Journal of Mathematical Inequalities
Volume 19, Number 2 (2025), 669–682
doi:10.7153/jmi-2025-19-42

APPROXIMATE ADDITIVE (ρ_1,ρ_2)–RANDOM OPERATOR INEQUALITY IN MENGER BANACH SPACES

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(*Communicated by I. Raşa*)

Abstract. In this paper, we attempt to solve an additive (ρ_1,ρ_2)-random operator inequality. We also obtain the Hyers-Ulam stability of such random operator inequality in Menger Banach spaces by using two different approaches.

1. Introduction

In 1940, Ulam [22] formulated the problem of stability for homomorphisms of metric groups which motivated the study of the stability problems of functional equations, and its solutions (for Banach spaces) was published a year later by Hyers [8]. The stability of functional equations has been also known as Hyers-Ulam stability. It was later generalized by Aoki [1], Găvruţă [6] and Rassias [14] for additive mappings and linear mappings, respectively. We refer the interested readers for more information on such problems to the papers (see [2, 3, 9, 10, 12, 13, 15, 17, 18, 23] and references therein).

In 2017, Yun and Shin [21] introduced and solved the following additive (ρ_1,ρ_2)-functional inequality

$$\begin{aligned}\|2f(\frac{x+y}{2})-f(x)-f(y)\| \leqslant \|\rho_1(f(x+y)+f(x-y)-2f(x))\|\\ +\|\rho_2(f(x+y)-f(x)-f(y))\|, \qquad (1.1)\end{aligned}$$

where ρ_1 and ρ_2 are fixed nonzero complex numbers with $\sqrt{2}|\rho_1|+|\rho_1|<1$. They established the Hyers-Ulam stability of the functional inequality (1.1) for mappings $f:X\to Y$, where X is a real or complex normed space, and Y is a complex Banach space.

In this article, let $(\Omega,\mathscr{U},\mu)$ be a probability measure space. Assume that U and V are Menger Banach spaces (briefly, MB-spaces), $(U,\mathscr{B}_U)$ and $(V,\mathscr{B}_V)$ are Borel measurable spaces, and $T:\Omega\times U\to V$ is a random operator. We first study the following

Mathematics subject classification (2020): 47H10, 39B82, 39B52, 46L57.
Keywords and phrases: Fixed point method, Hyers-Ulam stability, Menger Banach spaces, random operator inequality.

additive (ρ_1,ρ_2)-random operator inequality

$$\begin{aligned}&\xi_t^{2T(\omega,\frac{u+v}{2})-T(\omega,u)-T(\omega,v)}\\&\geqslant \mathscr{K}_M\left(\xi_t^{\rho_1(T(\omega,u+v)+T(w,u-v)-2T(\omega,u))},\xi_t^{\rho_2(T(\omega,u+v)-T(\omega,u)-T(\omega,v))}\right),\end{aligned} \tag{1.2}$$

in which ρ_1,ρ_2 are fixed and $\max\{\sqrt{2}|\rho_1|,|\rho_2|\}<1$. And we obtain a random approximation of the additive (ρ_1,ρ_2)-random operator inequality (1.2) in Menger Banach spaces by employing the direct and fixed point methods. The results improve and extend some stability results of the additive (ρ_1,ρ_2)-functional inequality (1.1) in complex Banach spaces.

2. Preliminaries

Following [7, 16, 19, 20], we present some definitions and preliminary results, which will help to investigate the Hyers-Ulam stability in Menger Banach spaces.

Let Δ^+ be the space of all probability distribution mappings, i.e., the space of all mappings $G:\mathbb{R}\cup\{-\infty,+\infty\}\to[0,1]$, writing G_t for $G(t)$, such that G is left continuous and non-decreasing on $\mathbb{R}$. $\mathscr{O}^+\subseteq\Delta^+$ includes all mappings $G\in\Delta^+$ for which $\ell^-G_{+\infty}=1$, where ℓ^-g_x denotes the left limit of the mapping g at the point x, that is, $\ell^-g_x=\lim_{t\to x^-}g_t$. Δ^+ is partially ordered by the usual point-wise ordered of mappings, i.e., $F\leqslant G$ if and only if $F_s\leqslant G_s$ for all $s\in\mathbb{R}$. Note that the function ϑ^u defined by

$$\vartheta_s^u=\begin{cases}0, \text{ if } s\leqslant u,\\ 1, \text{ if } s>u\end{cases}$$

is an element of Δ^+ and ϑ^0 is the maximal element in this space(see [16, 19, 20]).

DEFINITION 2.1. (cf. [7, 19]). A function $\mathscr{K}:[0,1]\times[0,1]\to[0,1]$ is a continuous triangular norm (briefly, a continuous t-norm) if $\mathscr{K}$ satisfies the following conditions:
(a) $\mathscr{K}(\varsigma,\tau)=\mathscr{K}(\tau,\varsigma)$ and $\mathscr{K}(\varsigma,\mathscr{K}(\tau,\nu))=\mathscr{K}(\mathscr{K}(\varsigma,\tau),\nu)$ for all $\varsigma,\tau,\nu\in[0,1]$;
(b) $\mathscr{K}$ is continuous;
(c) $\mathscr{K}(\varsigma,1)=\varsigma$ for all $\varsigma\in[0,1]$;
(d) $\mathscr{K}(\varsigma,\tau)\leqslant\mathscr{K}(\nu,\iota)$ whenever $\varsigma\leqslant\nu$ and $\tau\leqslant\iota$ for all $\varsigma,\tau,\nu,\iota\in[0,1]$.

