beta\in[0,\pi)$, $p\geqslant 2$ is an integer, Δ_p is the p-Laplacian defined by $\Delta_p y=([y']^{p-1})'$, $[y]^{p-1}=|y|^{p-2}y$, λ is the spectral parameter, q is the potential function and the weighted function w changes its sign on [0,1] in the sense that

$$mes\{x\in[0,1]:w(x)>0\}>0,\ \ mes\{x\in[0,1]:w(x)<0\}>0,$$

and q,w are real-valued functions satisfying

$$q,w\in L^1[0,1],\ w(x)\neq 0\ \text{ a.e. on }\ [0,1]. \tag{1.3}$$

Let $W_0^{1,p}=W_0^{1,p}(0,1)$ be the Sobolev space which is the completion of $C_0^\infty(0,1)$ with respect to the norm $\|y\|_{1,p}=(\int_0^1|y'|^p)^{\frac{1}{p}}$. Set

$$\mathscr{L}_q(y)=\int_0^1(|y'|^p+q|y|^p)\mathrm{d}x+(\cot^*\alpha)^{p-1}|y(0)|^p-(\cot^*\beta)^{p-1}|y(1)|^p,$$

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$$\mathscr{R}(y) = \int_0^1 w(x)|y(x)|^p \mathrm{d}x.$$

If there exist nonzero $g_1, g_2 \in W_0^{1,p}$ such that $\mathscr{L}_q(g_1) > 0$ and $\mathscr{L}_q(g_2) < 0$, then the problem (1.1)–(1.2) is called *left-indefinite*. The eigenvalue problem (1.1)–(1.2) is *right-indefinite* if w changes its sign on [0,1]. The problem is called *indefinite* if it is both left-indefinite and right-indefinite [3, 14].

The authors in [3] and [14] introduced the eigenvalue problem of the right definite p-Laplacian Sturm-Liouville problem in detail, which is real, discrete, and semi-bounded. However, if the p-Laplacian problem is indefinite, then the upper and lower bounds of the set of real eigenvalues are unbounded([3], Theorem 3.2), and non-real eigenvalues may exist.

The study of non-real eigenvalues for Sturm-Liouville problems with indefinite weights was first mentioned in the works of Haupt, Richardson et al. ([16]– [26]). In 1986, Mingarelli proposed some open questions about non-real eigenvalues [22], among which there are pre-estimates of the upper and lower bounds of the real and imaginary parts of non-real eigenvalues, respectively. Determining a priori estimates for non-real eigenvalues in indefinite Sturm-Liouville theory is a very interesting and challenging problem. For the classical regular indefinite S-L problem $(p = 2)$, this problem was solved by Xie and Qi in 2013, they gave an upper bound estimation of the non-real eigenvalues of the indefinite S-L problem and obtained sufficient conditions for the existence and nonexistence of the non-real eigenvalues [31]. Since then, such problems have been obtained various generalizations under different conditions, for regular indefinite S-L problems under different boundary conditions, see [7, 8, 18, 24, 25, 28]. For singular indefinite S-L problems with limiting point type as well as limiting circle type endpoints, see [9–12, 15, 29].

The p-Laplacian problem has important applications in many fields, such as in the flow of highly viscous fluids (see Ladyzhenskaya [19], Lions [20]). For more applications of the p-Laplacian problem, please refer to ([2,4,6,21,30,33]). A priori estimation of upper bounds on non-real eigenvalues for one-dimensional indefinite p-Laplacian problems was not given until 2015 by Xie, Qi, and Chen in [32] with a more complete research methodology and quantitative results. Using the method in ([23], [18]), Sun in [27] provided an upper bound on the non-real eigenvalues of indefinite p-Laplacian problems with Dirichlet boundary conditions. The existence of non-real eigenvalues for this indefinite p-Laplacian has been studied through the eigencurve method and the two parameter theory in [32]. For more studies on the one-dimensional p-Laplacian eigenvalue problem, including the prüfer transformation, eigenvalue existence, asymptotics, and vibrationality of eigenvalues, etc., see ([1, 2, 5, 13, 17]).

