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# COMPLEMENTARY AND REFINED INEQUALITIES FOR THE CAUCHY–SCHWARZ INEQUALITY INVOLVING MEANS

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*Abstract.* Let $(\mathscr{H},\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The well-known Cauchy-Schwarz inequality for the inner product asserts that $\left|\langle x,y\rangle\right| \leqslant \|x\|\|y\|$ for all $x,y\in\mathscr{H}$. In this paper, by using the consent of means, we obtain a refinement of the Cauchy-Schwarz inequality. Among other results, it is shown that, if $x,y\in\mathscr{H}$, $\mu,\nu\in[0,1]$, and $p,q>0$ with $\frac{1}{p}+\frac{1}{q}=1$, then

$$\left|\langle x,y\rangle\right| \leqslant \frac{1}{p}\left|\langle x,y\rangle\right|^{1-\mu}\|x\|^{\mu}\|y\|^{\mu}+\frac{1}{q}\left|\langle x,y\rangle\right|^{\nu}\|x\|^{(1-\nu)}\|y\|^{1-\nu} \leqslant \|x\|\|y\|.$$

Moreover, we present a refinement of the classical Cauchy-Schwarz inequality. Furthermore, we obtain some numerical radius inequalities for the product of operators, which are interpolations of some earlier inequalities. For instance, if $T$ is an operator on a Hilbert space $\mathscr{H}$, then we have

$$\begin{aligned} w^{2r}(T) &\leqslant \frac{1}{2^{\mu+1}p}w^{r(1-\mu)}(T^2)\left\||T|^{2r}+|T^*|^{2r}\right\|^{\mu} \\ &\qquad + \frac{1}{2^{2-\nu}p}w^{r\nu}(T^2)\left\||T|^{2r}+|T^*|^{2r}\right\|^{1-\nu}+\frac{1}{4}\left\||T|^{2r}+|T^*|^{2r}\right\| \\ &\leqslant \frac{1}{2}\left\||T|^{2r}+|T^*|^{2r}\right\| \end{aligned}$$

for $r\geqslant 1$, $\mu,\nu\in[0,1]$, and $p,q>0$ with $\frac{1}{p}+\frac{1}{q}=1$.

## 1. Introduction

A binary function $\sigma$ on $[0,+\infty)$ is called a mean if the following conditions are satisfied:

(1) If $a\leqslant b$, then $a\leqslant a\,\sigma\, b\leqslant b$;

(2) $a\leqslant c$ and $b\leqslant d$ imply $a\,\sigma\, b\leqslant c\,\sigma\, d$;

(3) $\sigma$ is continuous in both variables;

(4) $t(a\,\sigma b)\leqslant(ta)\sigma(tb)$ $(t>0)$.

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For instance, if $\mu \in (0,1)$, then the weighted geometric mean is $a \sharp_\mu b = a^{1-\mu} b^\mu$, the weighted arithmetic mean is $a \nabla_\mu b = (1-\mu)a + \mu b$, and the weighted harmonic mean is $a \,!_\mu\, b = \left((1-\mu)a^{-1} + \mu b^{-1}\right)^{-1}$, with

$$a \,!_\mu\, b \leqslant a \sharp_\mu b \leqslant a \nabla_\mu b \qquad \text{for all } a,b > 0.$$

A mean $\sigma$ is symmetric if $a \sigma b = b \sigma a$ for all positive numbers $a,b$. For more information about means, see [8] and references therein.

For a symmetric mean $\sigma$, a parametrized mean $\sigma_t$, $0 \leqslant t \leqslant 1$, is called an interpolational path for $\sigma$ if it satisfies

(1) $a \sigma_0 b = a$, $a \sigma_{1/2} b = a \sigma b$, and $a \sigma_1 b = b$;

(2) $(a \sigma_p b) \sigma (a \sigma_q b) = a \sigma_{\frac{p+q}{2}} b$ for all $p,q \in [0,1]$;

(3) The map $t \in [0,1] \mapsto a \sigma_t b$ is continuous for each $a$ and $b$;

(4) $\sigma_t$ is increasing in each of its components for $t \in [0,1]$.

It is easy to see that the set of all $r \in [0,1]$ satisfying

$$(a\sigma_p b)\sigma_r(a\sigma_q b) = a\sigma_{rp+(1-r)q} b$$

for all $p,q$ is a convex subset of $[0,1]$ including $0$ and $1$. For instance, the power means

$$a m_r b = \left(\frac{a^r + b^r}{2}\right)^{\frac{1}{r}} \qquad (r \in [-1,1])$$

are some typical interpolational means. Their interpolational paths are

$$a m_{r,t} b = ((1-t)a^r + t b^r)^{\frac{1}{r}} \qquad (t \in [0,1] \text{ and } r \in [-1,1]).$$

In particular, $a m_{1,t} b = a \nabla_t b$ and $a m_{-1,t} b = a !_t b$. By an easy calculation, we have

$$\lim_{r \to 0} ((1-t)a^r + t b^r)^{\frac{1}{r}} = a^{1-t} b^t \quad (a,b > 0),$$

and then $a m_{0,t} b := \lim_{r\to 0} \left(a m_{r,t} b\right) = a \sharp_t b$ for positive numbers $a$ and $b$.

The classical Cauchy–Schwarz inequality asserts that

$$\left(\sum_{j=1}^n x_j y_j\right) \leqslant \left(\sum_{j=1}^n x_j^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^n y_j^2\right)^{\frac{1}{2}},$$

where $x_j, y_j$ $(1 \leqslant j \leqslant n)$ are positive real numbers. One of the most basic, yet useful inequalities, is the Cauchy-Schwarz inequality for the inner product as follows

$$\left|\langle x,y\rangle\right| \leqslant \|x\|\|y\| \quad \text{for all } x,y \in \mathscr{H}. \tag{1}$$

Buzano [5] showed an extension of the Cauchy-Schwarz inequality as follows

$$\left|\langle x,e\rangle\langle e,y\rangle\right| \leqslant \frac{1}{2}\left(|\langle x,y\rangle| + \|x\|\|y\|\right) \leqslant \|x\|\|y\| \quad \text{for all } x,y,e \in \mathscr{H}, \tag{2}$$

where $\|e\| = 1$. Recently, the author of [2] proved a refinement of the Cauchy-Schwarz inequality as follows

$$\left|\langle x,y\rangle\right| \leqslant \beta\left|\langle x,y\rangle\right|^{\frac{1}{2}}\|x\|^{\frac{1}{2}}\|y\|^{\frac{1}{2}} + (1-\beta)\|x\|\|y\| \leqslant \|x\|\|y\| \tag{3}$$

for all $x,y \in \mathscr{H}$ and $0 \leqslant \beta \leqslant 1$. During the last decades, several generalizations, refinements, and applications of the Cauchy-Schwarz inequality in various settings have been given, and some results related to integral inequalities are presented. For other results and refinements of the Cauchy-Schwarz inequality, the reader can consult the works [2, 5] and references therein.

Let $(\mathscr{H},\langle\cdot,\cdot\rangle)$ be a complex Hilbert space, and let $\mathscr{B}(\mathscr{H})$ be the $C^*$-algebra of all bounded linear operators defined on $\mathscr{H}$. For a bounded linear operator $T$ on a Hilbert space $\mathscr{H}$, the numerical range $W(T)$ is the image of the unit sphere of $\mathscr{H}$ under the quadratic form $x \to \langle Tx,x\rangle$ associated with the operator $T$. More precisely, $W(T) = \{\langle Tx,x\rangle : x \in \mathscr{H}, \|x\| = 1\}$. Moreover, the numerical radius is defined by

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\} = \sup_{\|x\|=1} |\langle Tx,x\rangle|.$$

It is well known that $w(\cdot)$ defines a norm on $\mathscr{B}(\mathscr{H})$, which is equivalent to the usual operator norm of the form

$$\frac{1}{2}\|T\| \leqslant w(T) \leqslant \|T\|. \tag{4}$$

For more facts about the numerical radius, we refer the reader to [1, 4, 9, 10, 11, 13] and references therein.