Typical examples of continuous t-norms are the Lukasiewicz t-norm $\mathscr{K}_L$, where $\mathscr{K}_L(\varsigma,\tau)=\max(\varsigma+\tau-1,0),\forall\varsigma,\tau\in[0,1]$ and the t-norms $\mathscr{K}_P,\mathscr{K}_M,\mathscr{K}_D$, where $\mathscr{K}_P(\varsigma,\tau):=\varsigma\tau$, $\mathscr{K}_M(\varsigma,\tau):=\min(\varsigma,\tau)$,

$$\mathscr{K}_D(\varsigma,\tau):=\begin{cases}\min(\varsigma,\tau), \text{ if } \max(\varsigma,\tau)=1,\\ 0, \qquad\qquad \text{otherwise.}\end{cases}$$

DEFINITION 2.2. (cf. [20]). A Menger normed space (briefly, MN-space) is an ordered tuple $(V,\xi,\mathscr{K})$, where V is a linear space, $\mathscr{K}$ is a continuous t-norm and ξ is a mapping from V to $\mathscr{O}^+$ such that the following conditions hold:
(MN1) $\xi_t^v=\vartheta_t^0$ for all $t>0$ if and only if $v=0$;
(MN2) $\xi_t^{\alpha v}=\xi_{\frac{t}{|\alpha|}}^v$ for all $v\in V$, $t>0$ and $\alpha\in\mathbb{C}$ with $\alpha\neq 0$;
(MN3) $\xi_{t+s}^{u+v}\geqslant\mathscr{K}(\xi_t^u,\xi_s^v)$ for all $u,v\in V$ and $t,s>0$.
A Menger Banach space is a complete Menger normed space.

EXAMPLE 2.1. (cf. [11]). Let $(T,\|\cdot\|)$ be a linear normed space. Define

$$\xi_s^v=\begin{cases}0, & \text{if } s\leqslant 0,\\ \exp\left(-\frac{\|v\|}{s}\right), & \text{if } s>0.\end{cases}$$

Then ξ_s^v is a Menger norm on V and the ordered tuple $(V,\xi,\mathscr{K}_M)$ is an MN-space.

Let $(\Omega,\mathscr{U},\mu)$ be a probability measure space. Assume that $(U,\mathscr{B}_U)$ and $(V,\mathscr{B}_V)$ are Borel measurable spaces, where U and V are MB-spaces. A mapping $T:\Omega\times U\to V$ is said to be a random operator if $\{\omega:T(\omega,u)\in B\}\in\mathscr{U}$ for all $u\in U$ and $B\in\mathscr{B}_V$. Also, T is a random operator if $T(\omega,u)=v(\omega)$ is a V-valued random variable for all $u\in U$. A random operator $T:\Omega\times U\to V$ is called linear if $T(\omega,\alpha u_1+\beta u_2)=\alpha T(\omega,u_1)+\beta T(\omega,u_2)$ for all $u_1,u_2\in U$ and α,β are scalars and *Menger random bounded* (briefly, MR-bounded) if there exists a nonnegative real-valued random variable $M(\omega)$ such that

$$\xi_{M(\omega)t}^{T(\omega,u_1)-T(\omega,u_2)}\geqslant\xi_t^{u_1-u_2}$$

for all $u_1,u_2\in U$ and $t>0$.

3. Stability of additive (ρ_1,ρ_2)-random operator inequality: Direct method

From now on, let $(V,\xi,\mathscr{K}_M)$ be an MB-space. In this section, we investigate the Hyers-Ulam stability of the additive (ρ_1,ρ_2)-random operator inequality (1.2) in MB-spaces by using the direct method. At first, we solve the additive (ρ_1,ρ_2)-random operator inequality (1.2) as follows:

LEMMA 3.1. *Let $T:\Omega\times U\to V$ be a random operator satisfying $T(\omega,0)=0$ and*

$$\xi_t^{2T(\omega,\frac{u+v}{2})-T(\omega,u)-T(\omega,v)}\geqslant\mathscr{K}_M\left(\xi_t^{\rho_1(T(\omega,u+v)+T(\omega,u-v)-2T(\omega,u))},\xi_t^{\rho_2(T(\omega,u+v)-T(\omega,u)-T(\omega,v))}\right)\tag{3.1}$$

for all $u,v\in U$, $\omega\in\Omega$ and $t>0$. Then the random operator $T:\Omega\times U\to V$ is additive.