Motivated by the above results, in this paper, we study the upper bounds of non-real eigenvalues of indefinite p-Laplacian problems with general S-L type separated boundary conditions (1.1)–(1.3). We will give a priori upper bounds on non-real eigenvalues of indefinite problem (1.1)–(1.3) without any additional restrictions to the standard conditions (1.3). The main method is Ganelius Lemma in [23] to estimate $\|\phi\|_{1,p}^p$ in Lemma 4 along with bounded variation function, where ϕ is the eigenfunction of this indefinite p-Laplacian problem (1.1). It should be noted that this paper also employs the Ganelius lemma and utilizes the method of bounded variation to estimate the upper bounds of non-real eigenvalues compared with [28]. However, in contrast to [28], we generalize the Dirichlet boundary conditions to general separated boundary conditions. Although the boundary conditions in [31] are consistent with those in our work, the operators is different. Therefore, inspired by [31] and leveraging the research methods of [28], we further investigate and generalize the estimation of non-real eigenvalues for one-dimensional indefinite p-Laplacian problems.

An outline of this paper is as follows. In Section 2, we will provide upper bounds on non-real eigenvalues when the weight function changes sign once and multiple times. In Section 3, we prove Theorem 1 and Theorem 2 through some lemmas.

2. Main results

Consider the one dimensional p-Laplacian problem with indefinite weight

$$-\Delta_p y(x)+q(x)[y(x)]^{p-1}=\lambda w(x)[y(x)]^{p-1},\ \ x\in(0,1). \tag{2.1}$$

$$\begin{aligned} B_1y&:=\cos\alpha y(0)-\sin\alpha y'(0)=0,\\ B_2y&:=\cos\beta y(1)-\sin\beta y'(1)=0, \end{aligned} \tag{2.2}$$

where $\alpha,\beta\in[0,\pi)$, $p\geqslant 2$ is an integer, q, w are real-valued functions satisfying

$$q,w\in L^1[0,1],\ \ w(x)\neq 0\ \text{ a.e. on }\ [0,1],\ \ q_{\pm}(x)=\max\{\pm q,0\},\ \ \|q\|_1=\int_0^1|q|. \tag{2.3}$$

$$\cot^*\theta=\begin{cases}\cot\theta,\ \theta\in(0,\pi),\\ 0,\ \theta=0,\end{cases}\quad \text{where }\ \theta=\alpha\ \text{ or }\ \beta. \tag{2.4}$$

If $w(x)$ changes sign only once on [0,1], that is, there exists a point $x_0\in(0,1)$ such that

$$(x-x_0)w(x)>0\ \text{ a.e. on }\ [0,1]. \tag{2.5}$$

We choose $\varepsilon>0$ so small such that

$$\Omega(\varepsilon)=\{x\in[0,1]:(x-x_0)w(x)\leqslant\varepsilon\},\ \ 0<m(\varepsilon)=mes\,\Omega(\varepsilon)\leqslant\frac{1}{2}. \tag{2.6}$$

Now, we state the first estimate result of non-real eigenvalues for problem (2.1) and (2.2).

THEOREM 1. *Let λ be a non-real eigenvalue of (2.1). If there exists $x_0\in(0,1)$ such that (2.5) and (2.6) hold. Then the upper bounds of λ satisfy*

$$\begin{aligned} |Im\lambda|&\leqslant\frac{2}{\varepsilon}Q^{\frac{p-1}{p}},\\ |\lambda|&\leqslant\frac{2}{\varepsilon}(|\cot^*\alpha|^{p-1}+2|\cot^*\beta|^{p-1}+Q+\|q\|_1+Q^{\frac{p-1}{p}}), \end{aligned} \tag{2.7}$$

where $Q=2M+2\|q_-\|_1(p-1+2\|q_-\|_1)$, $M=|\cot^*\alpha|^{p-1}+|\cot^*\beta|^{p-1}$.