Kittaneh [10] proved a refinement of the second inequality in (4) as follows

$$w(T) \leqslant \frac{1}{2}\|\,|T| + |T^*|\,\| \qquad \text{for all } T \in \mathscr{B}(\mathscr{H}). \tag{5}$$

Another refinement and improvement of the second inequality in (4) has been given by the same author by showing that

$$w^2(T) \leqslant \frac{1}{2}\|\,|T|^2 + |T^*|^2\| \qquad \text{for all } T \in \mathscr{B}(\mathscr{H}), \tag{6}$$

which has been refined further in [12] as follows

$$w^2(T) \leqslant \frac{1}{6}\|\,|T|^2 + |T^*|^2\| + \frac{1}{3}w(T)\|\,|T| + |T^*|\,\| \qquad \text{for all } T \in \mathscr{B}(\mathscr{H}). \tag{7}$$

The inequality (7), was improved by Alomari [2] as follows:

$$\begin{aligned} w^2(T) &\leqslant \frac{1}{12}\|\,|T| + |T^*|\,\|^2 + \frac{1}{3}w(T)\|\,|T| + |T^*|\,\| \\ &\leqslant \frac{1}{6}\|\,|T|^2 + |T^*|^2\| + \frac{1}{3}w(T)\|\,|T| + |T^*|\,\| \qquad \text{for all } T \in \mathscr{B}(\mathscr{H}). \end{aligned}$$

Dragomir [6] obtained a numerical radius upper bound for the product of operators, which asserts that for $r \geqslant 1$,

$$w^r(S^*T) \leqslant \frac{1}{2}\left\| |T|^{2r} + |S|^{2r} \right\| \qquad \text{for all } S,T \in \mathscr{B}(\mathscr{H}). \tag{8}$$

The above inequality has been improved by the same author for $r = 2$ by showing that

$$w^2(S^*T) \leqslant \frac{1}{6}\| |T|^4 + |S|^4 \| + \frac{1}{3} w(S^*T) \| |T|^2 + |S|^2 \| \qquad \text{for all } S,T \in \mathscr{B}(\mathscr{H}). \tag{9}$$

Recently, Alomari [2] presented an extension of the inequalities (8) and (9) of the form

$$\begin{aligned} w^{2r}(S^*T) &\leqslant \frac{1}{4}\beta \left\| |T|^{2r} + |S|^{2r} \right\|^2 + \frac{1}{2}(1-\beta) w^r(S^*T) \left\| |T|^{2r} + |S|^{2r} \right\| \\ &\leqslant \frac{1}{2}\beta \left\| |T|^{4r} + |S|^{4r} \right\| + \frac{1}{2}(1-\beta) w^r(S^*T) \left\| |T|^{2r} + |S|^{2r} \right\| \\ &\leqslant \frac{1}{2}\left\| |T|^{4r} + |S|^{4r} \right\| \end{aligned} \tag{10}$$

for all $S,T \in \mathscr{B}(\mathscr{H})$, $r \geqslant 1$ and $0 \leqslant \beta \leqslant 1$.

In the present paper, we establish new numerical radius upper bounds for Hilbert space operators by providing a new generalization of the refined celebrated Cauchy-Schwarz inequality. In particular, our results generalize and refine the inequalities (7), (8), (9), and (10). Moreover, the obtained upper bounds have been compared with the previously known bounds to demonstrate their reliability.

## 2. Main results

In this section, we establish a generalization of the refined celebrated Cauchy-Schwarz inequality involving means. First, we obtain the following lemma involving means.

LEMMA 1. *Let $\sigma, \tau, \rho$ be three arbitrary means on $[0, +\infty)$. Then*

$$a \leqslant (a\sigma b)\rho(a\tau b) \leqslant b \tag{11}$$

*for all positive real numbers $a, b$ such that $a \leqslant b$.*

*Proof.* Using the definition of means, we get

$$a \leqslant a\sigma b \leqslant b \quad \text{and} \quad a \leqslant a\tau b \leqslant b \tag{12}$$

for all positive real numbers $a, b$ such that $a \leqslant b$. Now, if $a\sigma b \leqslant a\tau b$, then

$$(a \leqslant)\; a\sigma b \leqslant (a\sigma b)\rho(a\tau b) \leqslant a\tau b\; (\leqslant b) \tag{13}$$

and if $a\tau b \leqslant a\sigma b$, then

$$(a \leqslant)\; a\tau b \leqslant (a\sigma b)\rho(a\tau b) \leqslant a\sigma b\; (\leqslant b). \tag{14}$$

Combining the inequalities (12), (13) and (14), gives

$$a \leqslant (a\,\sigma\, b)\rho(a\,\tau\, b) \leqslant b,$$

as required. □

Now, we are in a position to present our first result that is a generalization of the refined Cauchy-Schwarz inequality.

THEOREM 1. *Let $\sigma, \tau, \rho$ be three arbitrary means on $[0,+\infty)$. Then*

$$\left|\langle x,y\rangle\right| \leqslant \left(\left|\langle x,y\rangle\right| \sigma \, \|x\|\|y\|\right) \rho \left(\left|\langle x,y\rangle\right| \tau \, \|x\|\|y\|\right) \leqslant \|x\|\|y\| \tag{15}$$

*for all $x,y \in \mathscr{H}$.*

*Proof.* Assume that $x,y \in \mathscr{H}$. The desired inequalities follow from the Cauchy-Schwarz inequality and replacing $a$ by $\left|\langle x,y\rangle\right|$ and $b$ by $\|x\|\|y\|$, respectively, in Lemma 1. □

COROLLARY 1. *Let $x,y \in \mathscr{H}$. Then*

$$\left|\langle x,y\rangle\right| \leqslant \frac{1}{p}\left|\langle x,y\rangle\right|^{1-\mu}\|x\|^{\mu}\|y\|^{\mu} + \frac{1}{q}\left|\langle x,y\rangle\right|^{\nu}\|x\|^{1-\nu}\|y\|^{1-\nu} \leqslant \|x\|\|y\| \tag{16}$$

*for all $\mu, \nu \in [0,1]$ and $p,q > 0$ with $\frac{1}{p}+\frac{1}{q}=1$.*

*Proof.* Assume that $\mu,\nu \in [0,1]$ and $p,q>0$ with $\frac{1}{p}+\frac{1}{q}=1$. If we take $\sigma = \sharp_{\mu}$, $\tau = \sharp_{1-\nu}$, and $\rho = \nabla_{\frac{1}{q}}$ in the inequalities (15), then we get the desired result. □

REMARK 1. Note that the inequalities (15) and (16) lead to a refinement of the Cauchy-Schwarz inequality (1) and a generalization of the inequality (3). To see this, put $\mu = \frac{1}{2}$, $\nu = 0$, and $\frac{1}{p} = \beta$ in (16). Then we have

$$\left|\langle x,y\rangle\right| \leqslant \beta\left|\langle x,y\rangle\right|^{\frac{1}{2}}\|x\|^{\frac{1}{2}}\|y\|^{\frac{1}{2}} + (1-\beta)\|x\|\|y\| \leqslant \|x\|\|y\|$$

for all $x,y \in \mathscr{H}$ and $0 \leqslant \beta \leqslant 1$.

REMARK 2. Utilizing the inequality (15) for two vectors $x = (x_1,\cdots,x_n)$ and $y = (y_1,\cdots,y_n)$, where $x_i, y_j$, $(j=1,\cdots,n)$ are positive numbers, $\sigma = \sharp_s$, $\rho = \nabla_{\frac{1}{p}}$, and $\tau = \sharp_t$, we have

$$\begin{aligned}
\sum_{j=1}^{n} x_j y_j \leqslant \frac{1}{p}&\left[\left(\sum_{j=1}^{n} x_j y_j\right)^{1-s}\left(\sum_{j=1}^{n} x_j^2\right)^{\frac{s}{2}}\left(\sum_{j=1}^{n} y_j^2\right)^{\frac{s}{2}}\right] \\
&+ \frac{1}{q}\left[\left(\sum_{j=1}^{n} x_j y_j\right)^{1-t}\left(\sum_{j=1}^{n} x_j^2\right)^{\frac{t}{2}}\left(\sum_{j=1}^{n} y_j^2\right)^{\frac{t}{2}}\right] \\
&\leqslant \left(\sum_{j=1}^{n} x_j^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n} y_j^2\right)^{\frac{1}{2}}
\end{aligned}$$

for all $0 \leqslant s,t \leqslant 1$ and $p,q>0$ with $\frac{1}{p}+\frac{1}{q}=1$, which is a refinement of the classical Cauchy-Schwarz inequality.