Proof. Letting $v=0$ in (3.1), we obtain

$$\xi_t^{2T(\omega,\frac{u}{2})-T(\omega,u)} \geqslant \vartheta_t^0$$

for all $u \in U$, $\omega \in \Omega$ and $t>0$. Then, we have

$$T\left(\omega,\frac{u}{2}\right)=\frac{1}{2}T(\omega,u) \tag{3.2}$$

for all $u \in U$ and $\omega \in \Omega$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} \xi_t^{T(\omega,u+v)-T(\omega,u)-T(\omega,v)} &= \xi_t^{2T(\omega,\frac{u+v}{2})-T(\omega,u)-T(\omega,v)} \\ &\geqslant \mathscr{K}_M\left(\xi_{\frac{t}{|\rho_1|}}^{T(\omega,u+v)+T(\omega,u-v)-2T(\omega,u)}, \xi_{\frac{t}{|\rho_2|}}^{T(\omega,u+v)-T(\omega,u)-T(\omega,v)}\right) \end{aligned} \tag{3.3}$$

and so

$$\xi_t^{T(\omega,u+v)-T(\omega,u)-T(\omega,v)} \geqslant \xi_{\frac{t}{|\rho_1|}}^{T(\omega,u+v)+T(\omega,u-v)-2T(\omega,u)} \tag{3.4}$$

for all $u,v \in U$, $\omega \in \Omega$ and $t>0$.

Putting $z=u+v$ and $w=u-v$ in (3.4), we obtain

$$\xi_t^{T(\omega,z+w)+T(\omega,z-w)-2T(\omega,z)} \geqslant \xi_{\frac{t}{2|\rho_1|}}^{T(\omega,z+w)-T(\omega,z)-T(\omega,w)} \tag{3.5}$$

for all $u,v \in U$, $\omega \in \Omega$ and $t>0$. It follows from (3.4) and (3.5) that

$$\xi_t^{T(\omega,u+v)-T(\omega,u)-T(\omega,v)} \geqslant \xi_{\frac{t}{2|\rho_1|^2}}^{T(\omega,u+v)-T(\omega,u)-T(\omega,v)} \tag{3.6}$$

for all $u,v \in U$, $\omega \in \Omega$ and $t>0$. Since $|\rho_1|<\frac{\sqrt{2}}{2}$, we have

$$T(\omega,u+v)=T(\omega,u)+T(\omega,v)$$

for all $u,v \in U$ and $\omega \in \Omega$, which implies that the random operator $T:\Omega\times U \to V$ is additive. This completes the proof of the lemma. □

THEOREM 3.1. *Assume that $\varphi : U^2 \to \mathscr{D}^+$ is a distribution function such that there exists $0<\beta<1$ with*

$$\varphi_{\frac{\beta t}{2}}^{\frac{u}{2},\frac{v}{2}} \geqslant \varphi_t^{u,v} \tag{3.7}$$

and

$$\lim_{n\to\infty} \varphi_{\frac{t}{2^n}}^{\frac{u}{2^n},\frac{v}{2^n}} = \vartheta_t^0 \tag{3.8}$$

for all $u,v\in U$ *and* $t>0$. *Suppose that* $T:\Omega\times U\to V$ *is a random operator satisfying* $T(\omega,0)=0$ *and*

$$\begin{aligned}&\xi_t^{2T(\omega,\frac{u+v}{2})-T(\omega,u)-T(\omega,v)}\\&\geqslant \mathscr{K}_M\left(\xi_t^{\rho_1(T(\omega,u+v)+T(\omega,u-v)-2T(\omega,u))},\xi_t^{\rho_2(T(\omega,u+v)-T(\omega,u)-T(\omega,v))},\varphi_t^{u,v}\right)\end{aligned}\tag{3.9}$$

for all $u,v\in U$, $\omega\in\Omega$ *and* $t>0$. *Then there exists a unique additive random operator* $A:\Omega\times U\to V$ *such that*

$$\xi_t^{T(\omega,u)-A(\omega,u)}\geqslant\varphi_{(1-\beta)t}^{u,\,0}\tag{3.10}$$

for all $u\in U$, $\omega\in\Omega$ *and* $t>0$.

Proof. Letting $v=0$ in (3.9), we get

$$\xi_t^{2T(\omega,\frac{u}{2})-T(\omega,u)}\geqslant\varphi_t^{u,\,0}\tag{3.11}$$

for all $u\in U$, $\omega\in\Omega$ and $t>0$. Replacing u by $\frac{u}{2^\ell}$ in (3.11) and applying (3.7), we have

$$\xi_t^{2^{\ell+1}T(\omega,\frac{u}{2^{\ell+1}})-2^\ell T(\omega,\frac{u}{2^{\ell+1}})}\geqslant\varphi_{\frac{t}{2^\ell}}^{\frac{u}{2^\ell},\,0}\geqslant\varphi_{\frac{t}{\beta^\ell}}^{u,\,0},\tag{3.12}$$

which implies that

$$\xi_{\sum_{k=1}^{\ell}\beta^{k-1}t}^{2^\ell T(\omega,\frac{u}{2^\ell})-T(\omega,u)}\geqslant\varphi_t^{u,\,0}\tag{3.13}$$

for all $u\in U$, $\omega\in\Omega$ and $t>0$.