If $w(x)$ is allowed to have more than one turning points, since $w(x) \neq 0$ $a.e.$ $x \in [0,1]$, we choose $\eta > 0$ so small such that

$$\Omega(\eta) = \{x \in [0,1] : w^2(x) \leqslant \eta\},\ 0 < m(\eta) = mes\Omega(\eta) \leqslant \frac{1}{2}. \tag{2.8}$$

Then we can state the second result of a priori bounds on the non-real eigenvalues for problem (2.1)–(2.2) as follows.

THEOREM 2. *Assume that* $w \in W_0^{1,p}$. *If there exist* $w_0 > 0$ *such that* $|w(x)| \leqslant w_0$ *a.e. on* $[0,1]$ *and* (2.8) *holds for* $\eta > 0$. *Then for any non-real eigenvalue* λ *of problem* (2.1), *it holds that*

$$\begin{aligned} |Im\lambda| &\leqslant \frac{2}{\eta}\|w\|_{1,p}Q^{\frac{p-1}{p}}, \\ |\lambda| &\leqslant \frac{2}{\eta}\{w_0(M+Q+\|q\|_1)+\|w\|_{1,p}Q^{\frac{p-1}{p}}\}, \end{aligned} \tag{2.9}$$

where Q *is consistent with the above description.*

3. The proof of Theorem 1 and Theorem 2

In order to prove the main results (Theorems 1 and 2), we firstly introduce some concepts and prepare some lemmas. Let f be a real-valued function defined on the closed, bounded interval $[a,b]$ and Δ: $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ be a partition of $[a,b]$. We define the variation of f with respect to Δ by [18]

$$Var_\Delta = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|,$$

and the total variation of f on $[a,b]$ by

$$\bigvee_a^b(f) = sup\{Var_\Delta : \Delta\ is\ an\ any\ partition\ of\ [a,b]\}.$$

A real-valued function f is said to be of bounded variation on the closed and bounded interval $[a,b]$ if $\bigvee_a^b(f) < \infty$. Now we prepare some lemmas in the following.

LEMMA 1. ([23]) *Let* $f \geqslant 0$ *and* σ *be functions of bounded variation on the closed interval* J. *Then*

$$\int_J f \mathrm{d}\sigma \leqslant (\inf_J f + Var_J f)(\sup_{K \subset J} \int_K \mathrm{d}\sigma), \tag{3.1}$$

where $Var_J f = \int_J |\mathrm{d}f(x)|$ *and the sup is taken over all compact subsets of J.*

With the help of Lemma 1, we have the following result.

LEMMA 2. ([27]) Let $f \in W_0^{1,p}$ and g be of bounded variation over all of $[0,1]$, that is, g satisfies the inequality $\int_0^x |\mathrm{d}g(x)| < \infty$. Then for all $x \in (0,1]$ and any $\theta > 0$ we have

$$\int_0^x |f(t)|^p |\mathrm{d}g(t)| \leqslant \gamma \left(\frac{1}{x} + p - 2 + \frac{\gamma}{\theta} \right) \int_0^x |f(t)|^p \mathrm{d}t + \theta \int_0^x |f'(t)|^p \mathrm{d}t, \tag{3.2}$$

where $0 < \gamma = \int_0^1 |\mathrm{d}g(x)|$.

Let λ be a non-real eigenvalue of (2.1)–(2.2) and $\phi \in W_0^{1,p}$ be the corresponding eigenfunction with $\int_0^1 |\phi(x)|^p \mathrm{d}x = 1$. That is $B_1\phi = 0, B_2\phi = 0$ and

$$-\Delta_p \phi + q(x)[\phi]^{p-1} = \lambda w(x)[\phi]^{p-1}. \tag{3.3}$$

LEMMA 3. ([27]) Let $q_- = \max\{-q, 0\}$ and ϕ, θ be defined as above. Then

$$\int_0^1 q_-(x)|\phi(x)|^p \mathrm{d}x \leqslant \|q_-\|_1 \left(p - 1 + \frac{\|q_-\|_1}{\theta} \right) \int_0^1 |\phi(x)|^p \mathrm{d}x + \theta \int_0^1 |\phi'(x)|^p \mathrm{d}x. \tag{3.4}$$

In what follows, we give the estimate of $\|\phi\|_{1,p}^p = \int_0^1 |\phi'(x)|^p$.