REMARK 3. Note that if $f$ is an increasing function on $[0,+\infty)$, then the inequality (15) can be extended to the following result

$$\begin{aligned} f\left(\left|\langle x,y\rangle\right|\right) &\leqslant \left[f\left(\left|\langle x,y\rangle\right|\right)\,\sigma\, f\left(\|x\|\|y\|\right)\right]\rho\left[f\left(\left|\langle x,y\rangle\right|\right)\,\tau\, f\left(\|x\|\|y\|\right)\right] \\ &\leqslant f\left(\|x\|\|y\|\right) \end{aligned} \tag{17}$$

for all three arbitrary means $\sigma,\tau,\rho$ and $x,y\in\mathscr{H}$. To see this, observe that from $f(|\langle x,y\rangle|)\leqslant f(\|x\|\|y\|)$ and Lemma 1, by replacing $a$ by $f(|\langle x,y\rangle|)$ and $b$ by $f(\|x\|\|y\|)$, we get the above inequality.

COROLLARY 2. *Let $x,y\in\mathscr{H}$ be such that $\langle x,y\rangle\neq 0$. Then*

$$\left|\langle x,y\rangle\right| \leqslant \left((1-\alpha)|\langle x,y\rangle|^r+\alpha\|x\|^r\|y\|^r\right)^{\frac{1}{r}}\sharp_\nu\left((1-\beta)|\langle x,y\rangle|^s+\beta\|x\|^s\|y\|^s\right)^{\frac{1}{s}}\leqslant\|x\|\|y\| \tag{18}$$

*for all $\alpha,\beta,\nu\in[0,1]$ and $s,r\in[-1,1]$.*

*In particular,*

$$\left|\langle x,y\rangle\right|\leqslant\left(\frac{\left|\langle x,y\rangle\right|^r+\|x\|^r\|y\|^r}{2}\right)^{\frac{1-\nu}{r}}\left(\frac{\left|\langle x,y\rangle\right|^s+\|x\|^s\|y\|^s}{2}\right)^{\frac{\nu}{s}}\leqslant\|x\|\|y\| \tag{19}$$

*for all $\nu\in[0,1]$ and $s,r\in[-1,1]$.*

*Proof.* Assume that $\alpha,\beta,\nu\in[0,1]$ and $s,r\in[-1,1]$. Applying $\sigma=m_{r,\alpha}$, $\rho=\sharp_\nu$, and $\tau=m_{s,\beta}$ in Theorem 1, we have the first inequalities. For the second result, put $\alpha=\beta=\frac{1}{2}$ in the first result. $\square$

EXAMPLE 1. Assume that $\mathbf{L}^2(\mathbb{R})$ is the Hilbert space with the inner product defined by $\langle f,g\rangle=\int_{\mathbb{R}}f(x)g(x)\,dx$ for two functions $f,g\in\mathbf{L}^2(\mathbb{R})$. Utilizing the inequality (19) for two positive functions $f,g\in\mathbf{L}(\mathbb{R})$ with $\langle f,g\rangle\neq 0$, we have a refinement of the Cauchy-Schwarz inequality for integrals as follows

$$\begin{aligned} \int_{\mathbb{R}}f(x)g(x)\,dx &\leqslant \left(\frac{\left(\int_{\mathbb{R}}f(x)g(x)\,dx\right)^r+\left(\int_{\mathbb{R}}f^2(x)\,dx\right)^{\frac{r}{2}}\left(\int_{\mathbb{R}}g^2(x)\,dx\right)^{\frac{r}{2}}}{2}\right)^{\frac{1-\nu}{r}} \\ &\qquad\times\left(\frac{\left(\int_{\mathbb{R}}f(x)g(x)\,dx\right)^s+\left(\int_{\mathbb{R}}f^2(x)\,dx\right)^{\frac{s}{2}}\left(\int_{\mathbb{R}}g^2(x)\,dx\right)^{\frac{s}{2}}}{2}\right)^{\frac{\nu}{s}} \\ &\leqslant\left(\int_{\mathbb{R}}f^2(x)\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}g^2(x)\,dx\right)^{\frac{1}{2}} \end{aligned}$$

for all $s,r,\nu\in[0,1]$.

In the next results, we present some refinements of the Buzano inequality.

THEOREM 2. *Let $\sigma,\tau,\rho$ be three arbitrary means on $[0,+\infty)$ and $f$ an increasing convex function on $[0,+\infty)$. Then*

$$\begin{aligned}
&f\left(\left|\langle x,e\rangle\langle e,y\rangle\right|\right)\\
&\leqslant f\left(\frac{1}{2}\left(|\langle x,y\rangle|+\|x\|\|y\|\right)\right)\\
&\leqslant \frac{1}{2}\Big(\left[f\left(\left|\langle x,y\rangle\right|\right)\sigma f\left(\|x\|\|y\|\right)\right]\rho\left[f\left(\left|\langle x,y\rangle\right|\right)\tau f\left(\|x\|\|y\|\right)\right]+f\left(\|x\|\|y\|\right)\Big)\\
&\leqslant f\left(\|x\|\|y\|\right)
\end{aligned}$$

*for all three arbitrary means $\sigma,\tau,\rho$ and $x,y,e\in\mathscr{H}$ with $\|e\|=1$.*

*Proof.* Assume that $f$ is an increasing convex function on $[0,+\infty)$. Then

$$\begin{aligned}
&f\left(\left|\langle x,e\rangle\langle e,y\rangle\right|\right)\\
&\leqslant f\left(\frac{1}{2}\left(|\langle x,y\rangle|+\|x\|\|y\|\right)\right)\\
&\qquad\qquad \text{(by the inequality (2))}\\
&\leqslant \frac{1}{2}\left[f\left(|\langle x,y\rangle|\right)+f\left(\|x\|\|y\|\right)\right]\\
&\qquad\qquad \text{(by the convexity of } f\text{)}\\
&\leqslant \frac{1}{2}\Big(\left[f\left(\left|\langle x,y\rangle\right|\right)\sigma f\left(\|x\|\|y\|\right)\right]\rho\left[f\left(\left|\langle x,y\rangle\right|\right)\tau f\left(\|x\|\|y\|\right)\right]+f\left(\|x\|\|y\|\right)\Big)\\
&\qquad\qquad \text{(by the inequality (17) )}\\
&\leqslant f\left(\|x\|\|y\|\right) \qquad \text{(by the inequality (17))}
\end{aligned}$$

for all three arbitrary means $\sigma,\tau,\rho$ and $x,y,e\in\mathscr{H}$ with $\|e\|=1$. This completes the proof. □

As a special case of Theorem 2 for $f(t)=t^r$ $(r\geqslant 1)$, we have the next result.