Replacing u by $\frac{u}{2^m}$ in (3.13), we get

$$\xi_t^{2^{\ell+m}T(\omega,\frac{u}{2^{\ell+m}})-2^m T(\omega,\frac{u}{2^m})}\geqslant\varphi_{\frac{t}{\sum_{k=m+1}^{\ell+m}\beta^{k-1}}}^{u,\,0},\tag{3.14}$$

which tends to ϑ_t^0 when m,ℓ tend to ∞, and so the sequence $\{2^\ell T(\omega,\frac{u}{2^\ell})\}$ is Cauchy in MB-space $(V,\xi,\mathscr{K}_M)$ and converges to a point $A(\omega,u)\in V$. Now, for $\varsigma>0$, we obtain

$$\begin{aligned}\xi_{t+\varsigma}^{T(\omega,u)-A(\omega,u)}&\geqslant\mathscr{K}_M\left(\xi_t^{T(\omega,u)-2^\ell T(\omega,\frac{u}{2^\ell})},\xi_\varsigma^{A(\omega,u)-2^\ell T(\omega,\frac{u}{2^\ell})}\right)\\&\geqslant\mathscr{K}_M\left(\varphi_{\frac{t}{\sum_{k=1}^{\ell}\beta^{k-1}}}^{u,\,0},\xi_\varsigma^{A(\omega,u)-2^\ell T(\omega,\frac{u}{2^\ell})}\right).\end{aligned}\tag{3.15}$$

When ℓ tends to ∞ in (3.15), we obtain

$$\xi_{t+\varsigma}^{T(\omega,u)-A(\omega,u)} \geqslant \varphi_{(1-\beta)t}^{u,0}. \tag{3.16}$$

Since $\varsigma > 0$ is arbitrary in (3.16), we get

$$\xi_t^{T(\omega,u)-A(\omega,u)} \geqslant \varphi_{(1-\beta)t}^{u,0}$$

for all $u \in U$, $\omega \in \Omega$ and $t > 0$.

It follows from (3.9) that

$$\begin{aligned}
\xi_t^{2A(\omega,\frac{u+v}{2})-A(\omega,u)-A(\omega,v)} &= \lim_{m\to\infty} \xi_t^{2^{m+1}T(\omega,\frac{u+v}{2^{m+1}})-2^mT(\omega,\frac{u}{2^m})-2^mT(\omega,\frac{v}{2^m})} \\
&\geqslant \lim_{m\to\infty} \mathscr{K}_M\Big(\xi_t^{\rho_1(2^mT(\omega,\frac{u+v}{2^m})+2^mT(\omega,\frac{u-v}{2^m})-2^{m+1}T(\omega,\frac{u}{2^m}))}, \\
&\qquad \xi_t^{\rho_2(2^mT(\omega,\frac{u+v}{2^m})-2^mT(\omega,\frac{u}{2^m})-2^mT(\omega,\frac{v}{2^m}))}, \varphi_{\frac{t}{2^m}}^{\frac{u}{2^m},\frac{v}{2^m}}\Big) \\
&= \mathscr{K}_M\Big(\xi_t^{\rho_1(A(\omega,u+v)+A(\omega,u-v)-2A(\omega,u))}, \xi_t^{\rho_2(A(\omega,u+v)-A(\omega,u)-A(\omega,v))}\Big)
\end{aligned} \tag{3.17}$$

for all $u,v \in U$, $\omega \in \Omega$ and $t > 0$, since $\lim_{m\to\infty} \varphi_{\frac{t}{2^m}}^{\frac{u}{2^m},\frac{v}{2^m}} = \vartheta_t^0$. So

$$\begin{aligned}
&\xi_t^{2A(\omega,\frac{u+v}{2})-A(\omega,u)-A(\omega,v)} \\
&\geqslant \mathscr{K}_M\Big(\xi_t^{\rho_1(A(\omega,u+v)+A(\omega,u-v)-2A(\omega,u))}, \xi_t^{\rho_2(A(\omega,u+v)-A(\omega,u)-A(\omega,v))}\Big)
\end{aligned}$$

for all $u,v \in U$, $\omega \in \Omega$ and $t > 0$. According to Lemma 3.1, the random operator $A : \Omega \times U \to V$ is additive.

Next, let A' be another additive random operator satisfying (3.10). For arbitrary $u \in U$ and $\omega \in \Omega$, we have that $2^mA(\omega,\frac{u}{2^m}) = A(\omega,u)$ and $2^mA'(\omega,\frac{u}{2^m}) = A'(\omega,u)$ for all natural numbers $m \in \mathbb{N}$. Using (3.10), we get

$$\begin{aligned}
\xi_t^{A(\omega,u)-A'(\omega,u)} &= \lim_{m\to\infty} \xi_t^{2^mA(\omega,\frac{u}{2^m})-2^mA'(\omega,\frac{u}{2^m})} \\
&\geqslant \lim_{m\to\infty} \mathscr{K}_M\Big(\xi_{\frac{t}{2}}^{2^mA(\omega,\frac{u}{2^m})-2^mT(\omega,\frac{u}{2^m})}, \xi_{\frac{t}{2}}^{2^mT(\omega,\frac{u}{2^m})-2^mA'(\omega,\frac{u}{2^m})}\Big) \\
&\geqslant \lim_{m\to\infty} \varphi_{\frac{(1-\beta)}{2\cdot 2^m}t}^{\frac{u}{2^m},0} \geqslant \lim_{m\to\infty} \varphi_{\frac{(1-\beta)}{2\cdot\beta^m}t}^{u,0} \to \vartheta_t^0,
\end{aligned} \tag{3.18}$$

which implies that $A(\omega,u) = A'(\omega,u)$ shows the uniqueness. This completes the proof of the theorem. □