LEMMA 4. *Let λ be a non-real eigenvalue with the corresponding eigenfunction ϕ defined as above. Assuming ϕ satisfies $\|\phi\|_\infty = 1$. Then*

$$\|\phi\|_{1,p}^p = \int_0^1 |\phi'(x)|^p \mathrm{d}x \leqslant Q. \tag{3.5}$$

where $Q = 2M + 2\|q_-\|_1(p - 1 + 2\|q_-\|_1)$, $M = |\cot^* \alpha|^{p-1} + |\cot^* \beta|^{p-1}$.

Proof. Multiplying both sides of (3.3) by $\overline{\phi}$ and integrating by parts over the interval $[0,1]$, then according to $B_1\phi = 0, B_2\phi = 0$, we have

$$-\int_0^1 \overline{\phi} d([\phi']^{p-1}) + \int_0^1 q|\phi|^p dx = \lambda \int_0^1 w|\phi|^p dx. \tag{3.6}$$

That is

$$(\cot^* \alpha)^{p-1}|\phi(0)|^p - (\cot^* \beta)^{p-1}|\phi(1)|^p + \int_0^1 |\phi'|^p \mathrm{d}x + \int_0^1 q(x)|\phi|^p \mathrm{d}x$$
$$= \lambda \int_0^1 w(x)|\phi|^p \mathrm{d}x. \tag{3.7}$$

Since $Im\lambda \neq 0$, q, w are real-valued, one sees that $\int_0^1 w(x)|\phi(x)|^p \mathrm{d}x = 0$,

$$\int_0^1 |\phi'(x)|^p \mathrm{d}x = (\cot^* \beta)^{p-1}|\phi(1)|^p - (\cot^* \alpha)^{p-1}|\phi(0)|^p - \int_0^1 q(x)|\phi(x)|^p \mathrm{d}x.$$

And because $\alpha, \beta \in [0, \pi)$, we're going to have a classified discussion:

(1) If $\alpha,\beta\in[0,\frac{\pi}{2}]$, then $(\cot^*\beta)^{p-1}|\phi(1)|^p-(\cot^*\alpha)^{p-1}|\phi(0)|^p\leqslant(\cot^*\beta)^{p-1}$.

(2) If $\alpha,\beta\in[\frac{\pi}{2},\pi)$, then $(\cot^*\beta)^{p-1}|\phi(1)|^p-(\cot^*\alpha)^{p-1}|\phi(0)|^p\leqslant-(\cot^*\alpha)^{p-1}$.

(3) If $\alpha\in[0,\frac{\pi}{2}],\beta\in[\frac{\pi}{2},\pi)$, then $(\cot^*\beta)^{p-1}|\phi(1)|^p-(\cot^*\alpha)^{p-1}|\phi(0)|^p\leqslant 0$.

(4) If $\alpha\in[\frac{\pi}{2},\pi),\beta\in[0,\frac{\pi}{2}]$, then

$$(\cot^*\beta)^{p-1}|\phi(1)|^p-(\cot^*\alpha)^{p-1}|\phi(0)|^p\leqslant(\cot^*\beta)^{p-1}-(\cot^*\alpha)^{p-1}.$$

By the definition of M, it can be seen that

$$(\cot^*\beta)^{p-1}|\phi(1)|^p-(\cot^*\alpha)^{p-1}|\phi(0)|^p\leqslant|\cot^*\alpha|^{p-1}+|\cot^*\beta|^{p-1}=M. \tag{3.8}$$