COROLLARY 3. *Let $\sigma,\tau,\rho$ be three arbitrary means on $[0,+\infty)$ and $r\geqslant 1$. Then*

$$\begin{aligned}
&\left|\langle x,e\rangle\langle e,y\rangle\right|^r\\
&\leqslant \frac{1}{2^r}\left(|\langle x,y\rangle|+\|x\|\|y\|\right)^r\\
&\leqslant \frac{1}{2}\Big(\left(\left|\langle x,y\rangle\right|^r\sigma\|x\|^r\|y\|^r\right)\rho\left(\left|\langle x,y\rangle\right|^r\tau\|x\|^r\|y\|^r\right)+\|x\|^r\|y\|^r\Big)\\
&\leqslant \|x\|^r\|y\|^r,
\end{aligned} \tag{20}$$

*where $x,y,e\in\mathscr{H}$ with $\|e\|=1$.*

COROLLARY 4. *Let* $x,y,e \in \mathscr{H}$ *with* $\|e\| = 1$ *and* $r \geqslant 1$. *Then*

$$\begin{aligned}
&\left|\langle x,e\rangle\langle e,y\rangle\right|^r \\
&\leqslant \frac{1}{2^r}\left(|\langle x,y\rangle| + \|x\|\|y\|\right)^r \\
&\leqslant \frac{1}{2}\left(\frac{1}{p}\left|\langle x,y\rangle\right|^{r(1-\mu)}\|x\|^{r\mu}\|y\|^{r\mu} + \frac{1}{q}\left|\langle x,y\rangle\right|^{r\nu}\|x\|^{r(1-\nu)}\|y\|^{r(1-\nu)} + \|x\|^r\|y\|^r\right) \\
&\leqslant \|x\|^r\|y\|^r
\end{aligned} \tag{21}$$

*for all* $\mu,\nu \in [0,1]$ *and* $p,q > 0$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

*Proof.* Assume that $\mu,\nu \in [0,1]$. If we take $\sigma = \sharp_\mu$, $\tau = \sharp_{1-\nu}$, and $\rho = \nabla_{\frac{1}{q}}$ in the inequalities (3), then we get the desired result. □

LEMMA 2. *Let* $\sigma,\tau,\rho$ *be three arbitrary means on* $[0,+\infty)$. *Then*

$$a \leqslant \sqrt{a(a\,\sigma\,b)}\rho\sqrt{b(a\,\tau\,b)} \leqslant b \tag{22}$$

*for all positive real numbers* $a,b$ *such that* $a \leqslant b$.

*Proof.* Using the definition of means, we get

$$a \leqslant a\,\tau\,b \leqslant b \quad \text{and} \quad a \leqslant a\,\sigma\,b \leqslant b \tag{23}$$

for all positive real numbers $a,b$ such that $a \leqslant b$. By multiplying by $a$ in the first inequalities and by $b$ in the second inequalities in (23), respectively, we get

$$(a^2 \leqslant) \quad ab \leqslant b(a\,\tau\,b) \leqslant b^2 \quad \text{and} \quad a^2 \leqslant a(a\,\sigma\,b) \leqslant ab \quad (\leqslant b^2). \tag{24}$$

Hence,

$$a \leqslant \sqrt{b(a\,\tau\,b)} \leqslant b \quad \text{and} \quad a \leqslant \sqrt{a(a\,\sigma\,b)} \leqslant b. \tag{25}$$

Therefore, for the mean $\rho$, we have

$$a \leqslant \sqrt{a(a\,\sigma\,b)}\rho\sqrt{b(a\,\tau\,b)} \leqslant b,$$

as required. □

Applying Lemma 2, we have another refinement of the Cauchy-Schwarz inequality.

THEOREM 3. *Let* $\sigma,\tau,\rho$ *be three arbitrary means on* $[0,+\infty)$. *Then*

$$\begin{aligned}
\left|\langle x,y\rangle\right| &\leqslant \sqrt{\left|\langle x,y\rangle\right|\left(\left|\langle x,y\rangle\right|\sigma\,\|x\|\|y\|\right)}\rho\sqrt{\|x\|\|y\|\left(\left|\langle x,y\rangle\right|\tau\,\|x\|\|y\|\right)} \\
&\leqslant \|x\|\|y\|.
\end{aligned}$$

*In particular, if* $x,y \in \mathscr{H}$ *, then*

$$\left|\langle x,y\rangle\right| \leqslant \frac{1}{p}\left|\langle x,y\rangle\right|^{1-\frac{\mu}{2}}\|x\|^{\frac{\mu}{2}}\|y\|^{\frac{\mu}{2}}+\frac{1}{q}\left|\langle x,y\rangle\right|^{\frac{\nu}{2}}\|x\|^{1-\frac{\nu}{2}}\|y\|^{1-\frac{\nu}{2}} \leqslant \|x\|\|y\| \tag{26}$$

*for all* $\mu,\nu \in [0,1]$ *and* $p,q>0$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.* Assume that $\mu,\nu \in [0,1]$ and $p,q>0$ with $\frac{1}{p}+\frac{1}{q}=1$. By replacing $a$ by $\left|\langle x,y\rangle\right|$ and $b$ by $\|x\|\|y\|$, and taking $\sigma=\sharp_{\mu}$, $\tau=\sharp_{1-\nu}$, and $\rho=\nabla_{\frac{1}{q}}$ in the inequalities (22), we get the desired result. □

REMARK 4. If $\sigma,\tau,\rho$ are three arbitrary means on $[0,+\infty)$, then with a similar strategy as in the proof Lemma 2, we get

$$a \leqslant \sqrt{a(a\,\sigma\, b)\rho b(a\,\tau\, b)} \leqslant b \tag{27}$$

for all positive real numbers $a,b$ such that $a \leqslant b$. To see this, observe that by using the inequalities in (24), we have

$$a^2 \leqslant a(a\,\sigma\, b)\rho b(a\,\tau\, b) \leqslant b^2$$

for all positive real numbers $a,b$ such that $a \leqslant b$. By squaring the above inequalities, we get the desired result. Moreover, the inequalities (27) yield that

$$\left|\langle x,y\rangle\right| \leqslant \sqrt{\left[\left|\langle x,y\rangle\right|\left(\left|\langle x,y\rangle\right|\sigma\,\|x\|\|y\|\right)\right]\rho\,\left[\|x\|\|y\|\left(\left|\langle x,y\rangle\right|\tau\,\|x\|\|y\|\right)\right]} \leqslant \|x\|\|y\|$$

for all $x,y \in \mathscr{H}$ and three arbitrary means $\sigma,\tau,\rho$ on $[0,+\infty)$.

REMARK 5. If $\sigma_1,\tau_1,\rho_1,\sigma_2,\tau_2,\rho_2,\omega$ are arbitrary means on $[0,+\infty)$, then by a similar proof to that of Lemma 1, the inequality (11) can be extended as follows

$$a \leqslant \big((a\,\sigma_1\, b)\rho_1(a\,\tau_1\, b)\big)\,\omega\big((a\,\sigma_2\, b)\rho_2(a\,\tau_2\, b)\big) \leqslant b$$

for all positive real numbers $a,b$ such that $a \leqslant b$. Hence, by using the consent of above inequality, the obtained results can be extended and refined.

## 3. Some applications for operators

In this section, we establish and generalize new numerical radius upper bounds for Hilbert space operators. For instance, we present a refinement of the second inequality in (4).

To prove our numerical radius inequalities, we need several known lemmas.

LEMMA 3. [3] *If* $S,T \in \mathscr{B}(\mathscr{H})$ *are positive, then*

1. $\|(S+T)^r\| \leqslant \|S^r+T^r\|$ *for* $0<r\leqslant 1$*;*

2. $\|(S+T)^r\| \leqslant 2^{r-1}\|S^r+T^r\|$ *for* $r \geqslant 1$.

The next lemma is McCarthy's inequality for positive operators.

LEMMA 4. [8] (McCarthy's inequality) *Let* $T \in \mathscr{B}(\mathscr{H})$ *be positive. Then for all unit vectors* $x \in \mathscr{H}$, *we have*

$$\langle Tx,x\rangle^r \leqslant \langle T^r x,x\rangle,$$

*where* $r \geqslant 1$. *This inequality is reversed for* $0 < r \leqslant 1$.

In the following lemma, we give the mixed Schwarz inequality, which can be found in [7].

LEMMA 5. *Let* $T \in \mathscr{B}(\mathscr{H})$ *and let* $x,y \in \mathscr{H}$. *Then*

$$|\langle Tx,y\rangle|^2 \leqslant \left\langle |T|^{2s}x,x\right\rangle \left\langle |T^*|^{2(1-s)}y,y\right\rangle \quad \textit{for all } 0 \leqslant s \leqslant 1.$$

Now, we obtain the first result of this section, which is a refinement of the inequality (8).