COROLLARY 3.1. *Let $p>1$ and $\theta>0$. Suppose that $T:\Omega\times U\to V$ is a random operator satisfying $T(\omega,0)=0$ and*

$$\xi_t^{2T(\omega,\frac{u+v}{2})-T(\omega,u)-T(\omega,v)} \geqslant \mathscr{K}_M\Bigg(\xi_t^{\rho_1(T(\omega,u+v)+T(\omega,u-v)-2T(\omega,u))}, \xi_t^{\rho_2(T(\omega,u+v)-T(\omega,u)-T(\omega,v))}, \frac{t}{t+\theta(\|u\|^p+\|v\|^p)}\Bigg) \tag{3.19}$$

for all $u,v\in U$, $\omega\in\Omega$ and $t>0$. Then there exists a unique additive random operator $A:\Omega\times U\to V$ such that

$$\xi_t^{T(\omega,u)-A(\omega,u)} \geqslant \frac{(2^p-2)t}{(2^p-2)t+2^p\theta\|u\|^p} \tag{3.20}$$

for all $u\in U$, $\omega\in\Omega$ and $t>0$.

Proof. The proof follows immediately by taking $\varphi_t^{u,v}=\frac{t}{t+\theta(\|u\|^p+\|v\|^p)}$ for all $u,v\in U$, $t>0$ and choosing $\beta=2^{1-p}$ in Theorem 3.1. This completes the proof of the corollary. □

THEOREM 3.2. *Assume that $\varphi:U^2\to\mathscr{O}^+$ is a distribution function such that there exists $0<\beta<1$ with*

$$\varphi_{2\beta t}^{2u,2v} \geqslant \varphi_t^{u,v} \tag{3.21}$$

and

$$\lim_{n\to\infty}\varphi_{2^nt}^{2^nu,\,2^nv}=\vartheta_t^0 \tag{3.22}$$

for all $u,v\in U$ and $t>0$. Suppose that $T:\Omega\times U\to V$ is a random operator satisfying $T(\omega,0)=0$ and (3.9). Then there exists a unique additive random operator $A:\Omega\times U\to V$ such that

$$\xi_t^{T(\omega,u)-A(\omega,u)} \geqslant \varphi_{\frac{(1-\beta)}{\beta}t}^{u,\,0} \tag{3.23}$$

for all $u\in U$, $\omega\in\Omega$ and $t>0$.

Proof. Letting $v=0$ in (3.9), we obtain

$$\xi_t^{\frac{T(\omega,2u)}{2}-T(\omega,u)} \geqslant \varphi_{\frac{t}{\beta}}^{u,\,0} \tag{3.24}$$

for all $u\in U$, $\omega\in\Omega$ and $t>0$. Replacing u by $2^\ell u$ in (3.24) and applying (3.21), we get

$$\xi_t^{\frac{T(\omega,2^{\ell+1}u)}{2^{\ell+1}}-\frac{T(\omega,2^\ell u)}{2^\ell}} \geqslant \varphi_{2^\ell t\cdot\frac{1}{\beta}}^{2^\ell u,\,0} \geqslant \varphi_{\frac{t}{\beta^{\ell+1}}}^{u,\,0}, \tag{3.25}$$

which implies that

$$\xi_{\sum\limits_{k=0}^{\ell-1}\beta^{k+1}t}^{\frac{T(\omega,2^{\ell}u)}{2^{\ell}}-T(\omega,u)} \geqslant \varphi_t^{u,0} \tag{3.26}$$

for all $u \in U$, $\omega \in \Omega$ and $t > 0$.

Replacing u by $2^m u$ in (3.26), we have

$$\xi_t^{\frac{T(\omega,2^{\ell+m}u)}{2^{\ell+m}}-\frac{T(\omega,2^m u)}{2^m}} \geqslant \varphi^{u,0}_{\frac{t}{\sum\limits_{k=m+1}^{\ell+m}\beta^{k+1}}}, \tag{3.27}$$

which tends to ϑ_t^0 when m,ℓ tend to ∞, and so the sequence $\{\frac{T(\omega,2^{\ell}u)}{2^{\ell}}\}$ is Cauchy in MB-space $(V,\xi,\mathscr{K}_M)$ and converges to a point $A(\omega,u) \in V$. Next, for $\varsigma > 0$, we obtain

$$\begin{aligned}\xi_{t+\varsigma}^{T(\omega,u)-A(\omega,u)} &\geqslant \mathscr{K}_M\left(\xi_t^{T(\omega,u)-\frac{T(\omega,2^{\ell}u)}{2^{\ell}}}, \xi_\varsigma^{A(\omega,u)-\frac{T(\omega,2^{\ell}u)}{2^{\ell}}}\right)\\ &\geqslant \mathscr{K}_M\left(\varphi^{u,0}_{\frac{t}{\sum\limits_{k=0}^{\ell-1}\beta^{k+1}}}, \xi_\varsigma^{A(\omega,u)-\frac{T(\omega,2^{\ell}u)}{2^{\ell}}}\right).\end{aligned} \tag{3.28}$$

When ℓ tends to ∞ in (3.28), we obtain

$$\xi_{t+\varsigma}^{T(\omega,u)-A(\omega,u)} \geqslant \varphi^{u,0}_{\frac{(1-\beta)}{\beta}t}. \tag{3.29}$$

Since $\varsigma > 0$ is arbitrary in (3.29), we get

$$\xi_t^{T(\omega,u)-A(\omega,u)} \geqslant \varphi^{u,0}_{\frac{(1-\beta)}{\beta}t}$$

for all $u \in U$, $\omega \in \Omega$ and $t > 0$.