And $q_-(x)=\max\{-q(x),0\}$, so $-\int_0^1 q(x)|\phi(x)|^p\mathrm{d}x\leqslant\int_0^1 q_-(x)|\phi(x)|^p\mathrm{d}x$. By Lemma 3 and (3.8), we have

$$\begin{aligned}\int_0^1|\phi'(x)|^p\mathrm{d}x&\leqslant M-\int_0^1 q|\phi|^p\mathrm{d}x\leqslant M+\int_0^1 q_-(x)|\phi(x)|^p\mathrm{d}x\\&\leqslant M+\|q_-\|_1(p-1+\frac{\|q_-\|_1}{\theta})\int_0^1|\phi(x)|^p\mathrm{d}x+\theta\int_0^1|\phi'(x)|^p\mathrm{d}x.\end{aligned} \tag{3.9}$$

Setting $\theta=\frac{1}{2}$ in (3.9) and from $\int_0^1|\phi(x)|^p\mathrm{d}x=1$, one can verity that

$$\int_0^1|\phi'(x)|^p\mathrm{d}x\leqslant 2M+2\|q_-\|_1(p-1+2\|q_-\|_1)=Q. \tag{3.10}$$

The proof of Lemma 4 is finished. □

With the help of the above results, we next prove Theorem 1 and Theorem 2.

The proof of Theorem 1. Multiplying both sides of (3.3) by $(x-x_0)\overline{\phi}$ and integrating by parts over the interval [0,1]. By (2.2), we get

$$\begin{aligned}&-\overline{\phi}(x)[\phi'(x)]^{p-1}(x-x_0)|_0^1+\int_0^1[\phi'(x)]^{p-1}\overline{\phi'}(x)(x-x_0)\mathrm{d}x\\&+\int_0^1[\phi'(x)]^{p-1}\overline{\phi}(x)\mathrm{d}x+\int_0^1 q(x)|\phi(x)|^p(x-x_0)\mathrm{d}x\\&=\lambda\int_0^1(x-x_0)w(x)|\phi(x)|^p\mathrm{d}x.\end{aligned} \tag{3.11}$$

that is,

$$\begin{aligned}&x_0[(\cot^*\beta)^{p-1}|\phi(1)|^p-(\cot^*\alpha)^{p-1}|\phi(0)|^p]-(\cot^*\beta)^{p-1}|\phi(1)|^p\\&+\int_0^1(x-x_0)|\phi'(x)|^p\mathrm{d}x+\int_0^1|\phi'(x)|^{p-2}\phi'(x)\overline{\phi}(x)\mathrm{d}x+\int_0^1 q(x)|\phi(x)|^p(x-x_0)\mathrm{d}x\\&=\lambda\int_0^1(x-x_0)w(x)|\phi(x)|^p\mathrm{d}x.\end{aligned} \tag{3.12}$$

Separating the imaginary parts yields

$$Im\lambda\int_0^1(x-x_0)w(x)|\phi(x)|^p\mathrm{d}x = Im\left(\int_0^1|\phi'(x)|^{p-2}\phi'(x)\overline{\phi}(x)\mathrm{d}x\right). \tag{3.13}$$

It follows from (3.5) in Lemma 4, $\int_0^1|\phi(x)|^p\mathrm{d}x=1$ and Cauchy-Schwarz inequality that

$$\begin{aligned}&\left|\int_0^1|\phi'(x)|^{p-2}\phi'(x)\overline{\phi}(x)\mathrm{d}x\right|\leqslant\int_0^1|\phi'(x)|^{p-1}|\overline{\phi}(x)|\mathrm{d}x\\&\leqslant\left(\int_0^1|\phi'(x)|^p\right)^{\frac{p-1}{p}}\left(\int_0^1|\overline{\phi}(x)|^p\right)^{\frac{1}{p}}\leqslant Q^{\frac{p-1}{p}}.\end{aligned} \tag{3.14}$$