THEOREM 4. *Let* $S,T \in \mathscr{B}(\mathscr{H})$. *Then*

$$\begin{aligned} w^r(S^*T) &\leqslant \frac{1}{2^{r\mu}p} w^{r(1-\mu)}(S^*T)\left\||T|^2+|S|^2\right\|^{r\mu} + \frac{1}{2^{r(1-\nu)}q} w^{r\nu}(S^*T)\left\||T|^2+|S|^2\right\|^{r(1-\nu)} \\ &\leqslant \frac{1}{2}\left\||T|^{2r}+|S|^{2r}\right\| \end{aligned}$$

*for all real numbers* $r \geqslant 1$, $\mu,\nu \in [0,1]$, *and* $p,q > 0$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.* Assume that $r \geqslant 1$.

$$\begin{aligned} w^r(S^*T) &= \frac{1}{p} w^{r(1-\mu)}(S^*T) w^{r\mu}(S^*T) + \frac{1}{q} w^{r\nu}(S^*T) w^{r(1-\nu)}(S^*T) \\ &\leqslant \frac{1}{2^{r\mu}p} w^{r(1-\mu)}(S^*T)\left\||T|^2+|S|^2\right\|^{r\mu} + \frac{1}{2^{r(1-\nu)}q} w^{r\nu}(S^*T)\left\||T|^2+|S|^2\right\|^{r(1-\nu)} \\ &\qquad \text{(by the inequality (8) for } r=1). \end{aligned}$$

Then, get the first inequality. For the second inequality, by using the inequality (8), we have

$$\begin{aligned} &\frac{1}{2^{r\mu}p} w^{r(1-\mu)}(S^*T)\left\||T|^2+|S|^2\right\|^{r\mu} + \frac{1}{2^{r(1-\nu)}q} w^{r\nu}(S^*T)\left\||T|^2+|S|^2\right\|^{r(1-\nu)} \\ &\leqslant \left(\frac{1}{2^{r\mu}p}\right) \frac{\left\||T|^2+|S|^2\right\|^{r(1-\mu)}}{2^{r(1-\mu)}} \left\||T|^2+|S|^2\right\|^{r\mu} \\ &\quad + \left(\frac{1}{2^{r(1-\nu)}q}\right) \frac{\left\||T|^2+|S|^2\right\|^{r\nu}}{2^{r\nu}} \left\||T|^2+|S|^2\right\|^{r(1-\nu)} \\ &\qquad \text{(by the inequality (8) for } r=1) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2^r}\left\| |T|^2+|S|^2 \right\|^r \\
&= \frac{1}{2^r}\left\| \left(|T|^2+|S|^2\right)^r \right\| \qquad \text{(by the functional calculus)} \\
&\leqslant \frac{1}{2}\left\| |T|^{2r}+|S|^{2r} \right\| \qquad \text{(by Lemma 3(2) for all } r\geqslant 1),
\end{aligned}
$$

as required. □

By taking $\mu = \nu = \frac{1}{2}$ in Theorem 4, we get the next result, which is a refinement of the inequality (8) for $r \geqslant 1$.

COROLLARY 5. *Let $S,T \in \mathscr{B}(\mathscr{H})$. Then*

$$
w^r(S^*T) \leqslant \frac{1}{2^{\frac{r}{2}}} w^{\frac{r}{2}}(S^*T)\left\| |T|^2+|S|^2 \right\|^{\frac{r}{2}} \leqslant \frac{1}{2}\left\| |T|^{2r}+|S|^{2r} \right\| \tag{28}
$$

*for all $r \geqslant 1$.*

In the next theorem, we present an interpolation of the inequality (8) as follows.

THEOREM 5. *Let $S,T \in \mathscr{B}(\mathscr{H})$ and $r \geqslant 1$. Then*

$$
\begin{aligned}
w^r(S^*T) &\leqslant \left((1-\alpha)w^{rt}(S^*T)+\frac{\alpha}{2^t}\left\| |T|^{2rt}+|S|^{2rt} \right\|\right)^{\frac{1}{t}} \sharp_\nu \\
&\quad \times \left((1-\beta)w^{rs}(S^*T)+\frac{\beta}{2^s}\left\| |T|^{2rs}+|S|^{2rs} \right\|\right)^{\frac{1}{s}} \\
&\leqslant \frac{1}{2}\left\| |T|^{2rt}+|S|^{2rt} \right\|^{\frac{1-\nu}{t}}\left\| |T|^{2rs}+|S|^{2rs} \right\|^{\frac{\nu}{s}}
\end{aligned}
$$

*for all $\alpha,\beta,\nu \in [0,1]$ and $0 \leqslant s,t \leqslant 1$.*

*Proof.* Assume that $x,y \in \mathscr{H}$ and $r \geqslant 1$. If we replace $|\langle x,y\rangle|$ by $|\langle x,y\rangle|^r$ and $\|x\|\|y\|$ by $\|x\|^r\|y\|^r$ in the first inequality in (18), respectively, we have

$$
|\langle x,y\rangle|^r \leqslant \left((1-\alpha)|\langle x,y\rangle|^{rt}+\alpha\|x\|^{rt}\|y\|^{rt}\right)^{\frac{1}{t}} \sharp_\nu \left((1-\beta)|\langle x,y\rangle|^{rs}+\beta\|x\|^{rs}\|y\|^{rs}\right)^{\frac{1}{s}}.
$$

Let $x \in \mathscr{H}$ be a unit vector. Now, replacing $x$ by $Tx$ and $y$ by $Sx$ in the above inequality, we get

$$
\begin{aligned}
&|\langle S^*Tx,x\rangle|^r \\
&\leqslant \left((1-\alpha)|\langle S^*Tx,x\rangle|^{rt}+\alpha\|Tx\|^{rt}\|Sx\|^{rt}\right)^{\frac{1}{t}} \sharp_\nu \left((1-\beta)|\langle S^*Tx,x\rangle|^{rs}+\beta\|Tx\|^{rs}\|Sx\|^{rs}\right)^{\frac{1}{s}}.
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
\|Tx\|^r\|Sx\|^r &= \langle |T|^2x,x\rangle^{\frac{r}{2}}\langle |S|^2x,x\rangle^{\frac{r}{2}} \\
&= \frac{1}{2}\left(\langle |T|^2x,x\rangle^r+\langle |S|^2x,x\rangle^r\right) \quad \text{(by the arithmetic geometric inequality)}
\end{aligned}
$$

$$\leqslant \frac{1}{2}\langle |T|^{2r}+|S|^{2r}x,x\rangle \quad \text{(by Lemma 4)}$$
$$\leqslant \frac{1}{2}\left\||T|^{2r}+|S|^{2r}\right\| \qquad \text{for all } r\geqslant 1. \tag{29}$$

Hence,

$$\begin{aligned}
&\left|\langle S^*Tx,x\rangle\right|^r\\
&\leqslant \Big((1-\alpha)|\langle S^*Tx,x\rangle|^{rt}+\alpha\|Tx\|^{rt}\|Sx\|^{rt}\Big)^{\frac{1}{t}}\sharp_\nu\Big((1-\beta)|\langle S^*Tx,x\rangle|^{rs}+\beta\|Tx\|^{rs}\|Sx\|^{rs}\Big)^{\frac{1}{s}}\\
&\leqslant \Big((1-\alpha)|\langle S^*Tx,x\rangle|^{rt}+\frac{\alpha}{2^t}\left\||T|^{2r}+|S|^{2r}\right\|^t\Big)^{\frac{1}{t}}\sharp_\nu\\
&\quad\times\Big((1-\beta)|\langle S^*Tx,x\rangle|^{rs}+\frac{\beta}{2^s}\left\||T|^{2r}+|S|^{2r}\right\|^s\Big)^{\frac{1}{s}}\\
&\qquad\qquad\qquad\qquad\text{(by the inequality (29))}\\
&\leqslant \Big((1-\alpha)w^{rt}(S^*T)+\frac{\alpha}{2^t}\left\||T|^{2r}+|S|^{2r}\right\|^t\Big)^{\frac{1}{t}}\sharp_\nu\\
&\quad\times\Big((1-\beta)w^{rs}(S^*T)+\frac{\beta}{2^s}\left\||T|^{2r}+|S|^{2r}\right\|^s\Big)^{\frac{1}{s}}\\
&=\Big((1-\alpha)w^{rt}(S^*T)+\frac{\alpha}{2^t}\left\|\left(|T|^{2r}+|S|^{2r}\right)^t\right\|\Big)^{\frac{1}{t}}\sharp_\nu\\
&\quad\times\Big((1-\beta)w^{rs}(S^*T)+\frac{\beta}{2^s}\left\|\left(|T|^{2r}+|S|^{2r}\right)^s\right\|\Big)^{\frac{1}{s}}\\
&\leqslant \Big((1-\alpha)w^{rt}(S^*T)+\frac{\alpha}{2^t}\left\||T|^{2rt}+|S|^{2rt}\right\|\Big)^{\frac{1}{t}}\sharp_\nu\\
&\quad\times\Big((1-\beta)w^{rs}(S^*T)+\frac{\beta}{2^s}\left\||T|^{2rs}+|S|^{2rs}\right\|\Big)^{\frac{1}{s}}\\
&\qquad\qquad\qquad\qquad\text{(by Lemma 3(1) for } 0\leqslant s,t\leqslant 1).
\end{aligned}$$