It follows from (3.9) that

$$\begin{aligned}&\xi_t^{2A(\omega,\frac{u+v}{2})-A(\omega,u)-A(\omega,v)} = \lim_{m\to\infty} \xi_t^{\frac{T(\omega,2^{m-1}(u+v))}{2^{m-1}}-\frac{T(\omega,2^m u)}{2^m}-\frac{T(\omega,2^m v)}{2^m}}\\ &\geqslant \lim_{m\to\infty} \mathscr{K}_M\Bigg(\xi_t^{\rho_1(\frac{T(\omega,2^m(u+v))}{2^m}+\frac{T(\omega,2^m(u-v))}{2^m}-\frac{2T(\omega,2^m u)}{2^m}}, \\ &\qquad \xi_t^{\rho_2(\frac{T(\omega,2^m(u+v))}{2^m}-\frac{T(\omega,2^m u)}{2^m}-\frac{T(\omega,2^m v)}{2^m}}, \varphi_{2^m t}^{2^m u,2^m v}\Bigg)\\ &= \mathscr{K}_M\left(\xi_t^{\rho_1(A(\omega,u+v)+A(\omega,u-v)-2A(\omega,u))}, \xi_t^{\rho_2(A(\omega,u+v)-A(\omega,u)-A(\omega,v))}\right)\end{aligned} \tag{3.30}$$

for all $u,v\in U$, $\omega\in\Omega$ and $t>0$, since $\lim_{n\to\infty}\varphi_{2^m t}^{2^m u,2^m v}=\vartheta_t^0$. So

$$\begin{aligned}&\xi_t^{2A(\omega,\frac{u+v}{2})-A(\omega,u)-A(\omega,v)}\\&\geqslant \mathscr{K}_M\left(\xi_t^{\rho_1(A(\omega,u+v)+A(\omega,u-v)-2A(\omega,u))},\xi_t^{\rho_2(A(\omega,u+v)-A(\omega,u)-A(\omega,v))}\right)\end{aligned}$$

for all $u,v\in U$, $\omega\in\Omega$ and $t>0$. According to Lemma 3.1, the random operator $A:\Omega\times U\to V$ is additive.

Next, let A' be another additive random operator satisfying (3.23). For arbitrary $u\in U$ and $\omega\in\Omega$, we have that $\frac{A(\omega,2^m u)}{2^m}=A(\omega,u)$ and $\frac{A'(\omega,2^m u)}{2^m}=A'(\omega,u)$ for all natural numbers $m\in\mathbb{N}$. Using (3.23), we get

$$\begin{aligned}\xi_t^{A(\omega,u)-A'(\omega,u)}&=\lim_{m\to\infty}\xi_t^{\frac{A(\omega,2^m u)}{2^m}-\frac{A'(\omega,2^m u)}{2^m}}\\&\geqslant\lim_{m\to\infty}\mathscr{K}_M\left(\xi_{\frac{t}{2}}^{\frac{A(\omega,2^m u)}{2^m}-\frac{T(\omega,2^m u)}{2^m}},\xi_{\frac{t}{2}}^{\frac{T(\omega,2^m u)}{2^m}-\frac{A'(\omega,2^m u)}{2^m}}\right)\\&\geqslant\lim_{m\to\infty}\varphi_{\frac{2^m(1-\beta)}{2\beta}t}^{2^m u,\,0}\geqslant\lim_{m\to\infty}\varphi_{\frac{(1-\beta)}{2\beta\cdot\beta^m}t}^{u,\,0}\to\vartheta_t^0,\end{aligned}$$

which implies that $A(\omega,u)=A'(\omega,u)$ shows the uniqueness. This completes the proof of the theorem. □

COROLLARY 3.2. *Let $p<1$ and $\theta>0$. Suppose that $T:\Omega\times U\to V$ is a random operator satisfying $T(\omega,0)=0$ and* (3.19)*. Then there exists a unique additive random operator $A:\Omega\times U\to V$ such that*

$$\xi_t^{T(\omega,u)-A(\omega,u)}\geqslant\frac{(2-2^p)t}{(2-2^p)t+2^p\theta\|u\|^p}$$

for all $u\in U$, $\omega\in\Omega$ and $t>0$.

Proof. The proof follows immediately by taking $\varphi_t^{u,\,v}=\frac{t}{t+\theta(\|u\|^p+\|v\|^p)}$ for all $u,v\in U$, $t>0$ and choosing $\beta=2^{p-1}$ in Theorem 3.2. This completes the proof of the corollary. □

4. Stability of additive (ρ_1,ρ_2)-random operator inequality: Fixed point method

In this section, we prove the Hyers-Ulam stability of the additive (ρ_1,ρ_2)-random operator inequality (1.2) in MB-spaces by using the fixed point method. For explicitly later use, we first recall the next lemma is due to Diaz and Margolis [5], which is extensively applied to the stability theory of functional equations and inequalities.