Choosing ε as in (2.6), one can verify that

$$\begin{aligned}&\int_0^1(x-x_0)w(x)|\phi(x)|^p\mathrm{d}x\geqslant\int_{[0,1]\setminus\Omega(\varepsilon)}(x-x_0)w(x)|\phi(x)|^p\mathrm{d}x\\&\geqslant\varepsilon\left(\int_0^1|\phi(x)|^p\mathrm{d}x-\int_{\Omega(\varepsilon)}|\phi(x)|^p\mathrm{d}x\right)=\varepsilon\left(1-\int_{\Omega(\varepsilon)}|\phi(x)|^p\mathrm{d}x\right)\\&\geqslant\varepsilon(1-\|\phi\|_\infty^p m(\varepsilon))\geqslant\varepsilon\left(1-\frac{1}{2}\right)=\frac{\varepsilon}{2}.\end{aligned} \tag{3.15}$$

This together with (3.13), (3.14) and (3.15) indicates that

$$\frac{\varepsilon}{2}|Im\lambda|\leqslant|Im\lambda|\int_0^1(x-x_0)w(x)|\phi|^p\mathrm{d}x\leqslant\left|Im\left(\int_0^1|\phi'(x)|^{p-2}\phi'(x)\overline{\phi}(x)\mathrm{d}x\right)\right|\leqslant Q^{\frac{p-1}{p}}. \tag{3.16}$$

Due to $x_0\in(0,1)$ and $\|\phi\|_\infty=1$,

$$\begin{aligned}&|x_0[(\cot^*\beta)^{p-1}|\phi(1)|^p-(\cot^*\alpha)^{p-1}|\phi(0)|^p]-(\cot^*\beta)^{p-1}|\phi(1)|^p|\\&\leqslant x_0[|\cot^*\beta|^{p-1}|\phi(1)|^p+|\cot^*\alpha|^{p-1}|\phi(0)|^p]+|\cot^*\beta|^{p-1}|\phi(1)|^p\\&\leqslant|\cot^*\alpha|^{p-1}+2|\cot^*\beta|^{p-1}.\end{aligned} \tag{3.17}$$

This fact yields that

$$\left|\int_0^1(x-x_0)(|\phi'(x)|^p+q(x)|\phi|^p)\mathrm{d}x\right|\leqslant\int_0^1|\phi'(x)|^p\mathrm{d}x+\|\phi\|_\infty^p\int_0^1|q(x)|\mathrm{d}x\leqslant Q+\|q\|_1 \tag{3.18}$$

by (3.5) in Lemma 4. This together with (3.12), (3.14), (3.15), (3.17), (3.18) yields that

$$\begin{aligned}\frac{\varepsilon}{2}|\lambda|&\leqslant|\lambda|\int_0^1(x-x_0)w(x)|\phi(x)|^p\mathrm{d}x\\&\leqslant|x_0[(\cot^*\beta)^{p-1}|\phi(1)|^p-(\cot^*\alpha)^{p-1}|\phi(0)|^p]\\&\quad-(\cot^*\beta)^{p-1}|\phi(1)|^p+\int_0^1(x-x_0)|\phi'(x)|^p\mathrm{d}x+\int_0^1|\phi'|^{p-2}\phi'\overline{\phi}\mathrm{d}x\\&\quad+\int_0^1q(x)|\phi|^p(x-x_0)\mathrm{d}x|\\&\leqslant|\cot^*\alpha|^{p-1}+2|\cot^*\beta|^{p-1}+Q+\|q\|_1+Q^{\frac{p-1}{p}}.\end{aligned} \tag{3.19}$$

Hence the inequalities in (2.7) hold through (3.16) and (3.19) immediately. □

The proof of Theorem 2. Multiplying both sides of (3.3) by $w\overline{\phi}$ and integrating by parts over the interval [0,1], we get

$$-w\overline{\phi}[\phi']^{p-1}|_0^1+\int_0^1 w|\phi'|^p+\int_0^1 w'|\phi'|^{p-2}\phi'\overline{\phi}+\int_0^1 wq|\phi|^p = \lambda\int_0^1 w^2|\phi|^p. \tag{3.20}$$

It follows from $|w(x)|\leqslant w_0$ and (3.5) in Lemma 4, $B_1\phi=0, B_2\phi=0$ and $w\in W_0^{1,p}$ that

$$\begin{aligned} &w(0)\overline{\phi}(0)[\phi'(0)]^{p-1}-w(1)\overline{\phi}(1)[\phi'(1)]^{p-1}\\ &=(\cot^*\alpha)^{p-1}|\phi(0)|^p w(0)-(\cot^*\beta)^{p-1}|\phi(1)|^p w(1). \end{aligned} \tag{3.21}$$