Then, by taking the supremum over all unit vectors $x\in\mathscr{H}$, we get the first inequality. For the second inequality, applying the inequality (8), we have

$$\begin{aligned}
&\leqslant \Big((1-\alpha)w^{rt}(S^*T)+\frac{\alpha}{2^t}\left\||T|^{2rt}+|S|^{2rt}\right\|\Big)^{\frac{1}{t}}\sharp_\nu\\
&\quad\times\Big((1-\beta)w^{rs}(S^*T)+\frac{\beta}{2^s}\left\||T|^{2rs}+|S|^{2rs}\right\|\Big)^{\frac{1}{s}}\\
&\leqslant \Big(\Big(\frac{1-\alpha}{2^t}\Big)\left\||T|^{2r}+|S|^{2r}\right\|^t+\frac{\alpha}{2^t}\left\||T|^{2rt}+|S|^{2rt}\right\|\Big)^{\frac{1}{t}}\sharp_\nu\\
&\quad\times\Big(\Big(\frac{1-\beta}{2^s}\Big)\left\||T|^{2r}+|S|^{2r}\right\|^s+\frac{\beta}{2^s}\left\||T|^{2rs}+|S|^{2rs}\right\|\Big)^{\frac{1}{s}}\\
&\qquad\qquad\qquad\qquad\text{(by the inequality (28) for } r\geqslant 1)
\end{aligned}$$

$$\leqslant \left( \left( \frac{1-\alpha}{2^t} \right) \left\| |T|^{2rt} + |S|^{2rt} \right\| + \frac{\alpha}{2^t} \left\| |T|^{2rt} + |S|^{2rt} \right\| \right)^{\frac{1}{t}} \sharp_\nu$$

$$\times \left( \left( \frac{1-\beta}{2^s} \right) \left\| |T|^{2rs} + |S|^{2rs} \right\| + \frac{\beta}{2^s} \left\| |T|^{2rs} + |S|^{2rs} \right\| \right)^{\frac{1}{s}}$$

(by Lemma 3(1) for $0 \leqslant s,t \leqslant 1$)

$$= \left( \frac{1}{2^t} \left\| |T|^{2rt} + |S|^{2rt} \right\| \right)^{\frac{1}{t}} \sharp_\nu \left( \frac{1}{2^s} \left\| |T|^{2rs} + |S|^{2rs} \right\| \right)^{\frac{1}{s}}$$

$$= \frac{1}{2} \left\| |T|^{2rt} + |S|^{2rt} \right\|^{\frac{1-\nu}{t}} \left\| |T|^{2rs} + |S|^{2rs} \right\|^{\frac{\nu}{s}},$$

as required. □

By taking $s = t = 1$ in Theorem 5, we have a refinement of the inequality (8) as follows.

COROLLARY 6. *Let $S,T \in \mathscr{B}(\mathscr{H})$ and $r \geqslant 1$. Then*

$$\begin{aligned} w^r(S^*T) &\leqslant \left( (1-\alpha) w^r(S^*T) + \frac{\alpha}{2} \left\| |T|^{2r} + |S|^{2r} \right\| \right) \sharp_\nu \\ &\quad \times \left( (1-\beta) w^r(S^*T) + \frac{\beta}{2} \left\| |T|^{2r} + |S|^{2r} \right\| \right) \\ &\leqslant \frac{1}{2} \left\| |T|^{2r} + |S|^{2r} \right\| \end{aligned}$$

*for all $\alpha, \beta, \nu \in [0,1]$.*

In the next theorem, we obtain an upper bound for the numerical radius, which is an extension of the inequality (6).

THEOREM 6. *Let $T \in \mathscr{B}(\mathscr{H})$ and $r \geqslant 1$. Then*

$$\begin{aligned} w^{2r}(T) &\leqslant \frac{1}{2p} w^{2r(1-\mu)}(T) \left\| |T|^{4rs\mu} + |T^*|^{4r(1-s)\mu} \right\| \\ &\quad + \frac{1}{2q} w^{2r(1-\nu)}(T) \left\| |T|^{4rs\nu} + |T^*|^{4r(1-s)\nu} \right\| \end{aligned}$$

*for all real numbers $\mu, \nu \in [0,1]$, $0 \leqslant s \leqslant 1$, and $p,q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$.*

*Proof.* Assume that $x \in \mathscr{H}$ is a unit vector, $0 \leqslant s \leqslant 1$ and $r \geqslant 1$. Then, utilizing Lemma 5, we have

$$|\langle Tx, x \rangle|^{2r} \leqslant \left\langle |T|^{2s} x, x \right\rangle^r \left\langle |T^*|^{2(1-s)} x, x \right\rangle^r.$$

Now, by replacing $a$ by $|\langle Tx,x\rangle|^{2r}$ and $b$ by $\left\langle |T|^{2s}x,x\right\rangle^{r}\left\langle |T^*|^{2(1-s)}x,x\right\rangle^{r}$, and taking $\sigma=\sharp_\mu$, $\rho=\nabla_{\frac{1}{q}}$, and $\tau=\sharp_\nu$ in Lemma 1, we get

$$\begin{aligned}|\langle Tx,x\rangle|^{2r} &\leqslant \frac{1}{p}|\langle Tx,x\rangle|^{2r(1-\mu)}\left\langle |T|^{2s}x,x\right\rangle^{r\mu}\left\langle |T^*|^{2(1-s)}x,x\right\rangle^{r\mu}\\ &\quad+\frac{1}{q}|\langle Tx,x\rangle|^{2r(1-\nu)}\left\langle |T|^{2s}x,x\right\rangle^{r\nu}\left\langle |T^*|^{2(1-s)}x,x\right\rangle^{r\nu}\\ &\leqslant \left\langle |T|^{2s}x,x\right\rangle^{r}\left\langle |T^*|^{2(1-s)}x,x\right\rangle^{r}. \end{aligned} \tag{30}$$

Moreover, we have

$$\begin{aligned}\left\langle |T|^{2s}x,x\right\rangle^{r\mu}\left\langle |T^*|^{2(1-s)}x,x\right\rangle^{r\mu} &\leqslant \left(\left\langle |T|^{2rs}x,x\right\rangle\left\langle |T^*|^{2r(1-s)}x,x\right\rangle\right)^{\mu}\\ &\qquad \text{(by Lemma 4)}\\ &\leqslant \frac{1}{2}\left(\left\langle |T|^{2rs}x,x\right\rangle^{2}+\left\langle |T^*|^{2r(1-s)}x,x\right\rangle^{2}\right)^{\mu}\\ &\qquad \text{(by the arithmetic-geometric mean inequality)}\\ &\leqslant \frac{1}{2}\left\langle |T|^{4rs}+|T^*|^{4r(1-s)}x,x\right\rangle^{\mu}\\ &\qquad \text{(by Lemma 4).}\end{aligned}$$

Similarly, we have

$$\left\langle |T|^{2s}x,x\right\rangle^{r\nu}\left\langle |T^*|^{2(1-s)}x,x\right\rangle^{r\nu}\leqslant \frac{1}{2}\left\langle |T|^{4rs}+|T^*|^{4r(1-s)}x,x\right\rangle^{\nu}.$$