LEMMA 4.1. ([5]). *Let (E,d) be a complete generalized metric space. Further let $J : E \to E$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each fixed element $x \in E$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that
(i) $d(J^n x, J^{n+1} x) < \infty, \quad \forall n \geqslant n_0$;
(ii) *the sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;*
(iii) *y^* is the unique fixed point of J in the set $E^* := \{y \in E \mid d(J^{n_0} x, y) < +\infty\}$;*
(iv) $d(y, y^*) \leqslant \frac{1}{1-L} d(y, Jy), \quad \forall y \in E^*$.

THEOREM 4.1. *Assume that $\varphi : U^2 \to \mathscr{D}^+$ is a distribution function such that there exists $0 < \beta < 1$ with*

$$\varphi_{\frac{t}{2}}^{\frac{u}{2}, \frac{v}{2}} \geqslant \varphi_{\frac{t}{\beta}}^{u,v} \tag{4.1}$$

for all $u, v \in U$ and $t > 0$. Suppose that $T : \Omega \times U \to V$ is a random operator satisfying $T(\omega, 0) = 0$ and (3.9). Then there exists a unique additive random operator $A : \Omega \times U \to V$ such that

$$\xi_t^{T(\omega,u)-A(\omega,u)} \geqslant \varphi_{(1-\beta)t}^{u,\,0} \tag{4.2}$$

for all $u \in U$, $\omega \in \Omega$ and $t > 0$.

Proof. Putting $v = 0$ in (3.9), we get

$$\xi_t^{2T(\omega,\frac{u}{2})-T(\omega,u)} \geqslant \varphi_t^{u,\,0} \tag{4.3}$$

for all $u \in U$, $\omega \in \Omega$ and $t > 0$.

Consider the set $S := \{F | F : \Omega \times U \to V, F(\omega, 0) = 0\}$, and introduce the generalized metric δ on S as follows:

$$\delta(F,K) := \inf\left\{ \mu \in \mathbb{R}_+ \middle| \xi_t^{F(\omega,u)-K(\omega,u)} \geqslant \varphi_{\frac{t}{\mu}}^{u,0}, \forall u \in U, \omega \in \Omega,\, t > 0 \right\}.$$

It is easy to prove that (S, δ) is a complete generalized metric space (cf. [4]). Now we define the mapping $\mathscr{J} : S \to S$ by

$$\mathscr{J} F(\omega, u) := 2F\left(\omega, \frac{u}{2}\right), \text{ for all } F \in S,\ u \in U \text{ and } \omega \in \Omega. \tag{4.4}$$

Let $F, K \in S$ and let $\mu \in \mathbb{R}_+$ be an arbitrary constant with $\delta(F,K) \leqslant \mu$. From the definition of δ, we get

$$\xi_t^{F(\omega,u)-K(\omega,u)} \geqslant \varphi_{\frac{t}{\mu}}^{u,0}$$

for all $u \in U$, $\omega \in \Omega$ and $t > 0$. Therefore, using (4.1), we get

$$\xi_t^{\mathscr{J}F(\omega,u)-\mathscr{J}K(\omega,u)} = \xi_t^{2F(\omega,\frac{u}{2})-2K(\omega,\frac{u}{2})} = \xi_{\frac{t}{2}}^{F(\omega,\frac{u}{2})-K(\omega,\frac{u}{2})} \geqslant \varphi_{\frac{t}{2\mu}}^{\frac{u}{2},0} \geqslant \varphi_{\frac{t}{\beta\mu}}^{u,0} \tag{4.5}$$

for all $u \in U$, $\omega \in \Omega$ and $t > 0$. Hence, it holds that $\delta(\mathscr{J}F, \mathscr{J}K) \leqslant \beta\mu$, that is, $\delta(\mathscr{J}F, \mathscr{J}K) \leqslant \beta\delta(F,K)$ for all $F,K \in S$. Thus, $\mathscr{J}$ is a strictly contractive self-mapping on S with Lipschitz constant $L = \beta < 1$.

Furthermore, by (4.3), we obtain $\delta(T, \mathscr{J}T) \leqslant 1$. Therefore, it follows from Lemma 4.1 that the sequence $\{\mathscr{J}^n T\}$ converges to a fixed point A of $\mathscr{J}$, that is,

$$A : \Omega \times U \to V, \quad \lim_{n\to\infty} 2^n T\left(\omega, \frac{u}{2^n}\right) = A(\omega,u)$$

for all $u \in U$, $\omega \in \Omega$ and

$$A(\omega,u) = 2A\left(\omega, \frac{u}{2}\right) \tag{4.6}$$

for all $u \in U$ and $\omega \in \Omega$. Meanwhile, A is the unique fixed point of $\mathscr{J}$ in the set $S^* = \{F \in S : \delta(T,F) < \infty\}$. Thus there exists a $\mu \in \mathbb{R}_+$ such that

$$\xi_t^{T(\omega,u)-A(\omega,u)} \geqslant \varphi_{\frac{t}{\mu}}^{u,0}$$

for all $u \in U$ and $\omega \in \Omega$. Also,

$$\delta(T,A) \leqslant \frac{1}{1-\beta}\delta(T, \mathscr{J}T) \leqslant \frac{1}{1-\beta}.$$