Therefore,

$$\begin{aligned} &|w(0)\overline{\phi}(0)[\phi'(0)]^{p-1}-w(1)\overline{\phi}(1)[\phi'(1)]^{p-1}|\\ &\leqslant|\cot^*\alpha|^{p-1}|\phi(0)|^p|w(0)|+|\cot^*\beta|^{p-1}|\phi(1)|^p|w(1)|\\ &\leqslant w_0(|\cot^*\alpha|^{p-1}+|\cot^*\beta|^{p-1})=w_0M. \end{aligned} \tag{3.22}$$

$$\begin{aligned} \int_0^1 w|\phi'|^p+\int_0^1 wq|\phi|^p &\leqslant w_0\left(\int_0^1|\phi'|^p+\int_0^1 q|\phi|^p\right)\\ &\leqslant w_0\left(\int_0^1|\phi'|^p+\|\phi\|_\infty^p\int_0^1|q|\right)\\ &\leqslant w_0(Q+\|q\|_1). \end{aligned} \tag{3.23}$$

By using the Cauchy-Schwarz inequality, it can be obtained that

$$\begin{aligned} \int_0^1 w'|\phi'|^{p-2}\phi'\overline{\phi} &\leqslant\|\phi\|_\infty\int_0^1|w'||\phi'|^{p-1}\\ &\leqslant\left(\int_0^1|w'|^p\right)^{\frac{1}{p}}\left(\int_0^1|\phi'|^p\right)^{\frac{p-1}{p}}\\ &\leqslant\|w\|_{1,p}Q^{\frac{p-1}{p}}. \end{aligned} \tag{3.24}$$

Choosing η as in (2.8), one sees that the right hand of (3.20) satisfies

$$\begin{aligned} \int_0^1 w^2|\phi|^p &\geqslant\int_{[0,1]\setminus\Omega(\eta)} w^2|\phi|^p\\ &\geqslant\eta\left(\int_0^1|\phi|^p-\int_{\Omega(\eta)}|\phi|^p\right)\\ &\geqslant\eta(1-\|\phi\|_\infty^p m(\eta))\geqslant\frac{1}{2}\eta. \end{aligned} \tag{3.25}$$

This fact together with (3.22), (3.23), (3.24) and (3.25) yields

$$\begin{aligned}\frac{\eta}{2}|\lambda| &\leqslant |\lambda|\int_0^1 w^2|\phi|^p \\ &\leqslant |w(0)\overline{\phi}(0)[\phi'(0)]^{p-1} - w(1)\overline{\phi}(1)[\phi'(1)]^{p-1} + \int_0^1 w|\phi'|^p \\ &\quad + \int_0^1 w'|\phi'|^{p-2}\phi'\overline{\phi} + \int_0^1 wq|\phi|^p| \\ &\leqslant w_0(M+Q+\|q\|_1) + \|w\|_{1,p}Q^{\frac{p-1}{p}}.\end{aligned} \tag{3.26}$$

Note that

$$Im\lambda \int_0^1 w^2|\phi|^p = Im\left(\int_0^1 w'|\phi'|^{p-2}\phi'\overline{\phi}\right) \tag{3.27}$$

by (3.20). Therefore, (3.24) and (3.25) lead to

$$\frac{\eta}{2}|Im\lambda| \leqslant |Im\lambda|\int_0^1 w^2|\phi|^p \leqslant \left|Im\left(\int_0^1 w'|\phi'|^{p-2}\phi'\overline{\phi}\right)\right| \leqslant \|w\|_{1,p}Q^{\frac{p-1}{p}}. \tag{3.28}$$

As a result, (3.26) and (3.28) yields the inequalities in (2.9). The proof is completed. □

Conflict of Interest. The authors have no conflicts to disclose.

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