Applying the first inequality in (30) and the above inequalities, we get

$$\begin{aligned}&|\langle Tx,x\rangle|^{2r}\\ &\leqslant \frac{1}{p}|\langle Tx,x\rangle|^{2r(1-\mu)}\left\langle |T|^{2s}x,x\right\rangle^{r\mu}\left\langle |T^*|^{2(1-s)}x,x\right\rangle^{r\mu}\\ &\qquad+\frac{1}{q}|\langle Tx,x\rangle|^{2r(1-\nu)}\left\langle |T|^{2s}x,x\right\rangle^{r\nu}\left\langle |T^*|^{2(1-s)}x,x\right\rangle^{r\nu}\\ &\leqslant \frac{1}{2p}|\langle Tx,x\rangle|^{2r(1-\mu)}\left\langle |T|^{4rs}+|T^*|^{4r(1-s)}x,x\right\rangle^{\mu}\\ &\qquad+\frac{1}{2q}|\langle Tx,x\rangle|^{2r(1-\nu)}\left\langle |T|^{4rs}+|T^*|^{4r(1-s)}x,x\right\rangle^{\nu}\\ &\leqslant \frac{1}{2p}w^{2r(1-\mu)}(T)\left\||T|^{4rs}+|T^*|^{4r(1-s)}\right\|^{\mu}+\frac{1}{2q}w^{2r(1-\nu)}(T)\left\||T|^{4rs}+|T^*|^{4r(1-s)}\right\|^{\nu}\\ &\leqslant \frac{1}{2p}w^{2r(1-\mu)}(T)\left\||T|^{4rs\mu}+|T^*|^{4r(1-s)\mu}\right\|+\frac{1}{2q}w^{2r(1-\nu)}(T)\left\||T|^{4rs\nu}+|T^*|^{4r(1-s)\nu}\right\|\\ &\qquad\qquad \text{(by Lemma 3(1) for } 0\leqslant\mu,\nu\leqslant 1). \end{aligned} \tag{31}$$

Then, by taking the supremum over all unit vectors $x\in\mathscr{H}$, we get the desired inequality. □

In the following theorem, by using Theorem 6, we obtain an interpolation of the second inequality in (4).

COROLLARY 7. *Let* $T \in \mathscr{B}(\mathscr{H})$ *and* $r \geqslant 1$. *Then*

$$\begin{aligned} w^{2r}(T) &\leqslant \frac{1}{2p} w^{2r(1-\mu)}(T) \left\| |T|^{2r\mu} + |T^*|^{2r\mu} \right\| + \frac{1}{2q} w^{2r(1-\nu)}(T) \left\| |T|^{2r\nu} + |T^*|^{2r\nu} \right\| \\ &\leqslant \frac{1}{4p} \left\| |T|^{2r(1-\mu)} + |T^*|^{2r(1-\mu)} \right\| \left\| |T|^{2r\mu} + |T^*|^{2r\mu} \right\| \\ &\quad + \frac{1}{4q} \left\| |T|^{2r(1-\nu)} + |T^*|^{2r(1-\nu)} \right\| \left\| |T|^{2r\nu} + |T^*|^{2r\nu} \right\| \end{aligned} \tag{32}$$

*for all real numbers* $r \geqslant 1$, $\mu, \nu \in [0,1]$ *such that* $\frac{1}{2} \leqslant r(1-\mu), r(1-\nu)$ *and* $p, q > 0$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

*Proof.* The first inequality in (32) follows from Theorem 6 for $s = \frac{1}{2}$. For the second inequality in (32), by applying the inequality (5) and Lemma 3, respectively, we have

$$w^{2r(1-\mu)}(T) \leqslant \frac{1}{2^{2r(1-\mu)}} \| |T| + |T^*| \|^{2r(1-\mu)} \leqslant \frac{1}{2} \| |T|^{2r(1-\mu)} + |T^*|^{2r(1-\mu)} \|$$

and

$$w^{2r\nu}(T) \leqslant \frac{1}{2^{2r(1-\nu)}} \| |T| + |T^*| \|^{2r(1-\nu)} \leqslant \frac{1}{2} \left\| |T|^{2r(1-\nu)} + |T^*|^{2r(1-\nu)} \right\|$$

for all real numbers $r \geqslant 1$, $\mu, \nu \in [0,1]$ such that $\frac{1}{2} \leqslant r(1-\mu), r(1-\nu)$. Hence,

$$\begin{aligned} &\frac{1}{2p} w^{2r(1-\mu)}(T) \left\| |T|^{2r\mu} + |T^*|^{2r\mu} \right\| + \frac{1}{2q} w^{2r\nu}(T) \left\| |T|^{2r(1-\nu)} + |T^*|^{2r(1-\nu)} \right\| \\ &\leqslant \frac{1}{4p} \left\| |T|^{2r(1-\mu)} + |T^*|^{2r(1-\mu)} \right\| \left\| |T|^{2r\mu} + |T^*|^{2r\mu} \right\| \\ &\quad + \frac{1}{4q} \left\| |T|^{2r(1-\nu)} + |T^*|^{2r(1-\nu)} \right\| \| |T|^{2r\nu} + |T^*|^{2r\nu} \|, \end{aligned}$$

as required. □

REMARK 6. Let $T \in \mathscr{B}(\mathscr{H})$ and $r \geqslant 1$. It follows from the inequality (32) for $\mu = \nu = \frac{1}{2}$ that

$$\begin{aligned} w^{2r}(T) &\leqslant \frac{1}{2p} w^r(T) \| |T|^r + |T^*|^r \| + \frac{1}{2q} w^r(T) \| |T|^r + |T^*|^r \| \\ &= \frac{1}{2} w^r(T) \| |T|^r + |T^*|^r \| \end{aligned}$$

$$\leqslant \frac{1}{4}\||T|^r+|T^*|^r\|^2 \quad \text{(by the inequality (8) for } r\geqslant 1)$$
$$\leqslant \frac{1}{2}\left\||T|^{2r}+|T^*|^{2r}\right\| \quad \text{(by Lemma 3(2))},$$

which is an extension of the inequalities (4) and (5) for the powers that are equal or bigger than $2$.

In the next theorem, we present an extension of the inequality (6) for $r \geqslant 2$.

THEOREM 7. *Let $T \in \mathscr{B}(\mathscr{H})$ and $r \geqslant 1$. Then*

$$\begin{aligned} w^{2r}(T) &\leqslant \frac{1}{2^{\mu+1}p} w^{r(1-\mu)}(T^2)\left\||T|^{2r}+|T^*|^{2r}\right\|^{\mu} + \frac{1}{2^{2-\nu}q} w^{r\nu}(T^2)\left\||T|^{2r}+|T^*|^{2r}\right\|^{1-\nu} \\ &\quad + \frac{1}{4}\left\||T|^{2r}+|T^*|^{2r}\right\| \\ &\leqslant \frac{1}{2}\left\||T|^{2r}+|T^*|^{2r}\right\| \end{aligned}$$

*for $\mu,\nu \in [0,1]$ and $p,q>0$ with $\frac{1}{p}+\frac{1}{q}=1$.*

*Proof.* Assume that $\mu,\nu\in[0,1]$ and $x\in\mathscr{H}$ is a unit vector. Applying the first inequality in (20) and replacing $x$ by $Tx$, $e$ by $x$, and $y$ by $T^*x$, we get

$$\begin{aligned} &\left|\langle Tx,x\rangle\right|^{2r} \\ &= \left|\langle Tx,x\rangle\langle x,T^*x\rangle\right|^{r} \\ &\leqslant \frac{1}{2}\Big(\frac{1}{p}\left|\langle Tx,T^*x\rangle\right|^{r(1-\mu)}\|Tx\|^{r\mu}\|T^*x\|^{r\mu} + \frac{1}{q}\left|\langle Tx,T^*x\rangle\right|^{r\nu}\|Tx\|^{r(1-\nu)}\|T^*x\|^{r(1-\nu)} \\ &\quad + \|Tx\|^r\|T^*x\|^r\Big) \\ &\leqslant \frac{1}{2}\Big(\frac{1}{2^{\mu}p} w^{r(1-\mu)}(T^2)\left\||T|^{2r}+|T^*|^{2r}\right\|^{\mu} + \frac{1}{2^{1-\nu}q} w^{r\nu}(T^2)\left\||T|^{2r}+|T^*|^{2r}\right\|^{1-\nu} \\ &\quad + \frac{1}{2}\left\||T|^{2r}+|T^*|^{2r}\right\|\Big) \qquad \text{(by the inequality (29))} \\ &= \frac{1}{2^{\mu+1}p} w^{r(1-\mu)}(T^2)\left\||T|^{2r}+|T^*|^{2r}\right\|^{\mu} + \frac{1}{2^{2-\nu}q} w^{r\nu}(T^2)\left\||T|^{2r}+|T^*|^{2r}\right\|^{1-\nu} \\ &\quad + \frac{1}{4}\left\||T|^{2r}+|T^*|^{2r}\right\|. \end{aligned}$$