This means that the inequality (4.2) holds. By the same reasoning as in the proof of Theorem 3.1, we can find the random operator $A : \Omega \times U \to V$ is additive. This completes the proof of the theorem. □

COROLLARY 4.1. *Let $p > 1$ and $\theta > 0$. Suppose that $T : \Omega \times U \to V$ is a random operator satisfying $T(\omega,0) = 0$ and*

$$\xi_t^{2T(\omega,\frac{u+v}{2})-T(\omega,u)-T(\omega,v)} \geqslant \mathscr{K}_M\Bigg(\xi_t^{\rho_1(T(\omega,u+v)+T(\omega,u-v)-2T(\omega,u))}, \xi_t^{\rho_2(T(\omega,u+v)-T(\omega,u)-T(\omega,v))}, \exp\left(-\frac{\theta(\|u\|^p+\|v\|^p)}{t}\right)\Bigg) \tag{4.7}$$

for all $u,v \in U$, $\omega \in \Omega$ and $t > 0$. Then there exists a unique additive random operator $A : \Omega \times U \to V$ such that

$$\xi_t^{T(\omega,u)-A(\omega,u)} \geqslant \exp\left(-\frac{2^p\theta\|u\|^p}{(2^p-2)t}\right) \tag{4.8}$$

for all $u \in U$, $\omega \in \Omega$ and $t > 0$.

Proof. The proof follows immediately by taking $\varphi_t^{u,v} = \exp(-\frac{\theta(\|u\|^p+\|v\|^p)}{t})$ for all $u,v \in U$, $t > 0$ and choosing $\beta = 2^{1-p}$ in Theorem 4.1. This completes the proof of the corollary. $\square$

THEOREM 4.2. *Assume that $\varphi : U^2 \to \mathscr{O}^+$ is a distribution function such that there exists $0 < \beta < 1$ with*

$$\varphi_{2t}^{2u,2v} \geqslant \varphi_{\frac{t}{\beta}}^{u,v} \tag{4.9}$$

for all $u,v \in U$ and $t > 0$. Suppose that $T : \Omega \times U \to V$ is a random operator satisfying $T(\omega,0) = 0$ and (3.9). Then there exists a unique additive random operator $A : \Omega \times U \to V$ such that

$$\xi_t^{T(\omega,u)-A(\omega,u)} \geqslant \varphi_{\frac{1-\beta}{\beta}t}^{u,0} \tag{4.10}$$

for all $u \in U$, $\omega \in \Omega$ and $t > 0$.

Proof. According to (3.9) and (4.9), we obtain

$$\xi_t^{\frac{T(\omega,2u)}{2}-T(\omega,u)} \geqslant \varphi_{\frac{t}{\beta}}^{u,0} \tag{4.11}$$

for all $u \in U$, $\omega \in \Omega$ and $t > 0$. And we introduce the definitions for S and δ as in the proof of Theorem 4.1 such that (S,δ) becomes complete generalized metric space. Now we consider the mapping $\mathscr{J} : S \to S$ defined by

$$\mathscr{J}F(\omega,u) := \frac{F(\omega,2u)}{2}, \text{ for all } F \in S,\ u \in U \text{ and } \omega \in \Omega.$$

Therefore, using (4.9), we get

$$\xi_t^{\mathscr{J}F(\omega,u)-\mathscr{J}K(\omega,u)} = \xi_t^{\frac{F(\omega,2u)}{2}-\frac{K(\omega,2u)}{2}} = \xi_{2t}^{F(\omega,2u)-K(\omega,2u)} \geqslant \varphi_{\frac{2t}{\mu}}^{2u,0} \geqslant \varphi_{\frac{t}{\beta\mu}}^{u,0}$$

for all $u \in U$, $\omega \in \Omega$ and $t > 0$. Hence, it holds that $\delta(\mathscr{J}F,\mathscr{J}K) \leqslant \beta\mu$, that is, $\delta(\mathscr{J}F,\mathscr{J}K) \leqslant \beta\delta(F,K)$ for all $F,K \in S$. Furthermore, by (4.11), we obtain $\delta(T,\mathscr{J}T) \leqslant \beta$.

The remaining assertion is similar to the corresponding part of Theorem 4.1. This completes the proof of the theorem. $\square$

COROLLARY 4.2. *Let $p < 1$ and $\theta > 0$. Suppose that $T : \Omega \times U \to V$ is a random operator satisfying $T(\omega,0) = 0$ and (4.7). Then there exists a unique additive random operator $A : \Omega \times U \to V$ such that*

$$\xi_t^{T(\omega,u)-A(\omega,u)} \geqslant \exp\left(-\frac{2^p\theta\|u\|^p}{(2-2^p)t}\right)$$

for all $u \in U$, $\omega \in \Omega$ and $t > 0$.

Proof. The proof follows immediately by taking $\varphi_t^{u,v} = \exp(-\frac{\theta(\|u\|^p+\|v\|^p)}{t})$ for all $u,v \in U$, $t > 0$ and choosing $\beta = 2^{p-1}$ in Theorem 4.2. This completes the proof of the corollary. □

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(Received October 29, 2024)

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