Taking the supremum over all unit vectors $x\in\mathscr{H}$, we get the first inequality. For the second inequality, applying the inequality (8), we have

$$
\begin{aligned}
&\frac{1}{2^{\mu+1}p} w^{r(1-\mu)}(T^2)\left\||T|^{2r}+|T^*|^{2r}\right\|^{\mu}+\frac{1}{2^{2-\nu}q} w^{r\nu}(T^2)\left\||T|^{2r}+|T^*|^{2r}\right\|^{1-\nu} \\
&\quad+\frac{1}{4}\left\||T|^{2r}+|T^*|^{2r}\right\| \\
&\leqslant\left(\frac{1}{2^{\mu+1}p}\right)\frac{\left\||T|^{2r}+|T^*|^{2r}\right\|^{1-\mu}}{2^{1-\mu}}\left\||T|^{2r}+|T^*|^{2r}\right\|^{\mu} \\
&\quad+\left(\frac{1}{2^{2-\nu}q}\right)\frac{\left\||T|^{2r}+|T^*|^{2r}\right\|^{\nu}}{2^{\nu}}\left\||T|^{2r}+|T^*|^{2r}\right\|^{1-\nu}+\frac{1}{4}\left\||T|^{2r}+|T^*|^{2r}\right\| \\
&=\frac{1}{4p}\left\||T|^{2r}+|T^*|^{2r}\right\|+\frac{1}{4q}\left\||T|^{2r}+|T^*|^{2r}\right\|+\frac{1}{4}\left\||T|^{2r}+|T^*|^{2r}\right\| \\
&=\frac{1}{2}\left\||T|^{2r}+|T^*|^{2r}\right\|,
\end{aligned}
$$

as required. □

Theorem 7, for $r=1$, yields a refinement of the inequality (6) as follows.

COROLLARY 8. *Let* $T\in\mathscr{B}(\mathscr{H})$. *Then*

$$
\begin{aligned}
w^2(T)&\leqslant\frac{1}{2^{\mu+1}p} w^{(1-\mu)}(T^2)\left\||T|^{2}+|T^*|^{2}\right\|^{\mu}+\frac{1}{2^{2-\nu}q} w^{\nu}(T^2)\left\||T|^{2}+|T^*|^{2}\right\|^{1-\nu} \\
&\quad+\frac{1}{4}\left\||T|^{2}+|T^*|^{2}\right\| \\
&\leqslant\frac{1}{2}\left\||T|^{2}+|T^*|^{2}\right\|
\end{aligned}
$$

*for* $\mu,\nu\in[0,1]$ *and* $p,q>0$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

REMARK 7. If $f$ is an increasing function on $[0,+\infty)$, then by using the inequality (8) for $r=1$, we have

$$
f\big(w(S^*T)\big)\leqslant f\left(\frac{\left\||T|^2+|S|^2\right\|}{2}\right)\qquad\text{for all } S,T\in\mathscr{B}(\mathscr{H}). \tag{33}
$$

Now, if we replace $a$ by $f\big(w(S^*T)\big)$ and $b$ by $f\left(\frac{\left\||T|^2+|S|^2\right\|}{2}\right)$ in the inequalities (11), then we get

$$
\begin{aligned}
f\big(w(S^*T)\big)&\leqslant\left[f\big(w(S^*T)\big)\,\sigma\, f\left(\frac{\left\||T|^2+|S|^2\right\|}{2}\right)\right]\rho\left[f\big(w(S^*T)\big)\,\tau\, f\left(\frac{\left\||T|^2+|S|^2\right\|}{2}\right)\right] \\
&\leqslant f\left(\frac{\left\||T|^2+|S|^2\right\|}{2}\right)
\end{aligned}
$$

for all three arbitrary means $\sigma,\tau,\rho$ and operators $S,T\in\mathscr{B}(\mathscr{H})$. In particular, for $f(t)=t^r\,(r\geqslant 1)$, we have

$$\begin{aligned} w^r(S^*T) &\leqslant \left[w^r(S^*T)\,\sigma\left(\frac{\||T|^2+|S|^2\|^r}{2^r}\right)\right]\rho\left[w^r(S^*T)\,\tau\left(\frac{\||T|^2+|S|^2\|^r}{2^r}\right)\right] \\ &\leqslant \left(\frac{\||T|^2+|S|^2\|}{2}\right)^r \\ &= \frac{\|(|T|^2+|S|^2)^r\|}{2^r} \qquad \text{(by the functional calculus)} \\ &\leqslant \frac{1}{2}\||T|^{2r}+|S|^{2r}\| \qquad \text{(by Lemma 3(2))}. \end{aligned}$$

These inequalities contain refinements and generalizations of the inequality (8).

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## REFERENCES

[1] D. AFRAZ, R. LASHKARIPOUR, AND M. BAKHERAD, *Further norm and numerical radius inequalities for sum of Hilbert space operators*, Filomat **38** (2024), no. 9, 3235–3242.

[2] M. W. ALOMARI, *On Cauchy-Schwarz type inequalities and applications to numerical radius inequalities*, Ric. Mat. **73** (2024), no. 3, 1493–1510.

[3] J. AUJLA AND F. SILVA, *Weak majorization inequalities and convex functions*, Linear Algebra Appl. **369** (2003), 217–233.

[4] P. BHUNIA, S. S. DRAGOMIR, M. S. MOSLEHIAN, AND K. PAUL, *Lectures on Numerical Radius Inequalities*, Infosys Science Foundation Series in Mathematical Sciences. Springer, Cham, 2022.

[5] M. L. BUZANO, *Generalizzazione della diseguaglianza di Cauchy-Schwarz* (Italian), Rend Sem Mat Univ e Politech Torino **31** (1974), 405–409.

[6] S. S. DRAGOMIR, *Power inequalities for the numerical radius of a product of two operators in Hilbert spaces*, Sarajevo J. Math. **5** (2009), 269–278.

[7] T. KATO, *Notes on some inequalities for linear operators*, Math. Ann., **125** (1952), 208–212.

[8] T. FURUTA, J. MIĆIĆ HOT, J. PEČARIĆ AND Y. SEO, *Mond Pečarić method in operator inequalities*, Zagreb, 2005.

[9] F. KITTANEH, *Notes on some inequalities for Hilbert space operators*, Publ. Res. Inst. Math. Sci. **24** (1988), 283–293.

[10] F. KITTANEH, *A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix*, Studia Math. **158** (2003), no. 1, 11–17.

[11] F. KITTANEH, *Numerical radius inequalities for Hilbert space operators*, Studia Math. **168** (2005), no. 1, 73–80.

[12] F. KITTANEH AND H. R. MORADI, *Cauchy-Schwarz type inequalities and applications to numerical radius inequalities*, Math. Inequal. Appl. **23** (2020), 1117–1125.

[13] R. LASHKARIPOUR, M. BAKHERAD, AND M. HAJMOHAMADI, *Normal shape and numerical range of a real 2-Toeplitz tridiagonal matrix*, Bull. Malays. Math. Sci. Soc. **46** (2023), no. 3, Paper No. 102, 20 pp.

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